On a Class of Guessing Models

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Young Set Theory Workshop
Bedlewo, Poland
May 15, 2014

Outline

1. Definition and some background information regarding guessing models

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2. Combinatorial consequences of guessing models

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3. Mitchell model and upperbounds

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4. The core model induction and lowerbounds
5. Some questions and open problems

Definition

Fix an uncountable cardinal θ. Let $R_\theta = H_\theta$ (or V_θ). Let $X \prec R_\theta$ and $\pi_X : M_X \to X$ be the uncollapse map with critical point μ_X. Let

$$\kappa_X = \min\{\alpha \in X \mid X \cap \alpha \neq \alpha\}.$$

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X is γ-*guessing* if whenever $z \in X$ and $b \subseteq z \cap X$, if for all $c \in \mathcal{P}_\gamma(X) \cap X$, $b \cap c \in X$, then there is some $d \in X$ such that $d \cap X = b$.

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We say the set b above a *bounded subset of X*. In the hypothesis of the definition, b is said to be *γ-approximated by X*. In the conclusion of the definition, b is said to be *X-guessed*.
Also, we can use the following alternative characterization of ξ-guessing: X is ξ-guessing if letting M_X be the transitive collapse of X, for any $b \subseteq a \in M_X$, if whenever $c \in \mathcal{P}_\xi(M_X) \cap M_X$, $c \cap b \in M_X$, then $b \in M_X$.

Some related backgrounds

For cardinals $\kappa \leq \lambda$, C. Weiss has defined the notion of a slender (κ, λ)-list, which generalizes the notion of a κ-tree. He also isolates the principle ISP(κ, λ) which states that every slender (κ, λ)-list has an ineffable branch.

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This principle in some sense captures the combinatorial essence of supercompactness. In fact, κ is supercompact if and only if κ is inaccessible and for all $\lambda \geq \kappa$, $\mathsf{ISP}(\kappa, \lambda)$. The point is the principle $\mathsf{ISP}(\kappa, \lambda)$ makes sense even if κ is not inaccessible. This is similar to the fact that the tree property can hold at successor cardinals, like ω_2.

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It turns out that the principle ISP$(\aleph_2, \lambda)$ is equivalent to the existence of stationary many $\aleph_1$-guessing models of size $\aleph_1$.

More definitions

Writing $\mathcal{P}^*_\kappa(R_\theta)$ for the set $Z \in \mathcal{P}_\kappa(R_\theta)$ such that $Z \cap \kappa \in \kappa$ and $(Z, \in) \prec (R_\theta, \in)$, we define for each $\xi \leq \kappa$,

$$\mathfrak{G}_{\kappa,\xi}(R_\theta) = \{Z \in \mathcal{P}^*_\kappa(R_\theta) \mid Z \text{ is } \xi\text{-guessing}\}.$$

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$$\mathfrak{S}_{\kappa,\xi}(R_\theta) = \{Z \in \mathcal{P}^*_\kappa(R_\theta) \mid Z \text{ is } \xi\text{-guessing}\}.$$

Definition

- $\mathrm{GM}_{\kappa,\xi}(\mathrm{R}_\theta)$ is the principle: the set $\mathfrak{S}_{\kappa,\xi}(R_\theta)$ is stationary.
- $\mathrm{GM}_{\kappa,\xi}$ is the principle: for all sufficiently large θ, $\mathrm{GM}_{\kappa,\xi}(\mathrm{R}_\theta)$.

$\aleph_0$-guessing models

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Lemma

(i) If X is $\aleph_0$-guessing then $\kappa_X, \kappa_X \cap X$ are strongly inaccessible.

(ii) $X \prec V_\delta$ is $\aleph_0$-guessing if and only if its transitive collapse is V_γ for some γ.

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(ii) $X \prec V_\delta$ is $\aleph_0$-guessing if and only if its transitive collapse is V_γ for some γ.

We only prove (ii). Suppose $X \prec V_\delta$ is $\aleph_0$-guessing. Let $\pi = \pi_X : M_X \to X$ be the uncollapse map. By induction on $\beta \in X$, suppose $X \cap V_\beta$ collapses to some V_{ξ_β}. If $a \subseteq V_{\xi_\beta}$ then $\pi[a] \subseteq V_\beta \in X$ is X-approximated. Hence $\pi[a] = X \cap b$ for some $b \in X$. Clearly, $b \in V_{\beta+1}$ and $\pi_X^{-1}[b] = a$. So $X \cap V_{\beta+1}$ collapses to $V_{\xi_\beta+1}$. The limit case is easy. A similar proof as above proves the converse.

$\aleph_0$-guessing models (cont.)

The following theorem is just a reformulation of Magidor's formulation of supercompactness in terms of $\aleph_0$-guessing models.

Theorem

κ is supercompact if and only if for every $\lambda > \kappa$, there is an $\aleph_0$-guessing model $M \prec V_\lambda$ with $\kappa_M = \kappa$.

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In short, $\aleph_0$-guessing models are not "interesting". From now on, we will focus on $\aleph_1$ (of size $\aleph_1$) and $\aleph_2$ guessing models (of size $\aleph_2$). These models in some sense capture combinatorial structures of supercompactness at ω_2 and ω_3 respectively.

Amenably closed hulls

X is amenably closed (at κ_X) if whenever $A \subseteq \mu_X$ is such that $A \cap \xi \in M_X$ (or equivalently in $A \cap \xi \in X$) for all $\xi < \mu_X$ then $A \in M_X$.

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Suppose X is ξ-guessing for some $\xi \leq \kappa_X$ and $|X| < \kappa_X$. Then X is amenably closed.

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Suppose X is ξ-guessing for some $\xi \leq \kappa_X$ and $|X| < \kappa_X$. Then X is amenably closed.

Proof. Let $\pi = \pi_X : M_X \to X$ be the uncollapse map. Let $A \subseteq \mu_X$ be such that for all $\alpha < \mu_X$, $A \cap \alpha \in M_X$. We show A is X-approximated. Let $b \in \mathcal{P}_\xi(X) \cap X$. We may assume $b \subset \kappa_X$. Since $b \in X$ and $|M| < \kappa_X$, b is bounded in κ_X and hence bounded in μ_X. Let $\alpha = \sup(b)$. Then $b \cap A = b \cap (A \cap \sup(b)) \in M_X$. So in fact $A \in M_X$.

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Failures of squares

Theorem (Weiss)

Suppose for all sufficiently large θ, $\mathrm{GM}_{\aleph_3,\aleph_2}(\mathrm{H}_\theta)$. Then for all $\lambda \geq \omega_1$, $\neg\square_\lambda$.

In fact, for all λ such that $\mathrm{cf}(\lambda) \geq \omega_2$, $\neg\square(\omega_3, \lambda)$ holds and hence $\neg\square(\lambda)$ holds. Similar conclusions hold for ξ-guessing models for other values of ξ.

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Proof. Now let $\gamma \geq \omega_2$ and suppose $\mathcal{C} = \langle C_\alpha \mid \alpha < \gamma^+\rangle$ is a $\Box_\gamma$-sequence and $C \in X \in \mathfrak{G}_{\aleph_3,\aleph_2}(H_\theta)$ for some $\theta > \gamma^+$. Let $\xi = \sup(X \cap \gamma^+)$. Note that $\xi < \gamma^+$.

It's not hard to show that $\mathrm{cf}(\xi) > \omega$. If $(b_n)_{n<\omega}$ is an increasing, cofinal in ξ sequence such that for each n, $b_n \in M$, then for any $b \in \mathcal{P}_{\omega_2}(X) \cap X$, $b \cap \{b_n \mid n < \omega\}$ is finite and hence in X. The point is that since γ^+ is regular and $b \cap \gamma^+ \in M$, $b \cap \gamma^+$ is bounded in ξ. So indeed there is some $e \in X$ such that $e \cap X = \{b_n \mid n < \omega\} = c$. Since $X \models$ "e is unbounded in γ^+" and $\omega_1 + 1 \subset M$, o.t.$(e) \geq \omega_1$, we have

Failures of squares (cont.)

$$\text{o.t.}(e \cap X) \geq \mathrm{o.t.}(\omega_1 \cap X) = \omega_1 > \omega = \mathrm{o.t.}(c).$$

Contradiction.

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Contradiction.
Now we show that C_ξ is M-guessed. Let $b \in \mathcal{P}_{\omega_2}(\gamma^+) \cap M$. Then b is bounded in γ^+ and hence bounded in ξ. Let $\alpha > \sup(b)$, $\alpha \in M \cap \lim(C_\xi)$, then $b \cap C_\xi = b \cap C_\xi \cap \alpha = b \cap C_\alpha \in M$. So there is some $e \in M$ such that $e \cap M = C_\xi \cap M$. Let $\pi : M_X \to X$ be the uncollapse map, then it's easy to see that $\pi^{-1}(e)$ is a thread through $\pi^{-1}(\mathcal{C})$.

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The proof of the stronger statement is similar. The main point is that if $\alpha \in M \cap \gamma^+$ and $C \subseteq \mathcal{P}(\alpha)$, $|C| < \omega_3$, then $C \subset M$.

Consistency

Theorem

(i) (Viale, Weiss) PFA implies $GM_{\aleph_2,\aleph_1}$

(ii) (T.) Suppose there is a supercompact cardinal. Then in a generic extension, $GM_{\aleph_3,\aleph_2}$ holds.

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We outline the proof of (ii). Let α be a supercompact cardinal and we may also assume $\omega_1^\omega = \omega_1$ and $\omega_2^\omega = \omega_2$. There is an iteration

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j \mid i \leq \alpha, j < \alpha \rangle,$$

such that

Consistency (cont.)

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(d) $\mathbb{P}_\alpha$ forces $\alpha = \omega_3$.

(e) $\forall \eta \leq \alpha$ such that η is inaccessible, letting $\mathbb{P}_\alpha = \mathbb{P}_\eta * \dot{\mathbb{Q}}$, then $\Vdash_{\mathbb{P}_\eta} \dot{\mathbb{Q}}$ satisfies the ω_2-approximation property, that is, whenever $G*H$ is V-generic for $\mathbb{P}_\eta * \dot{\mathbb{Q}}$, then if for all $x \in V[G][H]$, $x \subseteq V[G]$, it holds that if $x \cap z \in V[G]$ for all $z \in \mathcal{P}^{V[G]}_{\omega_2} V[G]$ then $x \in V[G]$.

Consistency (cont.)

In V, let $j : V \to M$ witness that α is H_θ-supercompact. Let $G * H \subseteq \mathbb{P}_\alpha * \mathbb{Q} \equiv \mathbb{P}_{j(\alpha)}$ be V-generic. Let $j^+ : V[G] \to M[G * H]$ be the canonical extension of j.

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Further observations:

- $\mathsf{CH} + 2^{\omega_1} = 2^{\omega_2} = \omega_3$ holds in $V[G]$.
- We can also get that in $V[G]$, if $\mathrm{cof}(\theta) > \omega$, then there are stationary many $\aleph_2$-guessing substructures X of H_θ of size $\aleph_2$ such that $X^\omega \subseteq X$. This is because $\mathbb{P}_\alpha$ is ω_1-closed, in particular, it doesn't add new ω-sequences of elements of V.

The size of the continuum

Proposition

If there is an $\aleph_1$-guessing model $X \prec H_\theta$ (for some $\theta > \omega_1$) of size $\aleph_1$ such that $X \cap Ord \in Ord$ then CH fails. Similarly, If there is an $\aleph_2$-guessing model of size $\aleph_2$ then $2^{\omega_1} > \omega_2$.

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Proof.

We prove this for the $\aleph_1$ case. Suppose CH holds. Let X be as in the hypothesis, so $\mathbb{R} \subseteq X$. Let $A \subseteq \omega_1$. We show that $A \in X$. For any $b \in \mathcal{P}_{\omega_1}(X) \cap X$, $A \cap b \in X$ because $\mathbb{R} \subseteq X$. This means A is $\aleph_1$-approximated by X, so A is X-guessed. But this means $A \in X$. So $\mathcal{P}(\omega_1) \subset X$. Contradiction. □

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Facts and consequences:

- Cummings and Unger have observed that $\mathrm{GM}_{\aleph_2,\aleph_1}$ is consistent relative to $2^\omega = \omega_2$ as well as $2^\omega > \omega_2$. (Similarly for $\mathrm{GM}_{\aleph_3,\aleph_2}$ wrt 2^{ω_1}). So $\mathrm{GM}_{\aleph_2,\aleph_1}$ doesn't decide the size of the continuum.

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- (T.) As mentioned before, in the Mitchell model of $\mathrm{GM}_{\aleph_3,\aleph_2}$, CH holds. $\mathrm{GM}_{\aleph_3,\aleph_2}$ is also consistent with $2^\omega = \omega_2$. Hence $\mathrm{GM}_{\aleph_3,\aleph_2}$ also doesn't decide the size of the continuum.

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- $\mathrm{GM}_{\aleph_2,\aleph_1}$ does not imply MRP. Since it is a consequence of PFA, it is consistent with failures of stationary reflection.

From K^c

Building on the theory of K^c by Andretta, Neeman, and Steel, and the theory of stacking mice developed by Jensen, Jensen, Schimmerling, Schindler, and Steel show.

Theorem (JSSS)

Suppose κ is regular, uncountable, countably closed cardinal. Suppose $\neg\square_\kappa$ and $\neg\square(\kappa)$. Then there is a sharp for a proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals (i.e. there is a non-domestic mouse). In particular, $\mathrm{GM}_{\aleph_2,\aleph_1}$ (and $\mathrm{GM}_{\aleph_2,\aleph_1}$) implies the existence of a non-domestic mouse.

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It is still open whether one can develop the theory of K^c under a weaker anti-large cardinal assumption (i.e. improving the A.N.S. construction). In terms of determinacy, a non-domestic mouse is roughly equiconsistent with the theory "$\mathrm{AD}_{\mathbb{R}} + \mathrm{DC}$" and is weaker than "$\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular".

The core model induction

The core model induction is a general method, pioneered by Woodin, further developed by Steel, Schindler, and others, used for mining strength of a given theory. In a typical core model induction, one constructs models of determinacy (inductively) that extend one another and this is achieved by constructing mouse operators that capture the relevant sets of reals.

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This relationship is given by a theorem of Neeman: if N is a (sufficiently iterable) mouse with iteration strategy Σ and $N \vDash \delta$ is Woodin. Suppose N *captures* $A \subseteq \mathbb{R}$, i.e. there is a $Col(\omega, \delta)$-term relation τ_A in N such that whenever $i : N \to M$ is according to Σ, letting $g \subseteq Col(\omega, i(\delta))$ be M-generic, then $A \cap M[g] = (i(\tau_A)_g)$.

The core model induction

The core model induction is a general method, pioneered by Woodin, further developed by Steel, Schindler, and others, used for mining strength of a given theory. In a typical core model induction, one constructs models of determinacy (inductively) that extend one another and this is achieved by constructing mouse operators that capture the relevant sets of reals.

This relationship is given by a theorem of Neeman: if N is a (sufficiently iterable) mouse with iteration strategy Σ and $N \vDash \delta$ is Woodin. Suppose N *captures* $A \subseteq \mathbb{R}$, i.e. there is a $Col(\omega, \delta)$-term relation τ_A in N such that whenever $i : N \to M$ is according to Σ, letting $g \subseteq Col(\omega, i(\delta))$ be M-generic, then $A \cap M[g] = (i(\tau_A)_g)$.

Examples of mouse operators are: $x \mapsto x^\sharp$, $x \mapsto \mathcal{M}_1^\sharp(x)$, a pair $(\mathcal{P}, \Sigma)$ where $\mathcal{P}$ is a fine structural mouse (or a hod mouse) and Σ is $\mathcal{P}$'s iteration strategy with nice condensation property (roughly, trees according to Σ collapses to trees according Σ).

The core model induction (cont.)

From now on, assume $(\dagger) \equiv$ "$2^{\omega_2} = \omega_3 + \forall\theta(\mathrm{cof}(\theta) > \omega \rightarrow \exists^* X \prec H_\theta$ $(X \text{ is } \aleph_2\text{-guessing}, |X| = \aleph_2, X^\omega \subseteq X))$".

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Schimmerling shows under $(\dagger)$ (in fact, just $\neg\Box_{\omega_3} + \neg\Box(\omega_3)$), for all n, the operators $x \mapsto \mathcal{M}_n^\sharp(x)$ is total on H_{ω_4}. In particular, PD holds.

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To go further in general, we need to organize our induction according to the pattern of scales. The key case is when we have already constructed a point class Γ such that

- Γ is inductive-like, i.e. Γ has the scales property, is closed under real quantifications, and non-self-dual.
- Γ is determined and $\Gamma \vDash \Theta = \theta_0$.
- Γ-MC holds, that is, for every $x, y \in \mathbb{R}$, $x \in OD^\Gamma(y)$ implies x is in a y-mouse with iteration strategy in Γ.

The core model induction (cont.)

We next construct a pair $(\mathcal{P}, \Sigma)$ that captures a countable, cofinal subset of $\mathbf{Env}(\Gamma)$ and Σ has condensation. The existence of $(\mathcal{P}, \Sigma)$ is hypothesis-dependent. It is not always possible to get such a pair and/or a cofinal, countable subset of $\mathbf{Env}(\Gamma)$.

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Hod mice are objects constructed from a (fine) extender sequence and a sequence of strategies of its own initial segments in a particular way. Hod mice satisfies GCH but does not satisfy full condensation like $L[E]$-mice; furthermore, it's not always possible to compare two hod mice. These objects are used to analyze HOD in an AD^+-model.

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To get past non-domestic, we need to be able to construct (fairly complicated) hod mice in our core model induction. (†) allows us to do just that.

The core model induction (cont.)

Theorem (T., 2013-2014)

Assume (†). Then in $V^{Col(\omega,\omega_2)}$, there are models of "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is regular", "$\mathsf{AD}_{\mathbb{R}} + \Theta$ is measurable", and much more.

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To go further, one seems to need to prove a general theorem about $\square$ in hod mice. At this point, it can be proved for fairly complicated class of hod mice but not for all. Main problems lie in lack of condensation and comparability of hod mice.

Questions and open problems

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- Chicken or egg, which exists first?

Thank you!