Applications of generic two-cardinal combinatorics

Piotr Koszmider

IM PAN, Warsaw

Definition

Let μ be a (κ, κ^+)-cardinal, then the following function $m_\mu = m\colon [\kappa^+]^2 \to \kappa$ is called a μ-coloring:

$$m(\alpha, \beta) = m(\{\alpha, \beta\}) = \min\{rank(X) : \ \alpha, \beta \in X \in \mu\}$$

Definition

Let μ be a (κ, κ^+)-cardinal, then the following function $m_\mu = m\colon [\kappa^+]^2 \to \kappa$ is called a μ-coloring:

$$m(\alpha, \beta) = m(\{\alpha, \beta\}) = \min\{rank(X) : \alpha, \beta \in X \in \mu\}$$

Theorem

Let $\alpha < \beta < \gamma < \kappa^+$, $\nu < \kappa$, $0 < \delta = \bigcup \delta < \epsilon < \kappa^+$, then the following conditions are satisfied:

(a) $|\{\xi < \alpha : m(\xi, \alpha) \leq \nu\}| < \kappa$

(b) $m(\alpha, \gamma) \leq \max\{m(\alpha, \beta), m(\beta, \gamma)\}$

(c) $m(\alpha, \beta) \leq \max\{m(\alpha, \gamma), m(\beta, \gamma)\}$

(d) *There is $\zeta < \delta$ such that $m(\xi, \epsilon) \geq m(\xi, \delta)$ for all $\zeta \leq \xi < \delta$.*

Definition

$f : [\omega_2]^2 \to \omega_1$ is called unbounded if and only if for every family $\{a_\xi : \xi < \omega_1\}$ of pairwise disjoint finite subsets of ω_2,

Definition

$f : [\omega_2]^2 \to \omega_1$ is called unbounded if and only if for every family $\{a_\xi : \xi < \omega_1\}$ of pairwise disjoint finite subsets of ω_2, for every $\delta < \omega_1$

Definition

$f : [\omega_2]^2 \to \omega_1$ is called unbounded if and only if for every family $\{a_\xi : \xi < \omega_1\}$ of pairwise disjoint finite subsets of ω_2, for every $\delta < \omega_1$ there are $\xi < \xi' < \omega_1$ such that

Definition

$f : [\omega_2]^2 \to \omega_1$ is called unbounded if and only if for every family $\{a_\xi : \xi < \omega_1\}$ of pairwise disjoint finite subsets of ω_2, for every $\delta < \omega_1$ there are $\xi < \xi' < \omega_1$ such that

$$f(\alpha, \beta) > \delta$$

for all $\alpha \in a_\xi$ and all $\beta \in a_{\xi'}$.

Definition

$f : [\omega_2]^2 \to \omega_1$ is called unbounded if and only if for every family $\{a_\xi : \xi < \omega_1\}$ of pairwise disjoint finite subsets of ω_2, for every $\delta < \omega_1$ there are $\xi < \xi' < \omega_1$ such that

$$f(\alpha, \beta) > \delta$$

for all $\alpha \in a_\xi$ and all $\beta \in a_{\xi'}$.

Theorem

Let $m : [\omega_2]^2 \to \omega_1$ be the μ-coloring. m is an unbounded function.

Definition

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ

Definition

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ if and only if for every uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2

Definition

A function $f : [\omega_2]^2 \rightarrow [\omega_2]^{\leq\omega}$ has the property Δ if and only if for every uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2 there are two elements $a, b \in \mathcal{A}$ such that

Definition

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ if and only if for every uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2 there are two elements $a, b \in \mathcal{A}$ such that for each $\tau \in a \cap b$ and $\alpha \in a - b$ and $\beta \in b - a$ we have

Definition

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ if and only if for every uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2 there are two elements $a, b \in \mathcal{A}$ such that for each $\tau \in a \cap b$ and $\alpha \in a - b$ and $\beta \in b - a$ we have

1. $\tau < \alpha, \beta \;\Rightarrow\; \tau \in f(\alpha, \beta)$,

Definition

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ if and only if for every uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2 there are two elements $a, b \in \mathcal{A}$ such that for each $\tau \in a \cap b$ and $\alpha \in a - b$ and $\beta \in b - a$ we have

1. $\tau < \alpha, \beta \ \Rightarrow \ \tau \in f(\alpha, \beta)$,
2. $\tau < \alpha \ \Rightarrow f(\tau, \beta) \subseteq f(\alpha, \beta)$,

Definition

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ if and only if for every uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2 there are two elements $a, b \in \mathcal{A}$ such that for each $\tau \in a \cap b$ and $\alpha \in a - b$ and $\beta \in b - a$ we have

1. $\tau < \alpha, \beta \Rightarrow \tau \in f(\alpha, \beta)$,
2. $\tau < \alpha \Rightarrow f(\tau, \beta) \subseteq f(\alpha, \beta)$,
3. $\tau < \beta \Rightarrow f(\tau, \alpha) \subseteq f(\alpha, \beta)$.

Definition

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ if and only if for every uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2 there are two elements $a, b \in \mathcal{A}$ such that for each $\tau \in a \cap b$ and $\alpha \in a - b$ and $\beta \in b - a$ we have

1. $\tau < \alpha, \beta \ \Rightarrow\ \tau \in f(\alpha, \beta)$,
2. $\tau < \alpha \ \Rightarrow f(\tau, \beta) \subseteq f(\alpha, \beta)$,
3. $\tau < \beta \ \Rightarrow f(\tau, \alpha) \subseteq f(\alpha, \beta)$.

Theorem

Let μ be (ω_1, ω_2)-cardinal, then $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ defined by

$$f(\alpha, \beta) = X \cap \min(\alpha, \beta),$$

where X is of minimal rank in μ which contains α, β has Δ-property.

Definition

A function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ has the property Δ if and only if for every uncountable Δ-system $\mathcal{A}$ of finite subsets of ω_2 there are two elements $a, b \in \mathcal{A}$ such that for each $\tau \in a \cap b$ and $\alpha \in a - b$ and $\beta \in b - a$ we have

1. $\tau < \alpha, \beta \ \Rightarrow \ \tau \in f(\alpha, \beta)$,
2. $\tau < \alpha \ \Rightarrow f(\tau, \beta) \subseteq f(\alpha, \beta)$,
3. $\tau < \beta \ \Rightarrow f(\tau, \alpha) \subseteq f(\alpha, \beta)$.

Theorem

Let μ be (ω_1, ω_2)-cardinal, then $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ defined by

$$f(\alpha, \beta) = X \cap \min(\alpha, \beta),$$

where X is of minimal rank in μ which contains α, β has Δ-property. Alternatively $f(\alpha, \beta) = \{\xi < \min(\alpha, \beta) : m(\xi, \alpha) \leq m(\alpha, \beta)\}$ for $\alpha < \beta$.

Definition

$(M_{\alpha,\beta})_{\alpha,\beta\in\omega_1}$ is called a universal matrix iff $M_{\alpha,\beta} \in \omega_1$ for each $\alpha, \beta \in \omega_2$ and

Definition

$(M_{\alpha,\beta})_{\alpha,\beta\in\omega_1}$ is called a universal matrix iff $M_{\alpha,\beta}\in\omega_1$ for each $\alpha,\beta\in\omega_2$ and

1. given any $k\in\mathbb{N}$

Definition

$(M_{\alpha,\beta})_{\alpha,\beta\in\omega_1}$ is called a universal matrix iff $M_{\alpha,\beta}\in\omega_1$ for each $\alpha,\beta\in\omega_2$ and

1. given any $k\in\mathbb{N}$
2. given any $k\times k$ matrix $(a_{i,j})_{i,j\leq k}$ with values in ω_1

Definition

$(M_{\alpha,\beta})_{\alpha,\beta\in\omega_1}$ is called a universal matrix iff $M_{\alpha,\beta} \in \omega_1$ for each $\alpha, \beta \in \omega_2$ and

1. given any $k \in \mathbb{N}$
2. given any $k \times k$ matrix $(a_{i,j})_{i,j\leq k}$ with values in ω_1
3. given any collection $\{\{\alpha_\xi(1), ..., \alpha_\xi(k)\} : \xi < \omega_1\}$ of pairwise disjoint k-tuples of elements of ω_2

Definition

$(M_{\alpha,\beta})_{\alpha,\beta\in\omega_1}$ is called a universal matrix iff $M_{\alpha,\beta}\in\omega_1$ for each $\alpha,\beta\in\omega_2$ and

1. given any $k\in\mathbb{N}$
2. given any $k\times k$ matrix $(a_{i,j})_{i,j\leq k}$ with values in ω_1
3. given any collection $\{\{\alpha_\xi(1),...,\alpha_\xi(k)\}:\xi<\omega_1\}$ of pairwise disjoint k-tuples of elements of ω_2

Definition

$(M_{\alpha,\beta})_{\alpha,\beta\in\omega_1}$ is called a universal matrix iff $M_{\alpha,\beta}\in\omega_1$ for each $\alpha,\beta\in\omega_2$ and

1. given any $k\in\mathbb{N}$
2. given any $k\times k$ matrix $(a_{i,j})_{i,j\leq k}$ with values in ω_1
3. given any collection $\{\{\alpha_\xi(1),...,\alpha_\xi(k)\}:\xi<\omega_1\}$ of pairwise disjoint k-tuples of elements of ω_2

There are $\xi<\xi'$ such that $M_{\alpha_\xi(i),\alpha_{\xi'}(j)}=a_{i,j}$ for all $i,j\leq k$.

Definition

Fix an unbounded function $m : [\omega_2]^2 \to \omega_1$. Let $\mathbb{P} \ni p = (a_p, M_p)$

Definition

Fix an unbounded function $m : [\omega_2]^2 \to \omega_1$. Let $\mathbb{P} \ni p = (a_p, M_p)$

1. $a^p \in [\omega_2]^{<\omega}$,

Definition

Fix an unbounded function $m : [\omega_2]^2 \to \omega_1$. Let $\mathbb{P} \ni p = (a_p, M_p)$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$

Definition

Fix an unbounded function $m : [\omega_2]^2 \to \omega_1$. Let $\mathbb{P} \ni p = (a_p, M_p)$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

Definition

Fix an unbounded function $m : [\omega_2]^2 \to \omega_1$. Let $\mathbb{P} \ni p = (a_p, M_p)$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

Definition

Fix an unbounded function $m : [\omega_2]^2 \to \omega_1$. Let $\mathbb{P} \ni p = (a_p, M_p)$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- $p \leq q$ iff $a^p \supseteq a^q$ and $M^p \supseteq M^q$

Definition

Fix an unbounded function $m : [\omega_2]^2 \to \omega_1$. Let $\mathbb{P} \ni p = (a_p, M_p)$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- $p \leq q$ iff $a^p \supseteq a^q$ and $M^p \supseteq M^q$

Definition

Fix an unbounded function $m : [\omega_2]^2 \to \omega_1$. Let $\mathbb{P} \ni p = (a_p, M_p)$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- $p \leq q$ iff $a^p \supseteq a^q$ and $M^p \supseteq M^q$

Theorem

$\mathbb{P}$ is c.c.c. and forces that $\bigcup\{M^p : p \in G\}$ is universal.

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha,\beta)$ for all $\{\alpha,\beta\} \in [a_p]^2$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- Find $q \in \mathbb{P}$ deciding $k \in \mathbb{N}$ and $k \times k$ matrix $(a_{i,j})_{i,j \leq k}$ with values in ω_1.

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- Find $q \in \mathbb{P}$ deciding $k \in \mathbb{N}$ and $k \times k$ matrix $(a_{i,j})_{i,j \leq k}$ with values in ω_1.

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- Find $q \in \mathbb{P}$ deciding $k \in \mathbb{N}$ and $k \times k$ matrix $(a_{i,j})_{i,j \leq k}$ with values in ω_1.

- For every $\xi < \omega_1$ find $p_\xi \leq q$ which decides k-tuple $\{\alpha_\xi(1), ..., \alpha_\xi(k)\}$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- Find $q \in \mathbb{P}$ deciding $k \in \mathbb{N}$ and $k \times k$ matrix $(a_{i,j})_{i,j \leq k}$ with values in ω_1.

- For every $\xi < \omega_1$ find $p_\xi \leq q$ which decides k-tuple $\{\alpha_\xi(1), ..., \alpha_\xi(k)\}$
- May assume that a^{p_ξ}s form an isomorphic Δ-system with root Δ and $\{\alpha_\xi(1), ..., \alpha_\xi(k)\} \subseteq a^{p_\xi}$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

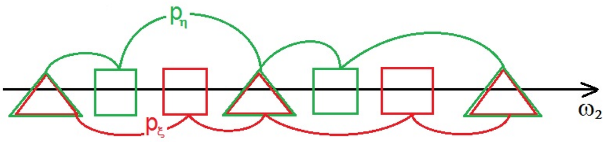
- Find $q \in \mathbb{P}$ deciding $k \in \mathbb{N}$ and $k \times k$ matrix $(a_{i,j})_{i,j \le k}$ with values in ω_1.

- For every $\xi < \omega_1$ find $p_\xi \le q$ which decides k-tuple $\{\alpha_\xi(1), ..., \alpha_\xi(k)\}$
- May assume that a^{p_ξ}s form an isomorphic Δ-system with root Δ and $\{\alpha_\xi(1), ..., \alpha_\xi(k)\} \subseteq a^{p_\xi}$
- Find $\xi < \eta < \omega_1$ such that $m(\alpha, \beta) > \delta = \max(a_{i,j \le k})$ for all $\alpha \in a^{p_\xi} \setminus a^{p_\eta}$ and $\alpha \in a^{p_\eta} \setminus a^{p_\xi}$

1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- Find $q \in \mathbb{P}$ deciding $k \in \mathbb{N}$ and $k \times k$ matrix $(a_{i,j})_{i,j \leq k}$ with values in ω_1.

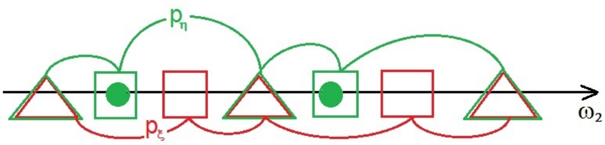
- For every $\xi < \omega_1$ find $p_\xi \leq q$ which decides k-tuple $\{\alpha_\xi(1), ..., \alpha_\xi(k)\}$
- May assume that a^{p_ξ}s form an isomorphic Δ-system with root Δ and $\{\alpha_\xi(1), ..., \alpha_\xi(k)\} \subseteq a^{p_\xi}$
- Find $\xi < \eta < \omega_1$ such that $m(\alpha, \beta) > \delta = \max(a_{i,j \leq k})$ for all $\alpha \in a^{p_\xi} \setminus a^{p_\eta}$ and $\alpha \in a^{p_\eta} \setminus a^{p_\xi}$



1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha, \beta)$ for all $\{\alpha, \beta\} \in [a_p]^2$

- Find $q \in \mathbb{P}$ deciding $k \in \mathbb{N}$ and $k \times k$ matrix $(a_{i,j})_{i,j \le k}$ with values in ω_1.

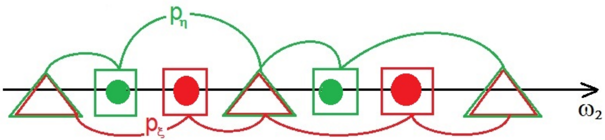
- For every $\xi < \omega_1$ find $p_\xi \le q$ which decides k-tuple $\{\alpha_\xi(1), ..., \alpha_\xi(k)\}$
- May assume that a^{p_ξ}s form an isomorphic Δ-system with root Δ and $\{\alpha_\xi(1), ..., \alpha_\xi(k)\} \subseteq a^{p_\xi}$
- Find $\xi < \eta < \omega_1$ such that $m(\alpha, \beta) > \delta = \max(a_{i,j \le k})$ for all $\alpha \in a^{p_\xi} \setminus a^{p_\eta}$ and $\alpha \in a^{p_\eta} \setminus a^{p_\xi}$



1. $a^p \in [\omega_2]^{<\omega}$,
2. $M^p : [a_p]^2 \to \omega_1$
3. $M^p_{\alpha,\beta} < m(\alpha,\beta)$ for all $\{\alpha,\beta\} \in [a_p]^2$

- Find $q \in \mathbb{P}$ deciding $k \in \mathbb{N}$ and $k \times k$ matrix $(a_{i,j})_{i,j\leq k}$ with values in ω_1.

- For every $\xi < \omega_1$ find $p_\xi \leq q$ which decides k-tuple $\{\alpha_\xi(1), ..., \alpha_\xi(k)\}$
- May assume that a^{p_ξ}s form an isomorphic Δ-system with root Δ and $\{\alpha_\xi(1), ..., \alpha_\xi(k)\} \subseteq a^{p_\xi}$
- Find $\xi < \eta < \omega_1$ such that $m(\alpha,\beta) > \delta = \max(a_{i,j\leq k})$ for all $\alpha \in a^{p_\xi} \setminus a^{p_\eta}$ and $\alpha \in a^{p_\eta} \setminus a^{p_\xi}$



General idea:

General idea:
Fix $F : [\omega_2]^2 \to$Countable sets (e.g., ω_1, $[\omega_2]^{\leq\omega}$) like an unbounded function or a function with Δ-property.

General idea:
Fix $F : [\omega_2]^2 \rightarrow$Countable sets (e.g., ω_1, $[\omega_2]^{\leq\omega}$) like an unbounded function or a function with Δ-property.
Consider forcing $\mathbb{P} \ni p = (a_p, \mathcal{S}_p)$ such that:

General idea:
Fix $F : [\omega_2]^2 \to$Countable sets (e.g., ω_1, $[\omega_2]^{\leq\omega}$) like an unbounded function or a function with Δ-property.
Consider forcing $\mathbb{P} \ni p = (a_p, \mathcal{S}_p)$ such that:

1. $a_p \in [\omega_2]^{<\omega}$,

General idea:
Fix $F : [\omega_2]^2 \to$Countable sets (e.g., ω_1, $[\omega_2]^{\leq\omega}$) like an unbounded function or a function with Δ-property.
Consider forcing $\mathbb{P} \ni p = (a_p, \mathcal{S}_p)$ such that:

1. $a_p \in [\omega_2]^{<\omega}$,
2. $\mathcal{S}_p$ is a structure with support a_p,

General idea:
Fix $F : [\omega_2]^2 \to$Countable sets (e.g., ω_1, $[\omega_2]^{\leq\omega}$) like an unbounded function or a function with Δ-property.
Consider forcing $\mathbb{P} \ni p = (a_p, \mathcal{S}_p)$ such that:

1. $a_p \in [\omega_2]^{<\omega}$,
2. $\mathcal{S}_p$ is a structure with support a_p,
3. F restricts the structure $\mathcal{S}_p$.

1. K compact, Hausdorff topological space.

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{\text{isolated points of } K\}$ (Cantor-Bendixson derivative),

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{\text{isolated points of } K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq \ldots K^{(\omega)} \supseteq \ldots$.

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{\text{isolated points of } K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq ... K^{(\omega)} \supseteq ....$
4. K is called scattered iff $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha = ht(K)$

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{\text{isolated points of } K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq \ldots K^{(\omega)} \supseteq \ldots.$
4. K is called scattered iff $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha = ht(K)$
5. $wd(K) = \sup\{|K^{(\alpha)} \setminus K^{(\alpha+1)}| : \alpha < ht(K)\}$,

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{$isolated points of $K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq ... K^{(\omega)} \supseteq ....$
4. K is called scattered iff $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha = ht(K)$
5. $wd(K) = \sup\{|K^{(\alpha)} \setminus K^{(\alpha+1)}| : \alpha < ht(K)\}$,
6. iff every nonempty subset of K has an isolated point

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{\text{isolated points of } K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq \ldots K^{(\omega)} \supseteq \ldots.$
4. K is called scattered iff $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha = ht(K)$
5. $wd(K) = \sup\{|K^{(\alpha)} \setminus K^{(\alpha+1)}| : \alpha < ht(K)\}$,
6. iff every nonempty subset of K has an isolated point
7. iff the Boolean algebra of clopen subsets of K is superatomic (every subalgebra has an atom)

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{\text{isolated points of } K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq \ldots K^{(\omega)} \supseteq \ldots.$
4. K is called scattered iff $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha = ht(K)$
5. $wd(K) = \sup\{|K^{(\alpha)} \setminus K^{(\alpha+1)}| : \alpha < ht(K)\}$,
6. iff every nonempty subset of K has an isolated point
7. iff the Boolean algebra of clopen subsets of K is superatomic (every subalgebra has an atom)
8. iff every Radon measure on K (signed, Borel, ctbly additive, regular) is atomic i.e., of the form $\Sigma_{n\in\mathbb{N}} a_n\delta_{x_n}$ for $x_n \in K$ and $\Sigma_{n\in\mathbb{N}}|a_n| < \infty$,

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{\text{isolated points of } K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq ... K^{(\omega)} \supseteq ....$
4. K is called scattered iff $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha = ht(K)$
5. $wd(K) = \sup\{|K^{(\alpha)} \setminus K^{(\alpha+1)}| : \alpha < ht(K)\}$,
6. iff every nonempty subset of K has an isolated point
7. iff the Boolean algebra of clopen subsets of K is superatomic (every subalgebra has an atom)
8. iff every Radon measure on K (signed, Borel, ctbly additive, regular) is atomic i.e., of the form $\Sigma_{n\in\mathbb{N}} a_n\delta_{x_n}$ for $x_n \in K$ and $\Sigma_{n\in\mathbb{N}}|a_n| < \infty$,
9. iff the Banach space $C(K)$ is Asplund (duals of separable subspaces are separable)

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{$isolated points of $K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq ... K^{(\omega)} \supseteq ....$
4. K is called scattered iff $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha = ht(K)$
5. $wd(K) = \sup\{|K^{(\alpha)} \setminus K^{(\alpha+1)}| : \alpha < ht(K)\}$,
6. iff every nonempty subset of K has an isolated point
7. iff the Boolean algebra of clopen subsets of K is superatomic (every subalgebra has an atom)
8. iff every Radon measure on K (signed, Borel, ctbly additive, regular) is atomic i.e., of the form $\Sigma_{n\in\mathbb{N}} a_n\delta_{x_n}$ for $x_n \in K$ and $\Sigma_{n\in\mathbb{N}}|a_n| < \infty$,
9. iff the Banach space $C(K)$ is Asplund (duals of separable subspaces are separable)
10. iff vector measures with values in the dual $C(K)^*$ satisfy the Radon-Nikodym theorem

1. K compact, Hausdorff topological space.
2. $K' = K \setminus \{\text{isolated points of } K\}$ (Cantor-Bendixson derivative),
3. $K^{(\alpha+1)} = (K^{(\alpha)})'$, $K^{(\lambda)} = \bigcap_{\alpha<\lambda} K^{(\alpha)}$, $K \supseteq K' \supseteq \dots K^{(\omega)} \supseteq \dots$.
4. K is called scattered iff $K^{(\alpha)} = \emptyset$ for some ordinal $\alpha = ht(K)$
5. $wd(K) = \sup\{|K^{(\alpha)} \setminus K^{(\alpha+1)}| : \alpha < ht(K)\}$,
6. iff every nonempty subset of K has an isolated point
7. iff the Boolean algebra of clopen subsets of K is superatomic (every subalgebra has an atom)
8. iff every Radon measure on K (signed, Borel, ctbly additive, regular) is atomic i.e., of the form $\Sigma_{n\in\mathbb{N}} a_n\delta_{x_n}$ for $x_n \in K$ and $\Sigma_{n\in\mathbb{N}}|a_n| < \infty$,
9. iff the Banach space $C(K)$ is Asplund (duals of separable subspaces are separable)
10. iff vector measures with values in the dual $C(K)^*$ satisfy the Radon-Nikodym theorem
11. iff every continuous convex function on any open convex subset U of $C(K)$ is differentiable on a dense G_δ set

Theorem (Juhasz-Weiss, 78)

There is a scattered compact space of countable width and height $\omega_1 + 1$.

Theorem (Juhasz-Weiss, 78)

There is a scattered compact space of countable width and height $\omega_1 + 1$*.*

Theorem (Just, 85)

It is consistent with $\neg$*CH that there is no scattered compact space of countable width and height* $\omega_2 + 1$*.*

Theorem (Juhasz-Weiss, 78)

There is a scattered compact space of countable width and height $\omega_1 + 1$.

Theorem (Just, 85)

It is consistent with $\neg$*CH that there is no scattered compact space of countable width and height* $\omega_2 + 1$.

Theorem (Baumgartner, Shelah, 87)

It is consistent that there is a scattered compact space of countable width and height $\omega_2 + 1$.

Theorem (Juhasz-Weiss, 78)

There is a scattered compact space of countable width and height $\omega_1 + 1$.

Theorem (Just, 85)

It is consistent with $\neg$CH that there is no scattered compact space of countable width and height $\omega_2 + 1$.

Theorem (Baumgartner, Shelah, 87)

It is consistent that there is a scattered compact space of countable width and height $\omega_2 + 1$.

Problems

Is it consistent that there is a scattered compact space of countable width and height $\omega_3 + 1$. Does Chang's conjecture implies that there is no scattered compact space of countable width and height $\omega_2 + 1$?.

Let $f^*:[\omega_2]^2 \to [\omega_2]^{\leq\omega}$ be fixed. Define P as the set of all $p = (x_p, \leqslant_p, i_p$ satisfying the following conditions:

(1) $x_p \in [T]^{<\omega}$.

(2) $\leqslant_p$ is a partial ordering of x_p with the property that if $s \in T_\alpha$, $t \in T_\beta$ and $s <_p t$, then $\alpha < \beta$.

(3) $i_p : [x_p]^2 \to [x_p]^{<\omega}$ is such that

(3.1) if $s \in T_\alpha$, $t \in T_\beta$, $s \neq t$ and $\alpha \leqslant \beta$, then

(3.1.1) if $\alpha = \beta$, then $i_p\{s, t\} = 0$,

(3.1.2) if $s <_p t$, then $i_p\{s, t\} = \{s\}$,

(3.1.3) if $\alpha < \beta$ and $s \not<_p t$, then

$$i_p\{s, t\} \subseteq x_p \cap \bigcup \{T_\tau : \tau \in f^*\{\alpha, \beta\}, \tau < \alpha\};$$

(3.2) if $\{s, t\} \in [x_p]^2$, then $\forall u \in i_p\{s, t\}\ u \leqslant_p s, t$, and if $v \leqslant_p s,t$, then $\exists u \in i_p\{s, t\}\ v \leqslant_p u$.

Set $p \leqslant q$ iff $x_p \supseteq x_q$, $\leqslant_p \restriction x_q = \leqslant_q$ and $i_p \restriction [x_q]^2 = i_q$.

Theorem (Shelah, 78)

($\diamondsuit$) There is a Banach space of density ω_1 where there are few operators in the sense that every operator is of the form $T = cI + S$, where S has separable range.

Theorem (Shelah, 78)

($\diamondsuit$) There is a Banach space of density ω_1 where there are few operators in the sense that every operator is of the form $T = cI + S$, where S has separable range.

Theorem (Shelah-Steprans, 88)

There is in ZFC a Banach space of density ω_1 where there are few operators in the above sense.

Theorem (Shelah, 78)

($\diamondsuit$) There is a Banach space of density ω_1 where there are few operators in the sense that every operator is of the form $T = cI + S$, where S has separable range.

Theorem (Shelah-Steprans, 88)

There is in ZFC a Banach space of density ω_1 where there are few operators in the above sense.

Theorem (Wark, 01)

There is a reflexive Banach space of density ω_1 where there are few operators in the above sense.

Theorem (Shelah, 78)

($\diamondsuit$) There is a Banach space of density ω_1 where there are few operators in the sense that every operator is of the form $T = cI + S$, where S has separable range.

Theorem (Shelah-Steprans, 88)

There is in ZFC a Banach space of density ω_1 where there are few operators in the above sense.

Theorem (Wark, 01)

There is a reflexive Banach space of density ω_1 where there are few operators in the above sense.

Theorem (Argyros-(Lopez-Abad)-Todorcevic, 05)

There is a reflexive Banach space of density ω_1 where there are few operators in the sense that every operator is of the form $T = Diag + S$ where S is strictly singular.

1. WCG Banach space: $X = \overline{span(K)}$, K weakly compact (Amir-Lindenstrauss)

1. WCG Banach space: $X = \overline{span(K)}$, K weakly compact (Amir-Lindenstrauss)
2. Nice generalization of separable Banach spaces and reflexive Banach spaces

1. WCG Banach space: $X = \overline{span(K)}$, K weakly compact (Amir-Lindenstrauss)
2. Nice generalization of separable Banach spaces and reflexive Banach spaces
3. (AL) Every WCG Banach space has a projectional resolution of identity: a decomposition into a continuous chain of 1-complemented subspaces of smaller densities

1. WCG Banach space: $X = \overline{span(K)}$, K weakly compact (Amir-Lindenstrauss)
2. Nice generalization of separable Banach spaces and reflexive Banach spaces
3. (AL) Every WCG Banach space has a projectional resolution of identity: a decomposition into a continuous chain of 1-complemented subspaces of smaller densities
4. A subspace is 1-complemented iff there is norm one projection onto it.

Definition

We say that a weakly compactly generated Banach space X has *few operators* if and only if there is a projectional resolution of identity $(P_\alpha : \omega \leq \alpha \leq \lambda)$ such that any operator $T: X \to X$ is of the form $P + S$ where P is in the closure of the linear span of countably many P_α's (in the strong operator topology) and S has a separable range.

Definition

We say that a weakly compactly generated Banach space X has *few operators* if and only if there is a projectional resolution of identity $(P_\alpha\colon \omega \le \alpha \le \lambda)$ such that any operator $T\colon X \to X$ is of the form $P + S$ where P is in the closure of the linear span of countably many P_α's (in the strong operator topology) and S has a separable range.

Theorem

It is consistent that there exists a WCG Banach space of density ω_2 which has few operators. For every cardinal λ it is consistent with ZFC that there exists a WCG Banach space of density λ where all operators are sums of a separable range operator and a diagonal operator with respect to a certain Marcuševič's basis.

Definition

We say that a weakly compactly generated Banach space X has *few operators* if and only if there is a projectional resolution of identity $(P_\alpha : \omega \leq \alpha \leq \lambda)$ such that any operator $T : X \to X$ is of the form $P + S$ where P is in the closure of the linear span of countably many P_α's (in the strong operator topology) and S has a separable range.

Theorem

It is consistent that there exists a WCG Banach space of density ω_2 which has few operators. For every cardinal λ it is consistent with ZFC that there exists a WCG Banach space of density λ where all operators are sums of a separable range operator and a diagonal operator with respect to a certain Marcuševič's basis.

Theorem

Chang's Conjecture implies that there is no WCG Banach space of density ω_2 which has few operators.

Definition

We say that a weakly compactly generated Banach space X has *few operators* if and only if there is a projectional resolution of identity $(P_\alpha\colon \omega \le \alpha \le \lambda)$ such that any operator $T\colon X \to X$ is of the form $P + S$ where P is in the closure of the linear span of countably many P_α's (in the strong operator topology) and S has a separable range.

Theorem

It is consistent that there exists a WCG Banach space of density ω_2 which has few operators. For every cardinal λ it is consistent with ZFC that there exists a WCG Banach space of density λ where all operators are sums of a separable range operator and a diagonal operator with respect to a certain Marcuševič's basis.

Theorem

Chang's Conjecture implies that there is no WCG Banach space of density ω_2 which has few operators. In ZFC there is no WCG Banach space of density $\ge \omega_3$ which has few operators.

Definition

We say that a weakly compactly generated Banach space X has *few operators* if and only if there is a projectional resolution of identity $(P_\alpha\colon \omega \leq \alpha \leq \lambda)$ such that any operator $T\colon X \to X$ is of the form $P + S$ where P is in the closure of the linear span of countably many P_α's (in the strong operator topology) and S has a separable range.

Theorem

It is consistent that there exists a WCG Banach space of density ω_2 which has few operators. For every cardinal λ it is consistent with ZFC that there exists a WCG Banach space of density λ where all operators are sums of a separable range operator and a diagonal operator with respect to a certain Marcuševič's basis.

Theorem

Chang's Conjecture implies that there is no WCG Banach space of density ω_2 which has few operators. In ZFC there is no WCG Banach space of density $\geq \omega_3$ which has few operators.

A1) $\mathcal{A} \subseteq [\lambda]^{<\omega}$.
A2) $\bigcup \mathcal{A} = \lambda$.
A3) $\mathcal{A}$ is closed under subsets.
B1) $\mathcal{B} \subseteq [\lambda]^{<\omega}$.
B2) $\mathcal{B}$ is closed under subsets.
AB1) $\mathcal{A} \cap \mathcal{B} \subseteq [\lambda]^1$.
AB2) For every family $\{a_\xi : \xi < \omega_1\} \subseteq [\lambda]^2$ of disjoint pairs and $k \in \omega$ there are $\xi_1 < \xi_2 < \ldots < \xi_k < \omega_1$ such that

$$\{a_{\xi_1}(1), \ldots, a_{\xi_k}(1)\} \in \mathcal{A},$$
$$\{a_{\xi_1}(0), \ldots, a_{\xi_k}(0)\} \in \mathcal{B}$$

where $a_\xi = \{a_\xi(0), a_\xi(1)\}$ for $\xi \in \omega_1$.
D1) $\mathcal{D} \subseteq [[\lambda]^2]^{<\omega}$ and for every $a, b \in D \in \mathcal{D}$ we have either $\max(a) < \min(b)$ $(a < b)$ or $\max(b) < \min(a)$ $(b < a)$; we express this property by saying that elements of $\mathcal{D}$ consist of consecutive pairs.
D2) If $D, D' \in \mathcal{D}$ are distinct, then there may be at most five pairs in D which intersect other than itself pair from D' i.e.,

$$|\{a \in D : a \cap (\bigcup (D' - \{a\})) \neq \emptyset\}| \leq 5.$$

D3) Whenever $\{d_\xi : \xi < \omega_1\} \subseteq [\lambda]^2$ is a collection of consecutive pairs and $k \in \omega$, then there are $\xi_1 < \xi_2, \ldots, < \xi_k < \omega_1$ such that $\{d_{\xi_i} : 1 \leq i \leq k\} \in \mathcal{D}$.
D4) Whenever $D \in \mathcal{D}$ and $\alpha < \lambda$, and $X \subseteq \lambda - \alpha$ is countable, there is $D' \in \mathcal{D}$ such that

$$(\bigcup D') \cap X = \emptyset, \quad D \cap [\alpha]^2 = D' \cap [\alpha]^2, \quad (\bigcup D) \cap \alpha = (\bigcup D') \cap \alpha.$$

AD1) Whenever $a \in \mathcal{A}$ and $D \in \mathcal{D}$, then $|a \cap (\bigcup D)| \leq 2$.

BD1) Whenever $a \in \mathcal{B}$ and $D \in \mathcal{D}$, then $|a \cap (\bigcup D)| \leq 2$.

F1) $\mathcal{F} \subseteq [\lambda]^\omega$ is cofinal in $[\lambda]^\omega$.

DF1) Suppose that $D \in \mathcal{D}$, $d, d', d'' \in D$ and $d < d' < d''$ and moreover that $X \in \mathcal{F}$ is such that $d' \cap X$ and $d'' \cap X$ are both nonempty. Then $d \subseteq X$.

This completes the list of properties of the families. The reader might have noted that if not for D3), the families $\mathcal{D} = \emptyset$ and $\mathcal{F} = [\lambda]^\omega$ work. This is exploited in the proof of Theorem 1.9.

Now, let us define our Banach space B. We start with the set ${}^\lambda\mathbb{R}$, that is, all functions from λ into the reals. Following [17] and [21], we define

$$||f||_\mathcal{A} = \sup\{\sqrt{\Sigma\{f(\alpha)^2 \colon \alpha \in a\}} \colon a \in \mathcal{A}\}$$

and

$$\nu_\mathcal{D}(f) = \sup\{\sqrt{\Sigma\{[f(\alpha) - f(\beta)]^2 \colon \{\alpha, \beta\} \in D\}} \colon D \in \mathcal{D}\}.$$

We put $B_*(\mathcal{A}, \mathcal{D}) = \{f \in {}^\lambda\mathbb{R} \colon \nu_\mathcal{D}(f) + ||f||_\mathcal{A} \text{ is finite}\}$. Using A1), A2) and D1) one can calculate that $(B_*(\mathcal{A}, \mathcal{D}), \nu_\mathcal{D} + ||\ ||_\mathcal{A})$ are Banach spaces. Namely they are clearly linear spaces and the usual triangle inequality for $l_2(\lambda)$ implies that they are normed spaces. Given a Cauchy sequence, one gets its uniform coordinate-wise limit by the completness of $l_\infty(\lambda)$, it has to belong to the spaces since $\nu_\mathcal{D} + ||\ ||_\mathcal{A}$ can be approximated on finite sets using terms of the sequence which must be norm-bounded.

For every $X \subseteq \lambda$, by 1_X we denote the characteristic function of X. By $\phi_{\{\alpha\}}$ we define the functional satisfying $\phi(f) = f(\alpha)$ for $\alpha \in \lambda$. For every $X \subseteq \lambda$ we define $B_X(\mathcal{A}, \mathcal{D})$ to be the closure of the linear span $\{1_{\{\alpha\}} \colon \alpha \in X\}$ in $B_*(\mathcal{A}, \mathcal{D})$ with respect to the norm $\nu_\mathcal{D} + ||\ ||_\mathcal{A}$. The main result will concern the space $(B_\lambda(\mathcal{A}, \mathcal{D}), \nu_\mathcal{D} + ||\ ||_\mathcal{A})$, however we will consider the space $B_X(\mathcal{A}, \mathcal{D})$ for $X \in \mathcal{F}$ as in F1) and for X an ordinal less than λ, in the latter case we will call this subspace an initial block. The projection on the initial block α means the restriction of a function to α.

choose $\mathcal{D}$ to be empty and $\mathcal{F} = [\lambda]^\omega$, that is we just need to add families $\mathcal{A}$ and $\mathcal{B}$ which satisfy the properties A1)–AB2). The c.c.c. forcing P required by the theorem consists of the conditions p of the form $(a_p, \mathcal{A}_p, \mathcal{B}_p)$ where

1) $a_p \in [\lambda]^{<\omega}$.
2) $\mathcal{A}_p \subseteq \mathcal{P}(a_p)$, $\mathcal{B}_p \subseteq \mathcal{P}(a_p)$.
3) $\mathcal{A}_p$, $\mathcal{B}_p$ are closed under subsets.
4) $\mathcal{A}_p \cap \mathcal{B}_p \subseteq [a_p]^1$.

The order is defined by $p \leq q$ if and only if $a_p \supseteq a_q$, $\mathcal{A}_p \supseteq \mathcal{A}_q$, $\mathcal{B}_p \supseteq \mathcal{B}_q$.

The conditions p of the forcing Q are of the form $p = (a_p, \mathcal{A}_p, \mathcal{B}_p, \mathcal{D}_p)$ where $(a_p, \mathcal{A}_p, \mathcal{B}_p) \in P$ from the beginning of the proof for $\lambda = \omega_2$ and additionally we have:

5) Elements of $\mathcal{D}_p$ are sets of consecutive pairs of a_p.
6) AD1) and BD1) of Section 2 are satisfied.
7) Whenever $D^1, D^2 \in \mathcal{D}_p$; $d_1^i, d_2^i \in D^i$, $d_1^i < d_2^i$, $\alpha \in d_1^1 \cap d_1^2$, $\beta \in d_2^1 \cap d_2^2$, then
   (a) $$\{d \in D^1 : d < d_1^1\} = \{d \in D^2 : d < d_1^2\},$$
   (b) $$\bigcup\{d \in D^1 : d < d_1^1\} = \bigcup\{d \in D^2 : d < d_1^2\} \subseteq F(\alpha, \beta).$$
   (Note that D^1 may be equal to D^2 above.)

Claim 3. It is consistent that there is a function $F\colon [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ with the following two properties:

P1) Whenever $(a_\xi\colon \xi < \omega_1)$ is a Δ-system of finite subsets of ω_2 with root $\Delta \subseteq \omega_2$ and $k \in \omega$, then there are $\xi_1 < \ldots < \xi_k < \omega_1$ such that

$$\forall i, j \leq k,\ i \neq j\ \forall \alpha \in a_{\xi_i} - \Delta\ \forall \beta \in a_{\xi_j} - \Delta$$

$$F(\alpha, \beta) \supseteq \bigcup \{a_{\xi_m} \cap \min(\alpha, \beta)\colon m < i, j\}.$$

P2) For every $\alpha < \omega_2$, for every finite $E \subseteq \omega_2$ and every countable $Z \subseteq \omega_2 - \alpha$ there is a finite $E' \subseteq \omega_2 - Z$ such that there is an order preserving bijection $\pi\colon E \to E'$ which is the identity on $E \cap E'$ and satisfies for every $\alpha, \beta \in E$:

$$\pi[F(\alpha, \beta) \cap E] = F(\pi(\alpha), \pi(\beta)) \cap E'. \qquad (**)$$

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that
$$\forall X, Y \in \mu \ \ X \subset Y \ \Rightarrow c(X) \in Y.$$

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \ \ X \subset Y \ \Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)*-cardinal,* $\mu \subseteq [\omega_2]^{\omega_1}$ *is a stationary coding set and* $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ *and* $X \in \mu$.

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in \mu$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \ \ X \subset Y \ \Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in \mu$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in \mu$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 1. $rank(X) = rank(M \cap \omega_2)$.

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \ \ X \subset Y \ \Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in \mu$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 1. $rank(X) = rank(M \cap \omega_2)$.
Coherence lemma gives $X \cap M = M \cap \beta$.

□

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)*-cardinal,* $\mu \subseteq [\omega_2]^{\omega_1}$ *is a stationary coding set and* $\mu \in M \prec H(\omega_3)$*,* $|M| = \omega$*,* $M \cap \omega_2 \in \mu$ *and* $X \in M$*. Then either* $X \cap M = M \cap \beta$ *for some* $\beta \in \omega_2$ *or* $X \cap M \in M$*.*

Proof.

Case 2. $rank(X) > rank(M \cap \omega_2)$.

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 2. $rank(X) > rank(M \cap \omega_2)$.
Let $X' \in \mu$, $rank(X') = rank(X)$, $M \cap \omega_2 \subseteq X'$.

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 2. $rank(X) > rank(M \cap \omega_2)$.
Let $X' \in \mu$, $rank(X') = rank(X)$, $M \cap \omega_2 \subseteq X'$.
$X \cap M =$

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$. Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 2. $rank(X) > rank(M \cap \omega_2)$.
Let $X' \in \mu$, $rank(X') = rank(X)$, $M \cap \omega_2 \subseteq X'$.
$X \cap M = X \cap (M \cap \omega_2) =$

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$. Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 2. $rank(X) > rank(M \cap \omega_2)$.
Let $X' \in \mu$, $rank(X') = rank(X)$, $M \cap \omega_2 \subseteq X'$.
$X \cap M = X \cap (M \cap \omega_2) = X \cap X' \cap (M \cap \omega_2) =$

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \ \ X \subset Y \ \Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$. Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 2. $rank(X) > rank(M \cap \omega_2)$.
Let $X' \in \mu$, $rank(X') = rank(X)$, $M \cap \omega_2 \subseteq X'$.
$X \cap M = X \cap (M \cap \omega_2) = X \cap X' \cap (M \cap \omega_2) = (X' \cap \beta) \cap (M \cap \omega_2) =$

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$. Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 2. $rank(X) > rank(M \cap \omega_2)$.
Let $X' \in \mu$, $rank(X') = rank(X)$, $M \cap \omega_2 \subseteq X'$.
$X \cap M = X \cap (M \cap \omega_2) = X \cap X' \cap (M \cap \omega_2) = (X' \cap \beta) \cap (M \cap \omega_2) = M \cap \beta.$

□

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

$$\forall X, Y \in \mu \ \ X \subset Y \ \Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 3. $rank(X) < rank(M \cap \omega_2)$.

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

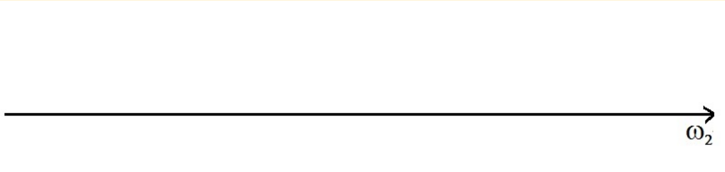
$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 3. $rank(X) < rank(M \cap \omega_2)$.



□

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

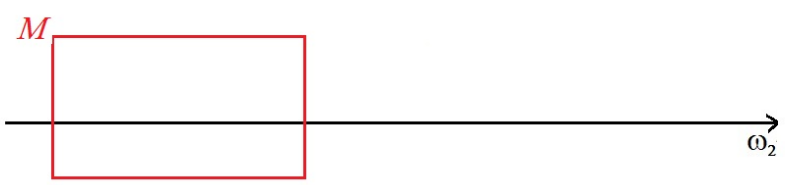
$$\forall X, Y \in \mu \ \ X \subset Y \ \Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)*-cardinal,* $\mu \subseteq [\omega_2]^{\omega_1}$ *is a stationary coding set and* $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ *and* $X \in M$*.*
Then either $X \cap M = M \cap \beta$ *for some* $\beta \in \omega_2$ *or* $X \cap M \in M$*.*

Proof.

Case 3. $rank(X) < rank(M \cap \omega_2)$.



□

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

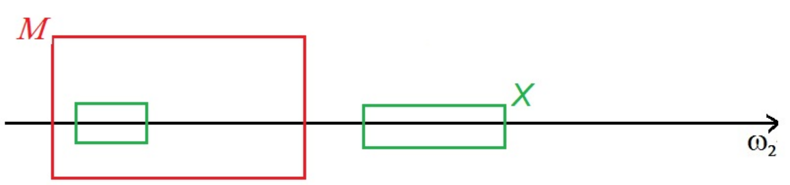
$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$. Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 3. $rank(X) < rank(M \cap \omega_2)$.



□

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

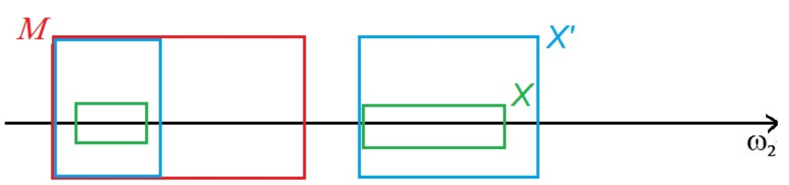
$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)*-cardinal,* $\mu \subseteq [\omega_2]^{\omega_1}$ *is a stationary coding set and* $\mu \in M \prec H(\omega_3)$*,* $|M| = \omega$*,* $M \cap \omega_2 \in \mu$ *and* $X \in M$*.*
Then either $X \cap M = M \cap \beta$ *for some* $\beta \in \omega_2$ *or* $X \cap M \in M$*.*

Proof.

Case 3. $rank(X) < rank(M \cap \omega_2)$.



□

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

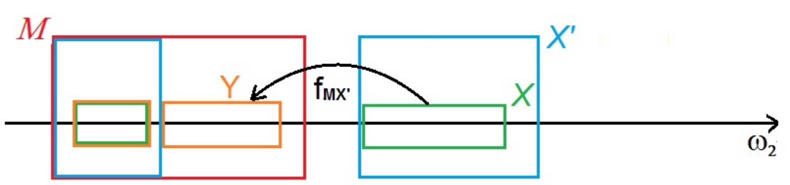
$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$.
Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 3. $rank(X) < rank(M \cap \omega_2)$.



□

Definition

μ is *stationary coding set* iff μ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mu \to \omega_2$ such that

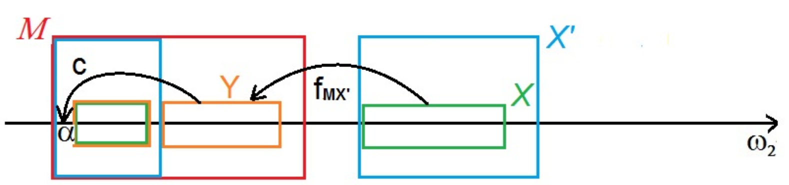
$$\forall X, Y \in \mu \;\; X \subset Y \;\Rightarrow c(X) \in Y.$$

Lemma

Suppose that an (ω_1, ω_2)-cardinal, $\mu \subseteq [\omega_2]^{\omega_1}$ is a stationary coding set and $\mu \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mu$ and $X \in M$. Then either $X \cap M = M \cap \beta$ for some $\beta \in \omega_2$ or $X \cap M \in M$.

Proof.

Case 3. $rank(X) < rank(M \cap \omega_2)$.



□