On properties of families of sets
Lecture 1

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7th Young Set Theory Workshop

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- Miller, Edwin W. On a property of families of sets. C. R. Soc. Sci. Varsovie 30, 31-38 (1937).

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- Erdős, Paul; Hajnal, András On a property of families of sets. Acta Math. Acad. Sci. Hung. 12, 87-123 (1961).

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Question:
Which families have property B?

Almost disjoint families

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Theorem (Balcar, Vojtáš, 1980)

Every non-principal ultrafilter $\mathcal{U}$ *on* ω *has an almost disjoint refinement, i.e. a family* $\mathcal{A}_{\mathcal{U}} = \{A_U : U \in \mathcal{U}\}$ *such that*

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Clearly $\chi(\mathcal{A}_{\mathcal{U}}) = \omega$: if $c : \omega \to n$, then there is $U \in \mathcal{U}$ with $c[U] = \{i\}$, and so $c[A_U] = \{i\}$.

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Theorem (Brendle, Balcar-Pazak, 2008)

If $V \subset W$ are models of ZFC, W contains a new real r, then $V \cap [\omega]^{\omega}$ has an almost disjoint refinement in W.

Thm: If $V \subset W$ are models of ZFC, W contains a new real r, and $\{x_\alpha : \alpha < \kappa\} = [\omega]^\omega \cap V$, then there is an almost disjoint family $\{a_\alpha : \alpha < \kappa\} \subset \left[\omega\right]^\omega$ in W s.t. $a_\alpha \subset x_\alpha$.

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- If $B(x)_y$ contains a perfect set then $|(B(x)_y)^W \setminus V| \geq \kappa = (2^\omega)^V$.

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- Hence $|B(x)_y| \leq \omega$ **OR** $B(x)_y$ contains a perfect set.
- If $B(x)_y$ contains a perfect set then $|(B(x)_y)^W \setminus V| \geq \kappa = (2^\omega)^V$.
- Contstruct $\{a_\alpha : \alpha < \kappa\}$ inductively. We will define γ_α, b_α as well

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Almost disjoint families with prescribed chromatic number

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Theorem ( )

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- $G_\alpha \cap G_\beta \subset \bigcup_{1 \leq i \leq n+1} (F^i_\alpha \cap F^i_\beta)$, and $F^i_\alpha \cap F^i_\beta$ is finite.
- $G_\alpha \subset D^j_\alpha$, so $(D^1_\alpha, \ldots, D^n_\alpha)$ can not prove $\chi(\mathcal{G}) \leq n$.

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- $\{F^i_\alpha : \alpha < 2^\omega\} \subset [A_i]^\omega$ almost disjont family for $1 \leq i \leq n+1$
- $\{(D^1_\alpha, \ldots, D^n_\alpha) : \alpha < 2^\omega)\}$ partitions of ω into n sets.
- for all $1 \leq i \leq n+1$ there is $1 \leq j \leq n$ s.t. $F^i_\alpha \cap D^j_\alpha$ is infinite
- there is $1 \leq j \leq n$ and there are $1 \leq i_1 < i_2 \leq n$ such that both $F^{i_1}_\alpha \cap D^j_\alpha$ and $F^{i_2}_\alpha \cap D^j_\alpha$ are infinite.
- Let $G_\alpha = D^j_\alpha \cap (F^{i_1}_\alpha \cup F^{i_2}_\alpha)$.

Claim: $\mathcal{G} = \{G_\alpha : \alpha < 2^\omega\}$ is almost disjoint and $\chi(\mathcal{G}) = n+1$.

- $G_\alpha \cap G_\beta \subset \bigcup_{1\leq i\leq n+1}(F^i_\alpha \cap F^i_\beta)$, and $F^i_\alpha \cap F^i_\beta$ is finite.
- $G_\alpha \subset D^j_\alpha$, so $(D^1_\alpha, \ldots, D^n_\alpha)$ can not prove $\chi(\mathcal{G}) \leq n$.
- $G_\alpha \cap A_{i_1} \supset D^j_\alpha \cap F^{i_1}_\alpha \neq \emptyset$ and $G_\alpha \cap A_{i_2} \supset D^j_\alpha \cap F^{i_1}_\alpha \neq \emptyset$

Almost disjoint families with prescribed chromatic number

Theorem (Erdős, Shelah, 1972)

$\forall n \geq 2$ there is an almost disjoint family $\mathcal{G} \subset [\omega]^{\omega}$ with $\chi(\mathcal{G}) = n+1$.

- Let $(A_1, \ldots, A_{n+1})$ be a partition of ω into infinite sets
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- So $(A^1, \ldots, A^{n+1}) \vdash \chi(\mathcal{G}) \leq n+1$

Disjoint sets in almost disjoint families

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Which almost disjoint families have property *B*?

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Theorem (Erdős, Shelah, 1972)
If $\mathcal{A} \subset [X]^{\geq\omega}$ is almost disjoint, and $A \cap A' \neq \emptyset$ for all $A, A' \in \mathcal{A}$, then $\chi(\mathcal{A}) = 2$.

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Proof:

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- Try to obtain a contradiction.

$\mathcal{A} \subset [X]^{\geq\omega}$ is a.d., $A \cap A' \neq \emptyset$ for $A \neq A' \in \mathcal{A}$, but $\chi(\mathcal{A}) > 2$.

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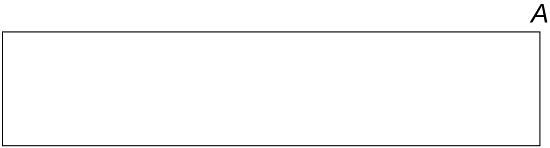

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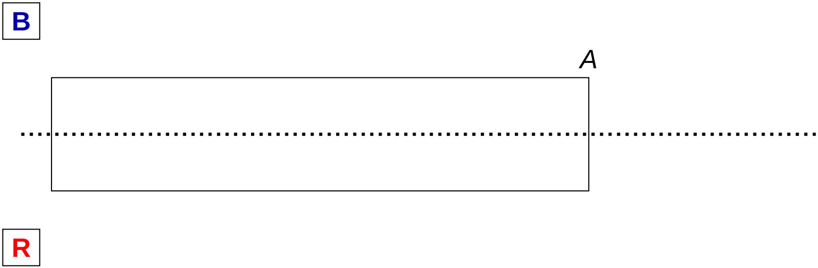

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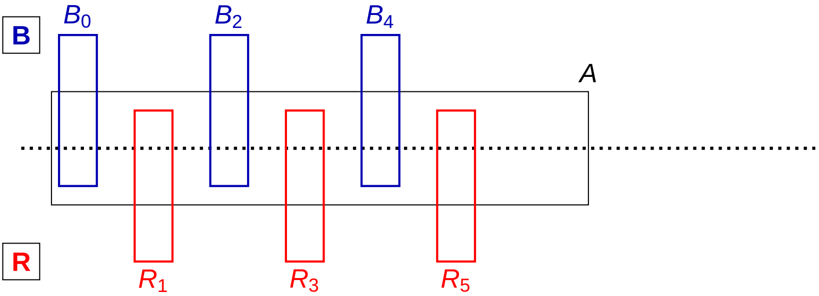

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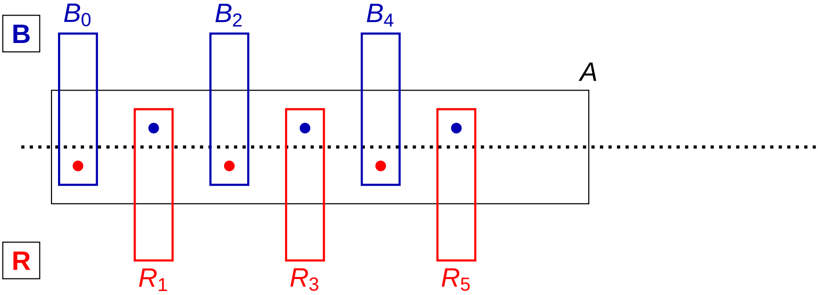

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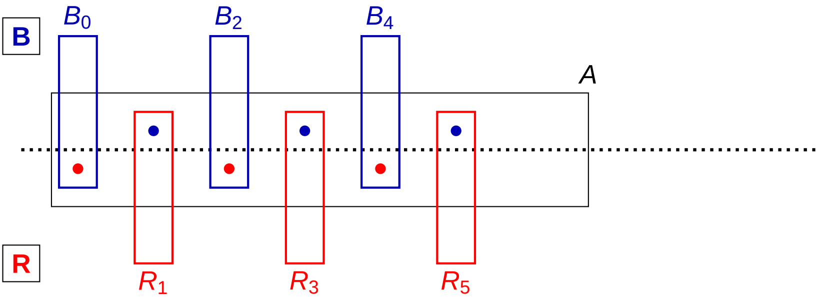

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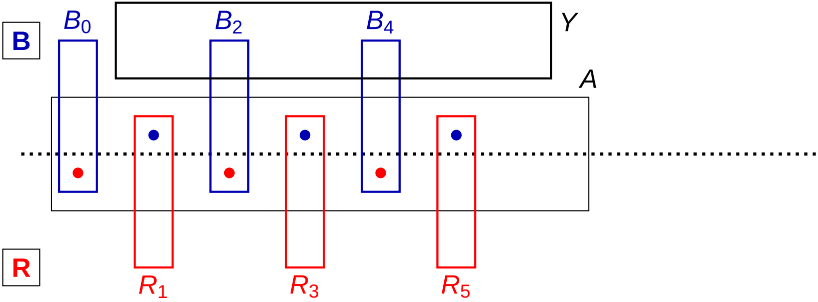

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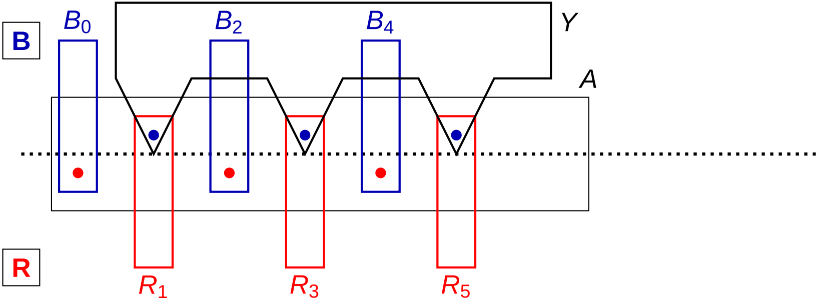

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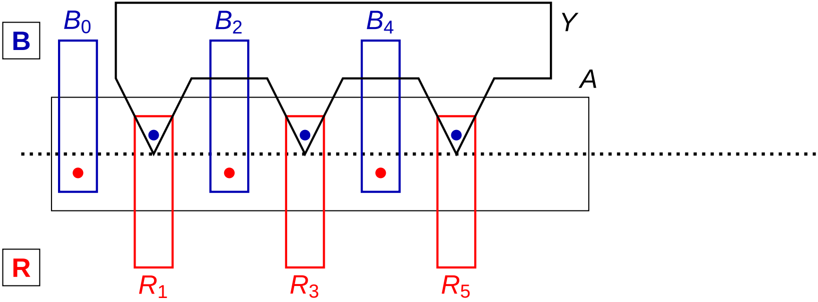

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- $\{b_{2m+1} : m \in \omega\} \subset Y$, so $Y \cap A$ is infinite. Contradiction.

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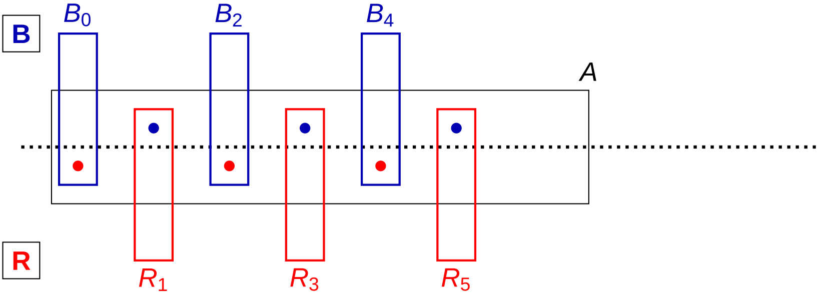

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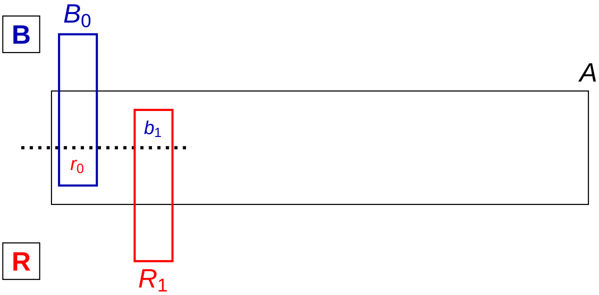

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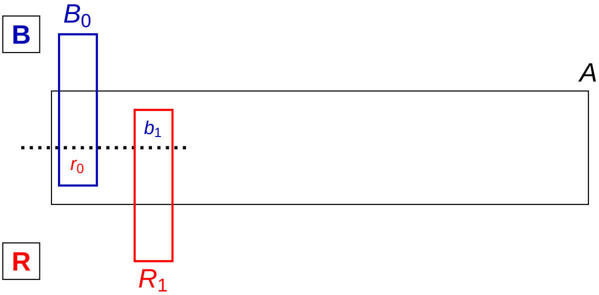

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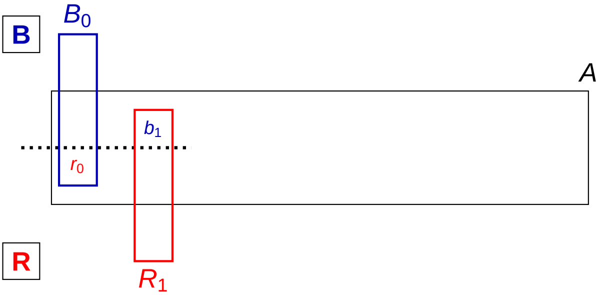

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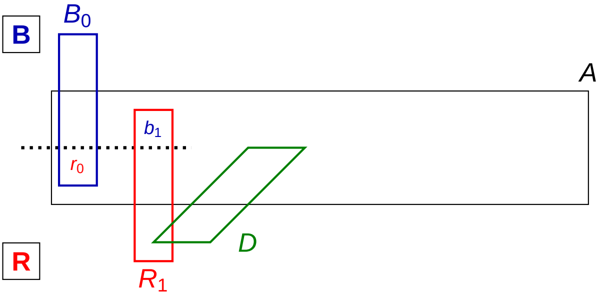

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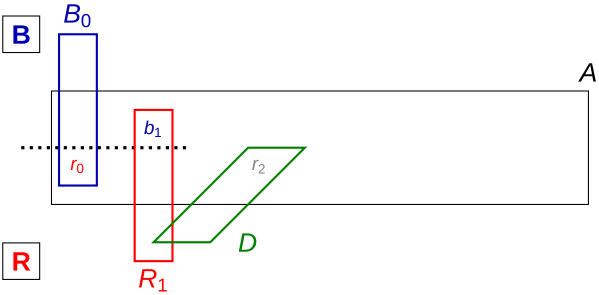

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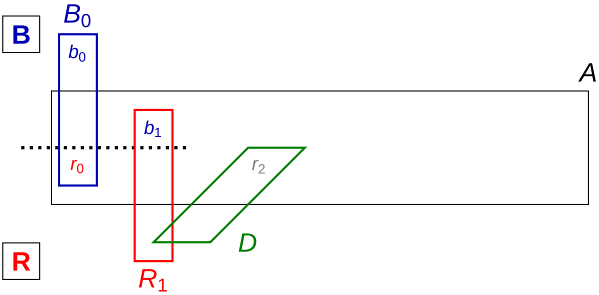

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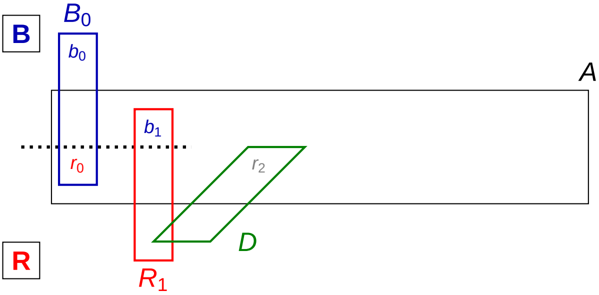

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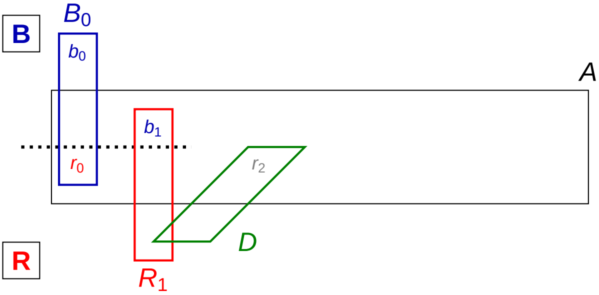

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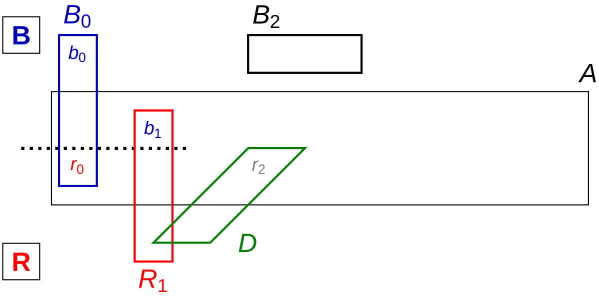

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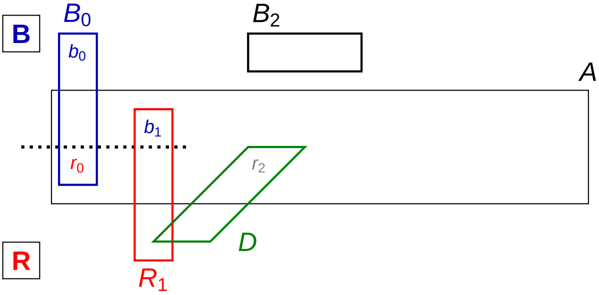

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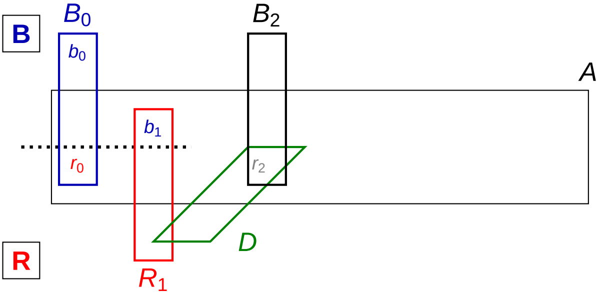

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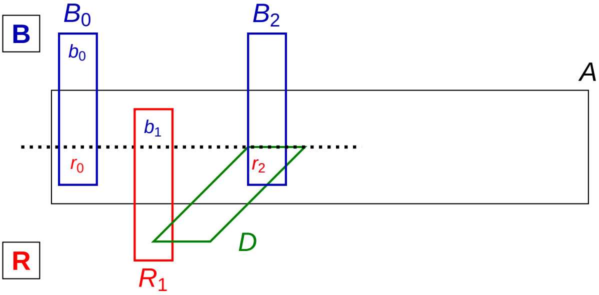

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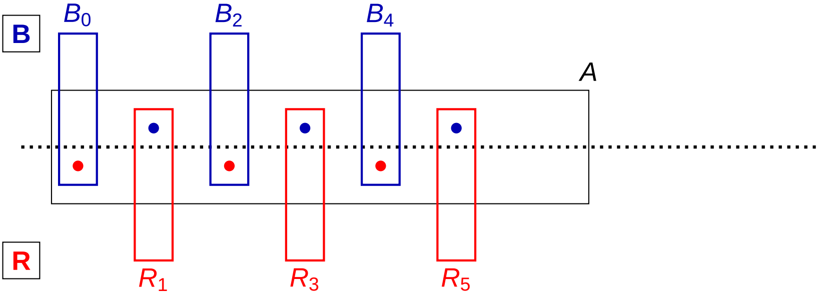

- Fix $A \in \mathcal{A}$. Color $X = \mathbf{B} \cup \mathbf{R}$, choose $B_0, R_1, B_2, R_2, \ldots$ from $\mathcal{A}$ pick r_0 and b_0 from B_0, r_1 and b_1 from R_1, ... s.t.
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- If $X \in [\mathbb{R}]^{\omega_1}$, then there is a dense-in-itself $P \in [X]^{\omega}$.
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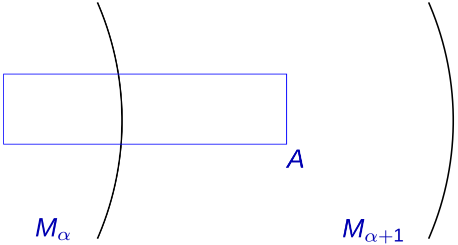

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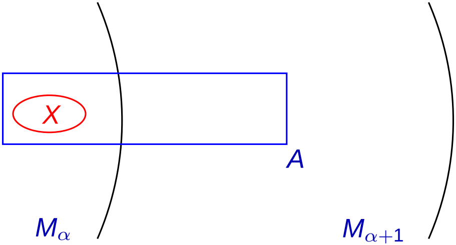

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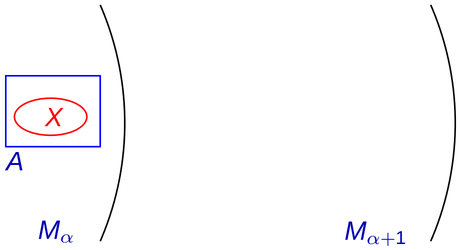

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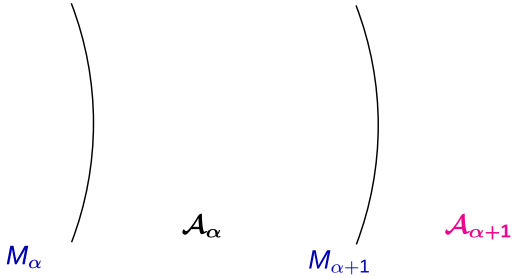

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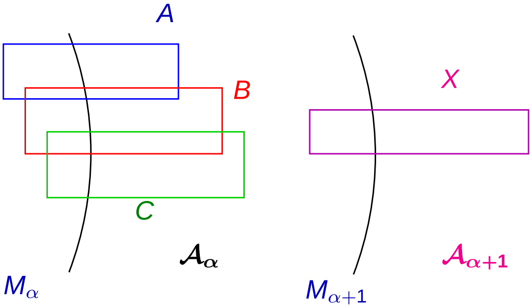

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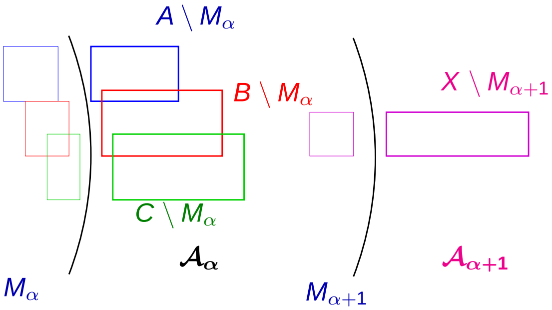

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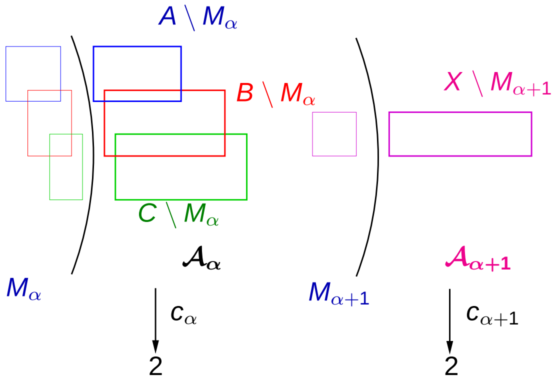

To be continued . . .