Infinite fields with large free automorphism groups

Philipp Moritz Lücke

Mathematisches Institut
Rheinische Friedrich-Wilhelms-Universität Bonn
http://www.math.uni-bonn.de/people/pluecke/

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The results to be presented in this talk are joint work with Saharon Shelah. They appeared in the following paper.

PHILIPP LÜCKE and SAHARON SHELAH.
Free groups and automorphism groups of infinite structures.
Forum of Mathematics, Sigma, 2.

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We start by presenting some known results related to this kind of problem.

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Given an infinite cardinal κ, we say that a group G is *the automorphism group of a κ-structure* if there is a first-order language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$ such that the cardinality of the signature of $\mathcal{L}$ and the cardinality of the domain of $\mathcal{M}$ are at most κ and the group $\mathrm{Aut}(\mathcal{M})$ is isomorphic to G.

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Lemma

Let κ be an infinite cardinal and G be a group. If G is the automorphism group of a κ-structure, then there is a connected graph Γ of cardinality κ such that G is isomorphic to $\mathrm{Aut}(\Gamma)$.

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Lemma

Let κ be an infinite cardinal and G be a group. If G is the automorphism group of a κ-structure, then there is a connected graph Γ of cardinality κ such that G is isomorphic to $\mathrm{Aut}(\Gamma)$.

Theorem (Fried & Kollár, 1982)

Let Γ be an infinite graph of cardinality κ. Then there is a field K of cardinality κ such that the groups $\mathrm{Aut}(K)$ and $\mathrm{Aut}(\Gamma)$ are isomorphic.

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Proof.

Define $\mathcal{L}_G$ to be the language with a binary relation symbol $\dot{R}_g$ for every $g \in G$ and let $\mathcal{M}_G$ denote the $\mathcal{L}_G$-structure with domain G and

$$\dot{R}_g^{\mathcal{M}_G} = \{\langle h, h \cdot g\rangle \mid h \in G\}$$

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for every $g \in G$. Fix $\pi \in \mathrm{Aut}(\mathcal{M}_G)$ and define $g_\pi = \pi(\mathbb{1}_G)$. Given $h \in H$, we have $\langle g_\pi, \pi(h)\rangle \in \dot{R}_h^{\mathcal{M}_G}$ and $\pi(h) = g_\pi \cdot h$.

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for every $g \in G$. Fix $\pi \in \mathrm{Aut}(\mathcal{M}_G)$ and define $g_\pi = \pi(\mathbb{1}_G)$. Given $h \in H$, we have $\langle g_\pi, \pi(h)\rangle \in \dot{R}_h^{\mathcal{M}_G}$ and $\pi(h) = g_\pi \cdot h$. Moreover, if $g \in G$, then there is a $\pi_g \in \mathrm{Aut}(\mathcal{M}_G)$ with $\pi_g(h) = g \cdot h$ for all $h \in G$. This shows that the function

$$\Phi : \mathrm{Aut}(\mathcal{M}_G) \longrightarrow G; \; \pi \longmapsto g_\pi$$

is an isomorphism of groups. □

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The following theorem shows that we can also find such groups of cardinality κ^{+}.

Theorem (De Bruijn, 1957)

If κ is an infinite cardinal, then the group $\mathrm{Fin}(\kappa^{+})$ consisting of all finite permutations of κ^{+} cannot be embedded into the group $\mathrm{Sym}(\kappa)$.

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If κ is an infinite cardinal, then the group $\mathrm{Fin}(\kappa^{+})$ consisting of all finite permutations of κ^{+} cannot be embedded into the group $\mathrm{Sym}(\kappa)$.

Next, we discuss results that show that there always is a subgroup of $\mathrm{Sym}(\kappa)$ that is not an automorphism group of a κ-structure.

Let $\mathcal{U}$ be an ultrafilter on an infinite set A. The *stabilizer of* $\mathcal{U}$ is the group

$$S_{\mathcal{U}} \;=\; \{\pi \in \mathrm{Sym}(A) \mid \forall X \subseteq A \, [X \in \mathcal{U} \longleftrightarrow \pi[X] \in \mathcal{U}]\}.$$

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Lemma

Let $\mathcal{U}_0$ and $\mathcal{U}_1$ be non-principal ultrafilters on an infinite cardinal κ. Then the groups $S_{\mathcal{U}_0}$ and $S_{\mathcal{U}_1}$ are isomorphic if and only if there is a $\sigma \in \mathrm{Sym}(\kappa)$ with

$$S_{\mathcal{U}_0} \;=\; \{\sigma \circ \pi \circ \sigma^{-1} \mid \pi \in S_{\mathcal{U}_1}\}.$$

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Corollary

If κ is an infinite cardinal, then there is a collection of $2^{2^{\kappa}}$-many pairwise non-isomorphic subgroups of $\mathrm{Sym}(\kappa)$.

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If κ is an infinite cardinal, then there is a subgroup of $\mathrm{Sym}(\kappa)$ that is not an automorphism group of a κ-structure.

The following result shows that, for certain cardinals κ, we can force a subgroup of $\mathrm{Sym}(\kappa)$ to be the automorphism group of a κ-structure in a generic extension of the ground model.

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Theorem (Just, Shelah & Thomas, 1999)

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. If G is a subgroup of $\mathrm{Sym}(\kappa)$, then there is a partial order $\mathbb{P}$ with the following properties.

- *Forcing with $\mathbb{P}$ preserves all cofinalities and the value of 2^κ.*
- *If F is $\mathbb{P}$-generic over V, then G is the automorphism group of a κ-structure in $\mathrm{V}[F]$.*

Remember that, given a set A, the *free group with basis* A is the group $\mathrm{F}(A)$ consisting of the set of all reduced words in the alphabet

$$\{\mathrm{x}_a^i \mid a \in A,\ i = \pm 1\}$$

equipped with the operation that sends two words to the unique reduced word that is equivalent to their concatenation.

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In the remainder of this talk, we focus on free groups and the following question.

Question

Given an infinite cardinal κ, is there a free group of rank greater than κ that is the automorphism group of a κ-structure?

Shelah showed that the above question has a negative answer for $\kappa = \aleph_0$. This answered a question of David Evans.

Theorem (Shelah, 2003)

A free group of uncountable rank is not the automorphism group of an $\aleph_0$*-structure.*

Shelah showed that the above question has a negative answer for $\kappa = \aleph_0$. This answered a question of David Evans.

Theorem (Shelah, 2003)

A free group of uncountable rank is not the automorphism group of an $\aleph_0$-structure.

The method developed in the proof of this result can also be used to answers the question in the negative for singular strong limit cardinals of countable cofinality.

Theorem (Shelah, 2003)

Let $\langle \kappa_n \mid n < \omega \rangle$ be a sequence of infinite cardinals with $2^{\kappa_n} < 2^{\kappa_{n+1}}$ for all $n < \omega$. Define $\kappa = \sum_{n<\omega} \kappa_n$ and $\mu = \sum_{n<\omega} 2^{\kappa_n}$. Then every free group of rank greater than μ is not the automorphism group of a κ-structure.

In contrast, it is possible to use the above forcing constriction to show that consistently the question has a positive answer for regular uncountable cardinals.

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If κ is an infinite cardinal, then $\mathrm{F}(2^\kappa)$ is isomorphic to a subgroup of $\mathrm{Sym}(\kappa)$.

Theorem (Just, Shelah & Thomas, 1999)

If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then there is a partial order $\mathbb{P}$ such that forcing with $\mathbb{P}$ preserves of cofinalities and the value of 2^κ and the free group of rank 2^κ is the automorphism group of a κ-structure in every $\mathbb{P}$-generic extension of the ground model.

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Theorem (L. & Shelah, 2014)

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Let κ be a cardinal with $\kappa = \kappa^{\aleph_0}$. Then the free group of rank 2^κ is the automorphism group of a κ-structure.

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The free group of rank $2^{2^{\aleph_0}}$ is the automorphism group of a $2^{\aleph_0}$-structure.

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The techniques developed in the proof of this result allow us to show that analogues of this statement hold for free objects in various varieties of groups. For example, the free abelian group of rank 2^κ is the automorphism group of a κ-structure whenever κ is a cardinal with $\kappa = \kappa^{\aleph_0}$.

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The techniques developed in the proof of this result allow us to show that analogues of this statement hold for free objects in various varieties of groups. For example, the free abelian group of rank 2^κ is the automorphism group of a κ-structure whenever κ is a cardinal with $\kappa = \kappa^{\aleph_0}$.

This contrasts a result of Solecki who showed that a free abelian group of uncountable rank is not the automorphism group of an $\aleph_0$-structure.

A combination of the above results allows us to simultaneously answer our question for all infinite cardinals under certain cardinal arithmetic assumptions.

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Corollary

Assume that the Continuum Hypothesis and the Singular Cardinal Hypothesis hold. Then the following statements are equivalent for every infinite cardinal κ.

- *There is a free group of rank greater than κ that is the automorphism group of a κ-structure.*
- *Either $\mathrm{cof}(\kappa) > \omega$ or there is a cardinal $\nu < \kappa$ with $2^\nu > \kappa$.*

The methods developed in the proof of the above theorem also allow us to show that the cardinal arithmetic assumption $\kappa = \kappa^{\aleph_0}$ is consistently not necessary for the existence of a free group of rank 2^κ that is the automorphism group of a κ-structure.

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This is a consequence of the following result.

Theorem (L. & Shelah, 2014)

Let κ be a cardinal of uncountable cofinality and $\nu > \kappa$ be a cardinal. If there is a tree of cardinality and height κ with ν-many cofinal branches, then there is a free group of rank ν that is the automorphism group of a κ-structure.

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Corollary

Let κ be a cardinal with $\mathrm{cof}(\kappa) > \omega$ and $\kappa = 2^{<\kappa}$ and let G be $\mathrm{Add}(\omega, \nu)$-generic over the ground model V for some cardinal ν. In $\mathrm{V}[G]$, there is a free group of rank greater than or equal to $(2^\kappa)^{\mathrm{V}}$ that is the automorphism group of a κ-structure.

The above theorem also shows that the existence of a cardinal κ of uncountable cofinality with the property that no free group of rank greater than κ is the automorphism group of a κ-structure implies the existence of large cardinals in inner models.

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Corollary

Let κ be a regular uncountable cardinal such that there is no free group of rank greater than κ that is the automorphism group of a κ-structure. Then κ^+ is an inaccessible cardinal in $\mathrm{L}[x]$ for every $x \subseteq \kappa$.

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Proof.

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Note that Mitchell used an inaccessible cardinal to construct a model of ZFC in which every tree of cardinality $\aleph_1$ and height ω_1 has at most $\aleph_1$-many cofinal branches.

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Assume, towards a contradiction, that there is no inner model with a Woodin cardinal. Then we can construct the *core model* K *below one Woodin cardinal*. It satisfies the *Generalized Continuum Hypothesis* and has the *covering property*.

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Let κ be a singular cardinal of uncountable cofinality such that there is no free group of rank greater than κ that is the automorphism group of a κ-structure. Then there is an inner model with a Woodin cardinal.

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Assume, towards a contradiction, that there is no inner model with a Woodin cardinal. Then we can construct the *core model* K *below one Woodin cardinal*. It satisfies the *Generalized Continuum Hypothesis* and has the *covering property*. In particular, we have $\kappa^+ = (\kappa^+)^{\mathrm{K}} = (2^\kappa)^{\mathrm{K}}$ and $(2^{<\kappa})^{\mathrm{K}} = \kappa$.

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Realizing inverse limits as automorphism groups

A *directed set* is a partial order $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$ with the property that $D \neq \emptyset$ and for all $p, q \in D$ there is an $r \in D$ with $p \leq_{\mathbb{D}} r$ and $q \leq_{\mathbb{D}} r$.

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Given a directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$, we call a pair

$$\mathbb{I} = \langle \langle A_p \mid p \in D \rangle, \langle f_{p,q} \mid p, q \in D, \ p \leq_{\mathbb{D}} q \rangle \rangle$$

an *inverse system of sets over* $\mathbb{D}$ if the following statements hold for all $p, q, r \in D$ with $p \leq_{\mathbb{D}} q \leq_{\mathbb{D}} r$.

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Given such an inverse system $\mathbb{I}$, we call the set

$$A_{\mathbb{I}} = \{(a_p)_{p \in D} \mid f_{p,q}(a_q) = a_p \textit{ for all } p, q \in D \textit{ with } p \leq_{\mathbb{D}} q\}$$

the *inverse limit of* $\mathbb{I}$.

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Then it is easy to see that

$$b : {}^{\kappa}2 \longrightarrow A_{\mathbb{I}(\kappa)}; \ x \longmapsto (x \restriction u)_{u \in [\kappa]^{\aleph_0}}$$

is a well-defined bijection between the sets ${}^{\kappa}2$ and $A_{\mathbb{I}(\kappa)}$.

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It is easy to see that the induced map

$$b : A_{\mathbb{I}(\mathbb{T})} \longrightarrow [\mathbb{T}];\; (t_\beta)_{\beta<\alpha} \longmapsto \{t_\alpha \mid \beta < \alpha\}$$

is a bijection between the inverse limit $A_{\mathbb{I}(\mathbb{T})}$ and the set $[\mathbb{T}]$ consisting of all cofinal branches through $\mathbb{T}$.

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is an inverse system of groups over $\mathbb{D}$ and there is a canonical homomorphism $u_{\mathbb{I}} : \mathrm{F}(A_{\mathbb{I}}) \longrightarrow G_{\mathbb{I}_{\mathrm{F}}}$ defined by

$$u_{\mathbb{I}}\left(\mathrm{x}_{(a_p)_{p\in D}}\right) \;=\; \left(\mathrm{x}_{a_p}\right)_{p\in D}.$$

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be an inverse system of groups over a directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}}\rangle$ such that $|D| \leq \kappa$ and $|G_p| \leq \kappa$ for all $p \in D$. Then $G_{\mathbb{I}}$ is the automorphism group of a κ-structure.

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If $\vec{g} = (g_p)_{p \in D} \in G_{\mathbb{I}}$, then there is a $\pi_{\vec{g}} \in \mathrm{Aut}(\mathcal{M}_{\mathbb{I}})$ with $\pi_{\vec{g}}(h, p) = \langle g_p \cdot h, p \rangle$ for all $p \in D$ and $h \in G_p$. This shows that Φ is an isomorphism. □

Proposition

If $\mathbb{I}$ *is an inverse system of sets over a directed set* $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$*, then the induced homomorphism* $u_{\mathbb{I}} : \mathrm{F}(A_{\mathbb{I}}) \longrightarrow G_{\mathbb{I}_{\mathrm{F}}}$ *is injective.*

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Proof.

Pick a non-trivial word w in $\mathrm{F}(A_{\mathbb{I}})$. Then there are $\vec{a}_0, \ldots, \vec{a}_n \in A_{\mathbb{I}}$ and $i_0, \ldots, i_n \in \mathbb{Z} \setminus \{0\}$ with

$$w \; = \; \mathrm{x}_{\vec{a}_0}^{i_0} \cdot \; \ldots \; \cdot \mathrm{x}_{\vec{a}_n}^{i_n}$$

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$$\mathrm{x}_{\vec{a}_0(p)}^{i_0} \cdot \;\ldots\; \cdot \mathrm{x}_{\vec{a}_n(p)}^{i_n} \;\neq\; \mathbb{1}_{G_p}.$$

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This shows that $u_{\mathbb{I}}(w) \neq \mathbb{1}_{G_{\mathbb{I}_{\mathrm{F}}}}$. □

We discuss properties of inverse systems of sets that imply that the induced homomorphism $u_{\mathbb{I}} : \mathrm{F}(A_{\mathbb{I}}) \longrightarrow G_{\mathbb{I}_{\mathbf{F}}}$ is an isomorphism.

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We say that a directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$ is *σ-directed* if for every $A \in [D]^{\aleph_0}$, there is a $q \in D$ with $p \leq_{\mathbb{D}} q$ for all $p \in A$.

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Let $\mathbb{D}$ be a directed set and $n : D \longrightarrow \omega$ be a function with $n(p) \leq n(q)$ for all $p, q \in D$ with $p \leq_{\mathbb{D}} q$. If $\mathbb{D}$ is σ-directed, then there is a $p \in D$ with $n(p) = n(q)$ for all $q \in D$ with $p \leq_{\mathbb{D}} q$.

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Theorem

Let $\mathbb{I}$ be an inverse system of sets over a directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$. If $\mathbb{D}$ is σ-directed, then the induced homomorphism $u_{\mathbb{I}} : \mathrm{F}(A_{\mathbb{I}}) \longrightarrow G_{\mathbb{I}_{\mathrm{F}}}$ is an isomorphism.

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Then there are homomorphisms $\langle \psi_q : \mathrm{F}(X_{p_*}) \longrightarrow G_p \mid p \in D \rangle$ such that

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Together with the above proposition, this shows that $u_{\mathbb{I}}$ is an isomorphism. □

We are now ready to prove our main results.

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Proof.

Let

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By the above results, $G_{\mathbb{I}(\kappa)_{\mathrm{F}}}$ is the automorphism group of a κ-structure and the canonical map

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Theorem

Let κ be a cardinal of uncountable cofinality, $\nu > \kappa$ be a cardinal and $\mathbb{T}$ be a tree of cardinality and height κ with ν-many cofinal branches. Then there is a free group of rank ν that is the automorphism group of a κ-structure.

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is an isomorphism. Since $|A_{\mathbb{I}(\mathbb{T})}| = |[T]| = \nu$, this yields the statement of the theorem. □

The above proofs actually yield more general results.

Definition

A functor F : SET $\longrightarrow$ GRP *induces a free construction* if the following statements hold for every set A.

- If $g \in F(A)$, then there is a unique finite subset $A(g)$ of A such that $g \in \mathrm{ran}(F(i_{A(g),A}))$ and $A(g) \subseteq \bar{A}$ for every finite subset $\bar{A}$ of A with $g \in \mathrm{ran}(F(i_{\bar{A},A}))$.
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Moreover, the construction of free objects in all non-trivial varieties of groups can be realized by a functor with these properties.

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Moreover, the construction of free objects in all non-trivial varieties of groups can be realized by a functor with these properties.

For example, the above statements are satisfied by the functor that sends a set A to the free abelian group with basis A and the functor that sends a set A to the A-fold free product of some fixed group G.

Theorem

Let F : SET $\longrightarrow$ GRP be a functor that induces a free construction. If κ is a cardinal with $\kappa = \kappa^{\aleph_0}$ and $|F(\kappa)| \leq \kappa$, then the group $F(2^\kappa)$ is the automorphism group of a κ-structure.

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Let $F : \mathsf{SET} \longrightarrow \mathsf{GRP}$ *be a functor that induces a free construction. If* κ *is a cardinal with* $\kappa = \kappa^{\aleph_0}$ *and* $|F(\kappa)| \leq \kappa$*, then the group* $F(2^\kappa)$ *is the automorphism group of a* κ*-structure.*

Theorem

Let $F : \mathsf{SET} \longrightarrow \mathsf{GRP}$ *be a functor that induces a free construction. If* κ *is a cardinal with* $|F(\kappa)| \leq \kappa$, $\nu > \kappa$ *is a cardinal and* T *is a tree of height and cardinality* κ *with* ν*-many cofinal branches, then the group* $F(\nu)$ *is the automorphism group of a* κ*-structure.*

We close this talk with questions raised by the above results.

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Question

Is it consistent with the axioms of ZFC that there is a cardinal κ of uncountable cofinality with the property that the free group of rank 2^κ is not the automorphism group of a κ-structure?

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Is it consistent with the axioms of ZFC that there is a cardinal κ of uncountable cofinality with the property that the free group of rank 2^κ is not the automorphism group of a κ-structure?

Question

Is it consistent with the axioms of ZFC that there is a cardinal κ of uncountable cofinality with the property that every free group of rank greater than κ is not the automorphism group of a κ-structure?

We close this talk with questions raised by the above results.

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Is it consistent with the axioms of ZFC that there is a cardinal κ of uncountable cofinality with the property that the free group of rank 2^κ is not the automorphism group of a κ-structure?

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Is it consistent with the axioms of ZFC that there is a cardinal κ of uncountable cofinality with the property that every free group of rank greater than κ is not the automorphism group of a κ-structure?

Question

Is it consistent with the axioms of ZFC that there is a singular cardinal κ of uncountable cofinality with the property that there is no tree of cardinality and height κ with more than κ-many branches of order-type κ?

Thank you for listening!