Borel Determinacy and the Word Problem for Finitely Generated Groups

Simon Thomas

Rutgers University

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The HNN Embedding Theorem

Theorem (Higman-Neumann-Neumann 1949)

If G is a countable group, then G can be embedded into a 2-generator group K_G.

Theorem

If $\varphi : \langle \mathcal{G}, \cong_{\mathcal{G}} \rangle \to \langle \mathcal{G}_{fg}, \cong_{\mathcal{G}_{fg}} \rangle$ is any Borel homomorphism, then there exists a group $G \in \mathcal{G}$ such that $G \not\hookrightarrow \varphi(G)$.

Heuristic Reason

Since $\cong_{\mathcal{G}}$ is much more complex than $\cong_{\mathcal{G}_{fg}}$, the Borel homomorphism must have a "large kernel" and hence "too many" groups $G \in \mathcal{G}$ will be mapped to a fixed $K \in \mathcal{G}_{fg}$.

The obvious follow-up question to the *HNN* Theorem

Question (Cherlin, Hrushovski, Sabok, ...)

Does there exist a Borel homomorphism $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_2$ such that $G \hookrightarrow \varphi(G)$ for all $G \in \mathcal{G}_{fg}$?

Definition

Let $\mathcal{G}_2$ be the space of 2-generator groups.

Theorem (Hjorth)

$\cong_{\mathcal{G}_2}$ *is a universal countable Borel equivalence relation.*

Talking to the wrong people ...

The Friedman Embedding Theorem

There exists a Borel homomorphism $\psi : \mathcal{G}_{fg} \to \mathcal{G}_2$ *such that* $G \hookrightarrow \psi(G)$ *for all* $G \in \mathcal{G}_{fg}$.

Question

What does Friedman know that the group theorists don't know ...
and that might conceivably be useful?

Answer

Absolutely nothing!

The word problem as a group-theoretic invariant

Proposition

If $(G, \bar{s}), (H, \bar{t}) \in \mathcal{G}_{fg}$ *and* $G \cong H$, *then* $\mathrm{Word}_{\bar{s}}(G) \equiv_T \mathrm{Word}_{\bar{t}}(H)$.

Definition

If $A \in 2^{\mathbb{N}}$, then $\mathrm{Rec}^A(\mathbb{N}) = \{ g \in \mathrm{Sym}(\mathbb{N}) \mid g \leq_T A \}$.

Proposition

(i) *If* $A \leq_T B$, *then* $\mathrm{Rec}^A(\mathbb{N}) \leqslant \mathrm{Rec}^B(\mathbb{N})$.

(ii) *If* $A \equiv_T B$, *then* $\mathrm{Rec}^A(\mathbb{N}) = \mathrm{Rec}^B(\mathbb{N})$.

(iii) *If* $(G, \bar{s}) \in \mathcal{G}_{fg}$ *and* $\mathrm{Word}_{\bar{s}}(G) \leq_T A$, *then* $G \hookrightarrow \mathrm{Rec}^A(\mathbb{N})$.

The word problem as a group-theoretic invariant

Theorem (Friedman)

There exists a Borel map $A \mapsto (\, g_A, h_A \,)$ *from* $2^{\mathbb{N}}$ *to* $\mathrm{Sym}(\mathbb{N}) \times \mathrm{Sym}(\mathbb{N})$ *such that:*

- $\mathrm{Rec}^A(\mathbb{N}) \hookrightarrow \langle\, g_A, h_A \,\rangle \in \mathcal{G}_2$.
- *If* $A \equiv_T B$, *then* $\{\, g_A, h_A \,\}$ *and* $\{\, g_B, h_B \,\}$ *generate the same subgroup of* $\mathrm{Sym}(\mathbb{N})$ *and so* $\langle\, g_A, h_A \,\rangle \cong \langle\, g_B, h_B \,\rangle$.

Corollary (Friedman)

Let $\psi : \mathcal{G}_{fg} \to \mathcal{G}_2$ *be the Borel homomorphism defined by*

$$(\, G, \bar{s} \,) \mapsto \mathrm{Word}_{\bar{s}}(G) \mapsto \langle\, g_{\mathrm{Word}_{\bar{s}}(G)}, h_{\mathrm{Word}_{\bar{s}}(G} \,\rangle.$$

Then $G \hookrightarrow \psi(G, \bar{s})$ *for all* $(\, G, \bar{s} \,) \in \mathcal{G}_{fg}$.

Friedman's Idea

Notation

If $A \in 2^{\mathbb{N}}$, then φ_i^A is the i-th partial A-recursive function and

$$\psi_i^A = \begin{cases} \varphi_i^A & \text{if } \varphi_i^A \in \mathrm{Sym}(\mathbb{N}); \\ \mathrm{id}_{\mathbb{N}} & \text{otherwise.} \end{cases}$$

Lemma (Friedman après Myhill)

If $A \equiv_T B$, then there exists a recursive permutation $\theta \in \mathrm{Sym}(\mathbb{N})$ such that $\psi_i^B = \psi_{\theta(i)}^A$ for all $i \in \mathbb{N}$.

Friedman's Idea

Definition

Define $\pi_A \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ by $\pi_A(i,j) = (i, \psi_i^A(j))$.

Lemma (Friedman)

If $A \equiv_T B$, then there exists a recursive permutation $\theta \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ such that $\theta^{-1}\pi_A\theta = \pi_B$.

Definition

Let $H_A \leqslant \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ be the subgroup generated by

$$\{\pi_A\} \cup \{\, \theta \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N}) \mid \theta \text{ is recursive} \,\}.$$

Remark

If $A \equiv_T B$, then $H_A = H_B$.

Friedman's Idea

Notation

For each $g \in \mathrm{Sym}(\mathbb{N})$, define $\tilde{g} \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N})$ by

$$\tilde{g}(i,j) = \begin{cases} (\,0, g(j)\,) & \text{if } i = 0. \\ (\,i, j\,) & \text{otherwise.} \end{cases}$$

Proposition (Friedman)

$\mathrm{Rec}^A(\mathbb{N}) \cong \{\, \tilde{g} \in \mathrm{Sym}(\mathbb{N} \times \mathbb{N}) \mid g \in \mathrm{Rec}^A \,\} \leqslant H_A$.

Corollary (Friedman)

If $(\, G, \bar{s}\,) \in \mathcal{G}_{fg}$ *and* $\mathrm{Word}_{\bar{s}}(G) \leq_T A$*, then* $G \hookrightarrow \mathrm{Rec}^A(\mathbb{N}) \hookrightarrow H_A$.

Galvin's Embedding Theorem

Notation

For each $\pi \in \mathrm{Sym}(\Omega)$, define $\hat{\pi} \in \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \Omega)$ by

$$\hat{\pi}(\, m, n, \omega \,) = \begin{cases} (\, 0, 0, \pi(\omega)\,) & \text{if } m = n = 0; \\ (\, m, n, \omega \,) & \text{otherwise.} \end{cases}$$

Theorem (Galvin)

If $K \leqslant \mathrm{Sym}(\Omega)$ is a countable subgroup, then there exists a 2-generator subgroup $T_K \leqslant \mathrm{Sym}(\mathbb{Z} \times \mathbb{Z} \times \Omega)$ such that $\{\, \hat{k} \mid k \in K \,\} \leqslant T_K$.

Definition

Let $\Omega = \mathbb{N} \times \mathbb{N}$ and let K be the group of recursive permutations of $\mathbb{N} \times \mathbb{N}$. Then G_A is the 3-generator group generated by $T_K \cup \{\, \hat{\pi}_A \,\}$.

And to get a 2-generator group? Work a little harder!

An Open Problem

Observation

The standard group-theoretic constructions (e.g. wreath products, free products with amalgamation, *HNN* extensions, ...) induce continuous homomorphisms $\varphi : \mathcal{G}_{fg} \to \mathcal{G}_{fg}$.

Conjecture

There does not exist a continuous homomorphism $\varphi : \mathcal{G}_3 \to \mathcal{G}_2$ such that $G \hookrightarrow \varphi(G)$ for all $G \in \mathcal{G}_3$.

Question (Kanovei)

Find nontrivial natural examples of Borel equivalence relations E, F such that $E \leq_B F$ but there is no continuous reduction from E to F.

Why are such examples hard to find?

Theorem (Folklore)

If X, Y are Polish spaces and $\varphi : X \to Y$ is a Borel map, then there exists a comeager subset $C \subseteq X$ such that $\varphi \upharpoonright C$ is continuous.

Theorem (Lusin)

Let X, Y be Polish spaces and let μ be any Borel probability measure on X. If $\varphi : X \to Y$ is a Borel map, then for every $\varepsilon > 0$, there exists a compact set $K \subseteq X$ with $\mu(K) > 1 - \varepsilon$ such that $\varphi \upharpoonright K$ is continuous.

Another notion of largeness ...

Definition

For each $z \in 2^{\mathbb{N}}$, the corresponding cone is $\mathcal{C}_z = \{ x \in 2^{\mathbb{N}} \mid z \leq_T x \}$.

- Suppose $z_n = \{ a_{n,\ell} \mid \ell \in \mathbb{N} \} \in 2^{\mathbb{N}}$ for each $n \in \mathbb{N}$ and define

$$\oplus z_n = \{ p_n^{a_{n,\ell}} \mid n, \ell \in \mathbb{N} \} \in 2^{\mathbb{N}},$$

  where p_n is the nth prime.

- Then $z_m \leq_T \oplus z_n$ for each $m \in \mathbb{N}$ and so $\mathcal{C}_{\oplus z_n} \subseteq \bigcap_n \mathcal{C}_{z_n}$.

Remark

It is well-known that if $\mathcal{C} \subsetneq 2^{\mathbb{N}}$ is a proper cone, then $\mathcal{C}$ is both null and meager.

Continuous maps on the Cantor space

Theorem (Folklore)

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, then the following are equivalent:

(a) *θ is continuous.*

(b) *There exists $C \in 2^{\mathbb{N}}$ and $e \in \mathbb{N}$ such that $\theta(A) = \varphi_e^{C \oplus A}$.*

Corollary

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is continuous, then there exists a cone $\mathcal{C}$ such that $\theta(A) \leq_T A$ for all $A \in \mathcal{C}$.

Continuous maps on $\mathcal{G}_{fg}$

Theorem

If $G \mapsto K_G$ is a continuous map from $\mathcal{G}_{fg}$ to $\mathcal{G}_{fg}$, then there exists a cone $\mathcal{C}$ such that if $\mathrm{Word}(G) \in \mathcal{C}$, *then* $\mathrm{Word}(K_G) \leq_T \mathrm{Word}(G)$.

Observation

If $\psi : \mathcal{G}_{fg} \to \mathcal{G}_2$ is the map given by the current proof of the Friedman Embedding Theorem, then $\mathrm{Word}(G)'' \leq_T \mathrm{Word}(\psi(G))$ for all $G \in \mathcal{G}_{fg}$.

Proof.

$\{ i \in \mathbb{N} \mid \varphi_i^A \in \mathrm{Sym}(\mathbb{N}) \smallsetminus \{ \mathrm{Id}_{\mathbb{N}} \} \} \equiv_T A''$. □

The "obvious" vs "nonobvious" Turing reductions ...

Definition

If $A, B \in 2^{\mathbb{N}}$, then *A is one-one reducible to B*, written $A \leq_1 B$, if there exists an injective recursive function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$n \in A \iff f(n) \in B.$$

Example

If $G, H \in \mathcal{G}_{fg}$ and $G \hookrightarrow H$, then $\text{Word}(G) \leq_1 \text{Word}(H)$.

Proof.

Suppose that $G = \langle a_1, \cdots, a_n \rangle$ and $H = \langle b_1, \cdots, b_m \rangle$. Let $\varphi : G \to H$ be an embedding and let $\varphi(a_i) = t_i(\bar{b})$. Then

$$w_k(a_1, \cdots, a_n) = 1 \iff w_k(t_1(\bar{b}), \cdots, t_n(\bar{b})) = 1.$$

□

Turing Equivalence vs. Recursive Isomorphism

Definition

The sets $A, B \in 2^{\mathbb{N}}$ are recursively isomorphic, written $A \equiv_1 B$, if both $A \leq_1 B$ and $B \leq_1 A$.

Theorem (Myhill)

If $A, B \in 2^{\mathbb{N}}$, then $A \equiv_1 B$ if and only if there exists a recursive permutation $\pi \in \mathrm{Sym}(\mathbb{N})$ such that $\pi[A] = B$.

Theorem (Folklore)

The map $A \mapsto A'$ is a Borel reduction from $\equiv_T$ to $\equiv_1$.

Observation

The Borel reduction $A \mapsto A'$ from $\equiv_T$ to $\equiv_1$ is certainly not continuous.

Turing Equivalence vs. Recursive Isomorphism

Definition

Let E, F be Borel equivalence relations on the Polish spaces X, Y. Then the Borel map $\varphi : X \to Y$ is a homomorphism from E to F if

$$x \, E \, y \Longrightarrow \varphi(x) \, F \, \varphi(y).$$

The Cone Theorem

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to $\equiv_1$, then there exists a cone $\mathcal{C}$ such that θ maps $\mathcal{C}$ into a single $\equiv_1$-class.

Corollary

There does not exist a continuous reduction from $\equiv_T$ to $\equiv_1$.

Turing Equivalence vs. Isomorphism on $\mathcal{G}_{fg}$

Theorem

There does not exist a continuous reduction from $\equiv_T$ to $\cong_{\mathcal{G}_{fg}}$.

Proof.

- Suppose $A \mapsto H_A$ is a continuous reduction from $\equiv_T$ to $\cong_{\mathcal{G}_{fg}}$.
- Note that $H \mapsto \mathrm{Word}(H)$ is an countable-to-one continuous homomorphism from $\cong_{\mathcal{G}_{fg}}$ to $\equiv_1$.
- Thus $A \mapsto \mathrm{Word}(H_A)$ is a countable-to-one continuous homomorphism from $\equiv_T$ to $\equiv_1$, which is a contradiction.

□

Determinacy

Definition

For each $X \subseteq 2^{\mathbb{N}}$, let $G(X)$ be the two player game

$$\begin{array}{lccccccccc} \color{red}{I} & s(0) & & s(2) & & s(4) & & s(6) & & \cdots \\ \color{red}{II} & & s(1) & & s(3) & & s(5) & & s(7) & \cdots \end{array}$$

where I wins if and only if $s = (\, s(0)\, s(1)\, s(2)\, s(3) \,\cdots\,) \in X$.

Definition

- A strategy is a map $2^{<\mathbb{N}} \to 2$ which tells the relevant player which move to make in a given position.
- The game $G(X)$ is determined if one of the players has a winning strategy.

Determinacy

Observation

If X is countable, then player II has a winning strategy in $G(X)$.

Theorem (*AC*)

There exists a subset $X \subseteq 2^{\mathbb{N}}$ such that $G(X)$ is not determined.

Borel Determinacy (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a Borel subset, then $G(X)$ is determined.

An easy application of Borel Determinacy

Definition

A subset $X \subseteq 2^{\mathbb{N}}$ is $\equiv_T$-invariant if it is a union of $\equiv_T$-classes.

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either X or $2^{\mathbb{N}} \smallsetminus X$ contains a cone.

Cf. Ergodicity ...

Proof of Martin's Theorem

- Suppose that $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset.
- Consider the two player game $G(X)$

$$s(0) \quad s(1) \quad s(2) \quad s(3) \quad \cdots$$

  where I wins if and only if $s = (\, s(0)\, s(1)\, s(2)\, \cdots) \in X$.

- Then the Borel game $G(X)$ is determined. Suppose, for example, that $\sigma : 2^{<\mathbb{N}} \to 2$ is a winning strategy for I.
- Let $\sigma \leq_T t \in 2^{\mathbb{N}}$ and consider the run of $G(X)$ where
  - II plays $t = (\, s(1)\, s(3)\, s(5)\, \cdots)$
  - I uses the strategy σ and plays $(\, s(0)\, s(2)\, s(4)\, \cdots)$.
- Then $s \in X$ and $s \equiv_T t$. Hence $t \in X$ and so $\mathcal{C}_\sigma \subseteq X$.

Some easy consequences of Martin's Theorem

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant Borel subset, then either X or $2^{\mathbb{N}} \smallsetminus X$ contains a cone.

Corollary

If $X \subseteq 2^{\mathbb{N}}$ is a $\equiv_T$-invariant $\leq_T$-cofinal Borel subset, then X contains a cone.

Corollary

If $X \subseteq 2^{\mathbb{N}}$ is an arbitrary $\leq_T$-cofinal Borel subset, then X contains representatives of a cone.

Pointed Trees

Definition

- A subset $S \subseteq 2^{<\mathbb{N}}$ is a tree if it is closed under taking initial segments.
- If S is a tree, then $[S] \subseteq 2^{\mathbb{N}}$ denotes the set of infinite branches through T.
- The tree S is perfect if for each $s \in S$, there exist incomparable $a, b \in S$ with $s \lessdot a, b$.
- The perfect tree S is pointed if $S \leq_T y$ for all $y \in [S]$.

Theorem (Martin)

If $X \subseteq 2^{\mathbb{N}}$ is a $\leq_T$-cofinal Borel subset, then there exists a pointed tree $S \subseteq 2^{<\mathbb{N}}$ such that $[S] \subseteq X$.

Proof of the Cone Theorem

The Cone Theorem

If $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous homomorphism from $\equiv_T$ to $\equiv_1$, then there exists a cone $\mathcal{C}$ such that θ maps $\mathcal{C}$ into a single $\equiv_1$-class.

- Let $\mathcal{A}$ be a cone such that $\theta(A) \leq_T A$ for all $A \in \mathcal{A}$.
- Then there exists a cone $\mathcal{C} \subseteq \mathcal{A}$ such that either
  (a) $\theta(A) <_T A$ for all $A \in \mathcal{C}$; or
  (b) $\theta(A) \equiv_T A$ for all $A \in \mathcal{C}$.

Case (a): suppose that $\theta(A) <_T A$ for all $A \in \mathcal{C}$.

Theorem (Slaman-Steel)

If $\mathcal{C}$ is a cone and $\theta : \mathcal{C} \to 2^{\mathbb{N}}$ is a Borel homomorphism from $\equiv_T \restriction \mathcal{C}$ to $\equiv_T$ such that $\theta(A) <_T A$ for all $A \in \mathcal{C}$, then there exists a cone $\mathcal{D} \subseteq \mathcal{C}$ such that θ maps $\mathcal{D}$ into a single $\equiv_T$-class.

- Thus θ maps a cone $\mathcal{D}$ into a single $\equiv_T$-class $\mathbf{a}$.
- Let $\mathbf{a} = \bigsqcup_{n \in \mathbb{N}} \mathbf{b}_n$ be the decomposition of $\mathbf{a}$ into $\equiv_1$-classes.
- For each $n \in \mathbb{N}$, let $\mathcal{B}_n = \theta^{-1}(\mathbf{b}_n)$.
- Then there exists $n \in \mathbb{N}$ such that $\mathcal{B}_n$ contains a cone, as required.

Case (b): suppose that $\theta(A) \equiv_T A$ for all $A \in \mathcal{C}$.

The Non-Selector Theorem

- *If $\mathcal{C}$ is a cone, then there does not exist a Borel homomorphism $\theta : \mathcal{C} \to \mathcal{C}$ from $\equiv_T \restriction \mathcal{C}$ to $\equiv_1 \restriction \mathcal{C}$ such that $\theta(A) \equiv_T A$ for all $A \in \mathcal{C}$.*
- *In other words, if $\mathcal{C}$ is a cone, then there does not exist a Borel map which selects an $\equiv_1$-class within each $\equiv_T$-class.*

Proof of the Non-Selector Theorem

- Suppose $\theta : \mathcal{C} \to \mathcal{C}$ selects a $\equiv_1$-class within each $\equiv_T$-class.
- Then $\theta[\mathcal{C}]$ is a $\leq_T$-cofinal Borel subset of $2^{\mathbb{N}}$.
- By Martin's Theorem, there exists a pointed tree $S \subseteq 2^{<\mathbb{N}}$ such that $[S] \subseteq \theta[\mathcal{C}]$.
- Note that if $x, y \in [S]$, then $x \equiv_T y$ iff $x \equiv_1 y$.
- We can suppose that $(\pi_n \mid n \in \mathbb{N}) \leq_T S$, where $\{\pi_n \mid n \in \mathbb{N}\}$ is the group of recursive permutations.
- Let $x \in [S]$ be the left-most branch, so that $x \equiv_T S$.
- Then we can construct a branch $y \leq_T S$ such that $\pi_n(y) \neq x$ for all $n \in \mathbb{N}$.
- But then $y \equiv_T x$ and $y \not\equiv_1 x$, which is a contradiction!

Proof of the Main Theorem

Main Theorem

There does not exist a Borel homomorphism $A \mapsto G_A$ from $\equiv_T$ to $\cong$ such that $\mathrm{Word}(G_A) \equiv_T A$ *for all* $A \in 2^{\mathbb{N}}$.

- Suppose that $A \mapsto G_A$ is a Borel homomorphism from $\equiv_T$ to $\cong$ such that $\mathrm{Word}(G_A) \equiv_T A$ for all $A \in 2^{\mathbb{N}}$.
- Consider the Borel map $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ defined by $A \mapsto \mathrm{Word}(G_A)$.
- If $A \equiv_T B$, then $G_A \cong G_B$ and so $\mathrm{Word}(G_A) \equiv_1 \mathrm{Word}(G_B)$.
- Thus $\theta : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a Borel map which selects an $\equiv_1$-class within each $\equiv_T$-class, which is a contradiction!

The End