On properties of families of sets
Lecture 3

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7th Young Set Theory Workshop

Applications of elementary submodels

Introduction

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- Easy applications
- Simplified proofs
- Davies trees
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Stephen G. Simpson, *Model theoretic proof a partition theorem*, Abstracts of Contributed Papers, Notices of AMS, 17 (1970), no 6 p 964.

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- C is closed $\implies \eta \in C$. Contradiction because $\eta \in S$

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- If $|G| = \omega_1$: partition G into pieces $\{G_\alpha : \alpha < \omega_1\}$ s.t. G_α is countable and NW.
- $G \in M \prec \mathcal{H}(\theta)$ and $|M| = \omega$.
- chain of elementary submodels.
- $G \upharpoonright M = \langle V \cap M, E \cap M\rangle$ is NW.

Simplified proofs: Nash-Williams theorem

A cut of a graph $G = \langle V, E \rangle$ is $E \cap [A, V \setminus A]$ for some $\emptyset \neq A \subsetneq V$.

Theorem (Nash-Williams)

A graph G is decomposable into circles if and only if it has no odd cut.

- Def: G is NW iff it does not have odd cuts.
- If G is finite: trivial.
- If G is countable: straightforward.
- If $|G| = \omega_1$: partition G into pieces $\{G_\alpha : \alpha < \omega_1\}$ s.t. G_α is countable and NW.
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- $G \restriction M = \langle V \cap M, E \cap M \rangle$ is NW.
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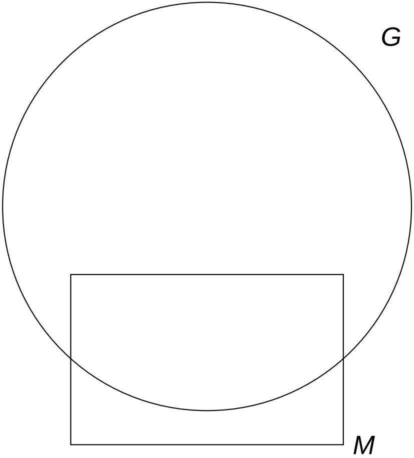

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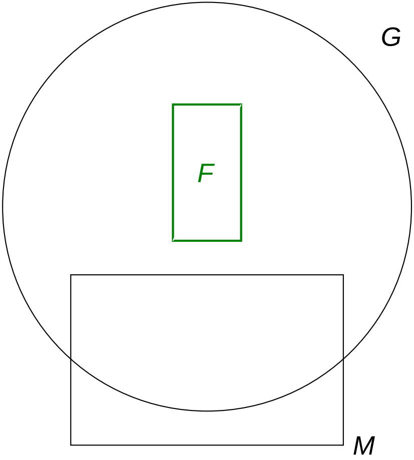

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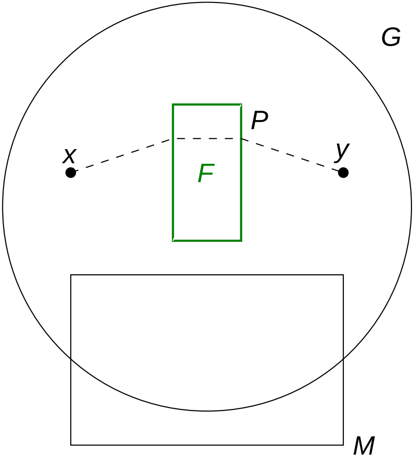

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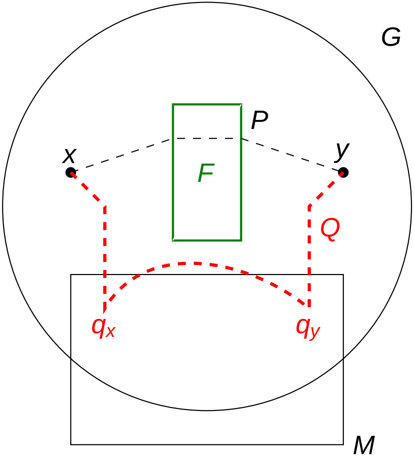

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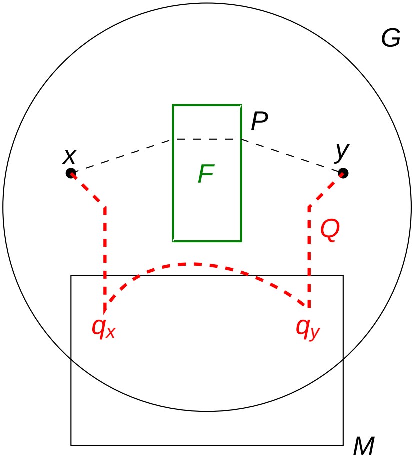

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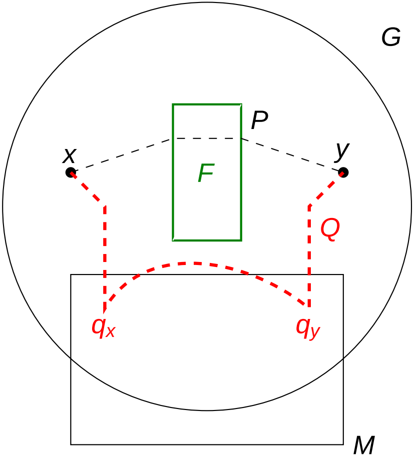

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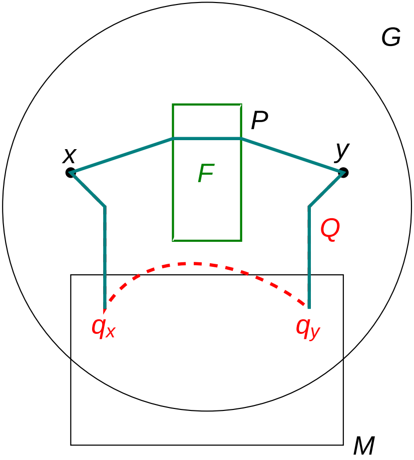

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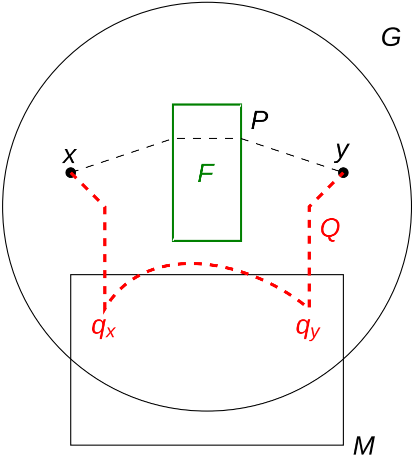

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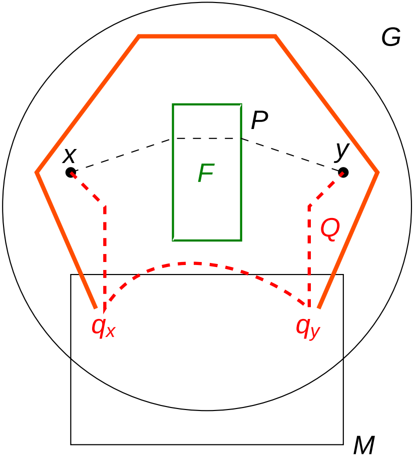

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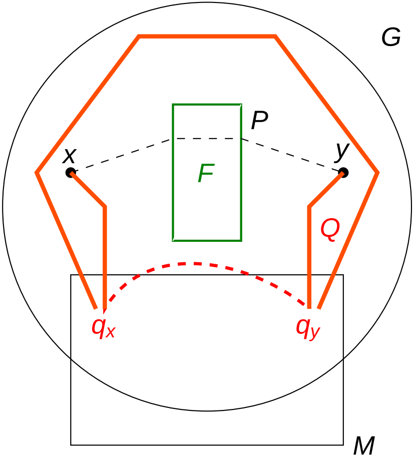

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- G_α is decomposable into circles.
- $E(G) = \cup^* \{E(G_\alpha) : \alpha < \omega_1\} \Longrightarrow G$ is decomposable into circles

Davies trees: the beginning

Covering of the plain

Theorem (Davies, 1963)

$\mathbb{R}^2$ is the union of countably many rotations of functions.

If $\alpha_0, \alpha_1, \ldots$ are pairwise different angles between 0 and π, then there are function $f_0, f_1 \ldots$ such that

$$\mathbb{R}^2 = \bigcup_{n \in \omega} R_{(\alpha_n)}[f_n],$$

where $R_{(\alpha)} : \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation by α degree around the origin.

Davies Trees

Davies Trees

Definition
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- $\kappa \subset \bigcup_{\alpha<\kappa} M_\alpha$, *and* $x \in M_\alpha$ *for all* $\alpha < \kappa$.
- *For all* $\alpha < \kappa$ *there is a natural number* $n(\alpha)$ *s.t.*

$$M_{<\alpha} = \bigcup_{i<n(\alpha)} N_i, \text{ for some } N_i \prec \mathcal{H}(\theta),$$

where $M_{<\alpha} = \bigcup_{\zeta<\alpha} M_\zeta$.

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Theorem (Davies)

$\forall \kappa > \omega \; \forall x$ there is a Davies sequence for κ over x.

If $\kappa = \omega_n$, then we can assume that $n(\alpha) = n$ for all $\alpha < \omega_n$, i.e. $M_{<\alpha} = \bigcup_{i<n} N_i$, for some $N_i \prec \mathcal{H}(\theta)$.

Davies trees

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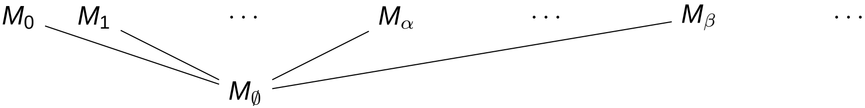

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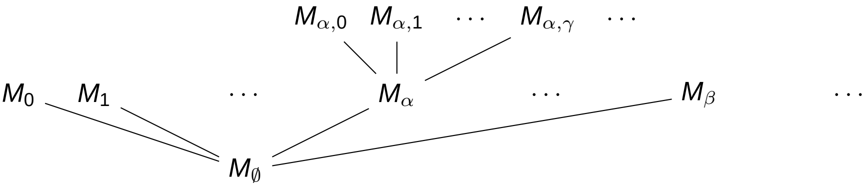

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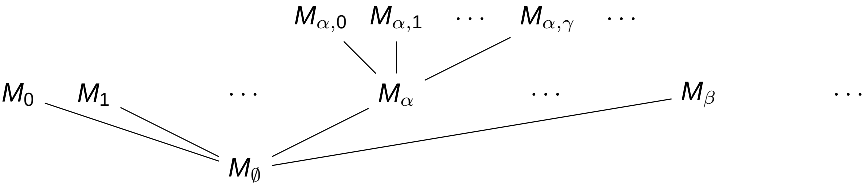

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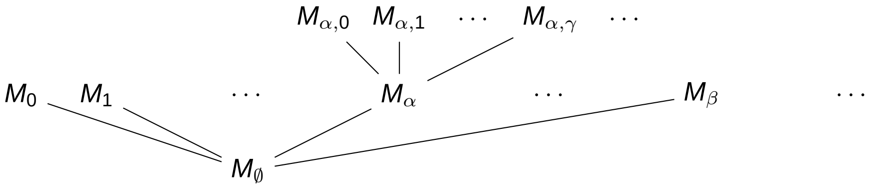

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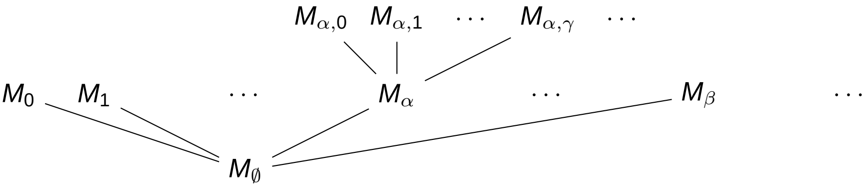

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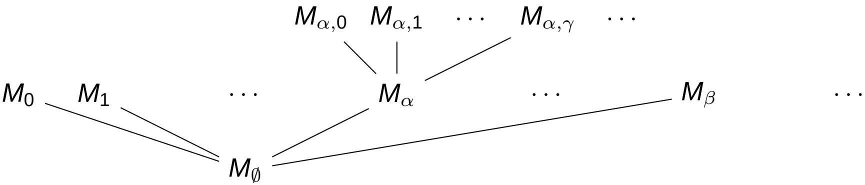
- Claim: $\langle M_{t_\alpha} : \alpha < \kappa \rangle$ is a Davies sequence for κ.



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- We defined a tree T and for all $t \in T$ we defined $M_t \prec \mathcal{H}(\theta)$
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⋆ $M_t = \bigcup_{t \subseteq t' \in leaf(T)} M_{t'}$

- Write $s <_{lex} t$ iff s is lexicographically smaller than s.
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n-almost disjoint families: sharper theorems

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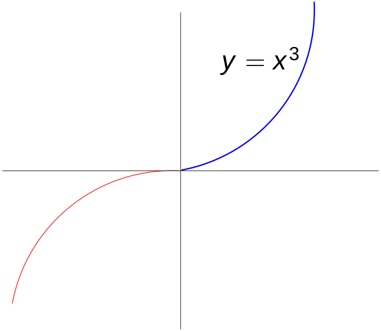

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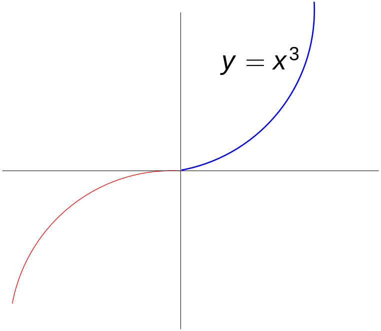

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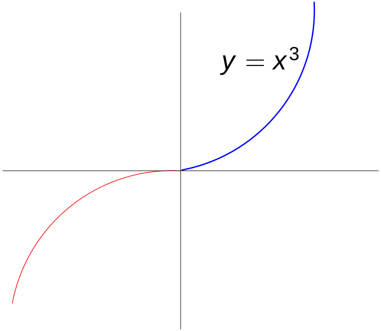

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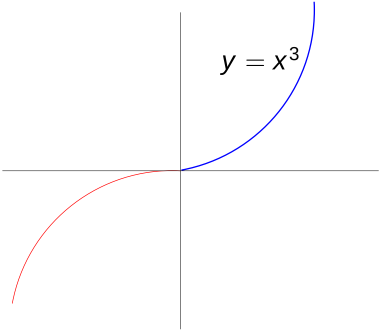

Let $\mathcal{E}_3$ be the family of lines in $\mathbb{R}^3$.

$\chi_{\mathrm{CF}}(\mathcal{E}_3) = 3$, but no constructive solution is known

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If m, d are natural numbers, then

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Yes, we need: $\chi_{\mathrm{CF}}(\left[\beth_\omega\right]^{\omega}, 2\text{-a.d.}) = \omega$.

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Conflict free coloring

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σ-Davies trees

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If GCH holds and $\square^{**}_\mu$ is true for all $\mu > \mathrm{cf}(\mu) = \omega$, then for all cardinal κ and set x there is a σ-Davies tree for κ over x.

The very weak square

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Definition (Foreman and Magidor, 1997)

For a cardinal μ, the very weak square principle for μ holds if there is a sequence $(C_\alpha)_{\alpha<\mu^+}$ and a club $D \subseteq \mu^+$ such that for every $\alpha \in D$

(v1) $C_\alpha \subseteq \alpha$, *$C_\alpha$ is unbounded in α;*

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Definition

Let $\mu > \mathrm{cf}(\mu) = \omega$. Let $\theta = \mathrm{cf}(\theta) \gg \mu$, $x \in \mathcal{H}(\theta)$. A matrix $\langle M_{\alpha,n} : \alpha < \mu^+, n < \omega \rangle$ of elementary submodels of $\mathcal{H}(\theta)$ is a strong μ-dominating matrix over x iff

(j1) *$x \in M_{\alpha,n}$, and $|M_{\alpha,n}| < \mu$ for all $\alpha < \mu^+$ and $n < \omega$;*

(j2) *$\langle M_{\alpha,n} : n < \omega \rangle$ is an increasing for each $\alpha < \mu^+$;*

(j3) $\forall \alpha < \mu^+ \quad \forall^\infty n \quad [M_{\alpha,n}]^\omega \subset M_{\alpha,n}$,

For $\alpha < \mu^+$, let $M_\alpha = \bigcup_{n<\omega} M_{\alpha,n} \prec \mathcal{H}(\theta)$.

(j4) *$\langle M_\alpha : \alpha < \mu^+ \rangle$ is continuously increasing and $\mu^+ \subseteq \bigcup_{\alpha<\mu^+} M_\alpha$.*

Very weak squares and dominating matrices

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Theorem (Fuchino, S, 1997)

If GCH + and $\square^{**}_\mu$ holds, then for any $\theta \gg \mu$ and $x \in \mathcal{H}(\theta)$, there is a strong μ-dominating matrix over x.

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Application of σ-Davies Trees

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Definition (Freese and Nation; Heindorf, Shapiro)

A poset $\langle P, \leq \rangle$ *has the* ***weak Freese-Nation property***
iff there is $f : P \to [P]^{\omega}$ *such that for any* $p, q \in P$
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Theorem (S, 2014)

If $\kappa^{\omega} = \kappa$ and there is a σ-Davies tree for κ over $[\kappa]^{\omega}$, then the poset $\langle [\kappa]^{\omega}, \subset \rangle$ has the wFN-property.

Thm: If $\kappa^\omega = \kappa$, and $\langle M_\alpha : \alpha < \kappa \rangle$ is a σ-Davies tree for κ over $\left[\kappa\right]^\omega$, then there is function $f : \left[\kappa\right]^\omega \to \left[\left[\kappa\right]^\omega\right]^\omega$ s.t. if $X \subset Y \in \left[\kappa\right]^\omega$ then there is $Z \in f(X) \cap f(Y)$ with $X \subseteq Z \subseteq Y$.

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- Since $[A_\zeta^\alpha]^\omega \subset M_\alpha$, so $\beta \le \alpha$.
- If $\beta = \alpha$, then $A^\alpha_{\min(\eta,\zeta)} \in f(A_\eta^\alpha) \cap f(A_\zeta^\alpha)$.
- If $\beta < \alpha$, then $A_\eta^\beta \in N_i^\alpha$ for some $i < \omega$.
- $A_\eta^\beta \subset N_i^\alpha \cap A_\zeta^\alpha \in [N_i^\alpha]^\omega \subset N_i^\alpha \subset M_{<\alpha}$.

Thm: If $\kappa^\omega = \kappa$, and $\langle M_\alpha : \alpha < \kappa \rangle$ is a σ-Davies tree for κ over $[\kappa]^\omega$, then there is function $f : [\kappa]^\omega \to [[\kappa]^\omega]^\omega$ s.t.

if $X \subset Y \in [\kappa]^\omega$ then there is $Z \in f(X) \cap f(Y)$ with $X \subseteq Z \subseteq Y$.

- $M_{<\alpha} = \bigcup_{i<\omega} N_i^\alpha$ s.t. $[N_i^\alpha]^\omega \subset N_i^\alpha$
- Let $\{A_\zeta^\alpha : \zeta < \omega_1\}$ be an enumeration of $(M_\alpha \setminus M_{<\alpha}) \cap [\kappa]^\omega$.
- Let

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- $A_\eta^\beta \subset N_i^\alpha \cap A_\zeta^\alpha \in [N_i^\alpha]^\omega \subset N_i^\alpha \subset M_{<\alpha}$.
- So there is $Z \in f(A_\eta^\beta) \cap f(A_\zeta^\alpha \cap N_i^\alpha)$ with $A_\eta^\beta \subseteq Z \subseteq A_\zeta^\alpha \cap N_i^\alpha$.

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- So there is $Z \in f(A_\eta^\beta) \cap f(A_\zeta^\alpha \cap N_i^\alpha)$ with $A_\eta^\beta \subseteq Z \subseteq A_\zeta^\alpha \cap N_i^\alpha$.
- Then $Z \in f(A_\zeta^\alpha)$.

To be continued . . .