Ramsey Cardinals and the HNN Embedding Theorem

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May 15th 2014

The HNN Embedding Theorem

Theorem (Higman-Neumann-Neumann 1949)

If G is a countable group, then G can be embedded into a 2-generator group K_G.

Notation

- $\mathcal{G}$ denotes the Polish space of countably infinite groups.
- $\mathcal{G}_{fg}$ denotes the Polish space of finitely generated groups.

Theorem

There does not exist a Borel map $G \mapsto K_G$ from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that for all $G, H \in \mathcal{G}$,

- $G \hookrightarrow K_G$; *and*
- *if* $G \cong H$, *then* $K_G \cong K_H$.

Acknowledging the existence of large cardinals ...

Main Theorem (*LC*)

- *Suppose that $G \mapsto K_G$ is any Borel map from $\mathcal{G}$ to $\mathcal{G}_{fg}$ such that $G \hookrightarrow K_G$ for all $G \in \mathcal{G}$.*
- *Then there exists an uncountable Borel family $\mathcal{F} \subseteq \mathcal{G}$ of pairwise isomorphic groups such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to relative constructibility; i.e., if $G \neq H \in \mathcal{F}$, then $K_G \notin L[K_H]$ and $K_H \notin L[K_G]$.*

Remarks

- (*LC*): There exists a Ramsey cardinal κ.
- In *ZFC*, we can find an uncountable Borel family $\mathcal{F}$ such that the groups $\{ K_G \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to embeddability.

Towards a proof of the Main Theorem ...

Question

What is known about the kernels of homomorphisms from complete analytic equivalence relations to countable Borel equivalence relations?

Answer (Kechris)

Not a lot!

Definition

- $\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ *is the Polish space of all injective maps* $z : \mathbb{N} \to 2^{\mathbb{N}}$.
- E_{cntble} *is the Borel equivalence relation on* $\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ *defined by*

$$z \, E_{cntble} \, z' \iff \{\, z(n) \mid n \in \mathbb{N} \,\} = \{\, z'(n) \mid n \in \mathbb{N} \,\}.$$

Homomorphisms have large kernels

Theorem

- *Let E be a countable Borel equivalence relation on the standard Borel space X and suppose that $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ is a Borel homomorphism from E_{cntble} to E.*
- *There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \mathrm{range}(z)$ such that $\theta(z) = x$.*

Corollary

If E is a countable Borel equivalence relation, then $E_{cntble} \not\leq_B E$.

Remark

There is no known "classical proof" of the Theorem.

Countable Quasi-orders

Definition

The relation $\preceq$ on the Polish space X is a countable quasi-order if:

(a) $\preceq$ *is reflexive and transitive.*

(b) *For all $x \in X$, the set $\{ y \in X \mid y \preceq x \}$ is countable.*

Some countable Borel quasi-orders

- The embeddability relation on $\mathcal{G}_{fg}$.
- The Turing reducibility relation $\leq_T$ on $2^{\mathbb{N}}$.

A countable Σ^1_2 quasi-order (LC)

The relative constructibility relation $\leq_c$ on $2^{\mathbb{N}}$ defined by

$$x \leq_c y \iff x \in L[y].$$

The Main Lemma

Main Lemma

Suppose that X is a Polish space and that $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ is any Borel map. Then at least one of the following must hold:

(a) *There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \mathrm{range}(z)$ such that $\theta(z) = x$.*

(b) *For each countable Borel quasi-order $\preccurlyeq$ on X, there exists a perfect subset $P \subseteq \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that*

  (i) *$y \; E_{cntble} \; z$ for all $y, z \in P$; and*

  (ii) *$\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.*

Moreover, if (LC) holds, then the conclusion also holds with respect to the quasi-order $\leq_c$ of relative constructibility.

We shall also make use of ...

Theorem (B.H. Neumann 1937)

There exists a Borel family $\{H_r \mid r \in 2^{\mathbb{N}}\} \subseteq \mathcal{G}$ *of pairwise nonisomorphic 2-generator groups.*

Proof.

- For each strictly increasing sequence $\mathbf{d} = \langle d_n \mid n \in \omega \rangle$ of odd integers with $d_0 \geq 5$, let $X^n_{\mathbf{d}} = \{x^n_1, x^n_2, \cdots, x^n_{d_n}\}$.
- And let $\Gamma_{\mathbf{d}}$ be the subgroup of $\prod_{n\in\omega} \mathrm{Alt}(X^n_{\mathbf{d}})$ generated by

$$\alpha_{\mathbf{d}} = \prod_{n\in\omega} (\, x^n_1 \;\; x^n_2 \;\; x^n_3 \;\; \cdots \;\; x^n_{d_n} \,) \quad \text{and} \quad \beta_{\mathbf{d}} = \prod_{n\in\omega} (\, x^n_1 \;\; x^n_2 \;\; x^n_3 \,).$$

- Then $\Gamma_{\mathbf{d}}$ has a normal subgroup isomorphic to $\mathrm{Alt}(m)$ iff $m = d_n$ for some $n \in \mathbb{N}$.

□

The Proof of the Main Theorem

- Suppose that $\varphi : \mathcal{G} \to \mathcal{G}_{fg}$ is a Borel map such that $G \hookrightarrow \varphi(G)$ for all $G \in \mathcal{G}$.
- Let $\{ H_r \mid r \in 2^{\mathbb{N}} \} \subseteq \mathcal{G}$ be a Borel family of pairwise nonisomorphic 2-generator groups.
- Let $\psi : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}$ be the injective Borel map defined by

$$\psi(z) = H_{z(0)} \oplus H_{z(1)} \oplus \cdots \oplus H_{z(n)} \oplus \cdots$$

  and consider $\theta = \varphi \circ \psi : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to \mathcal{G}_{fg}$.
- First suppose that there exists a group $G \in \mathcal{G}_{fg}$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that $r \in \mathrm{range}(z)$ and $\theta(z) = G$.
- Then H_r embeds into G for all $r \in 2^{\mathbb{N}}$, which is impossible since G has only countably many 2-generator subgroups!

The Proof of the Main Theorem

- Let $\preceq$ be either the embeddability relation or the relative constructibility relation on $\mathcal{G}_{fg}$.
- Then there exists a perfect subset $P \subseteq \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that
  (i) $y \; E_{cntble} \; z$ for all $y, z \in P$; and
  (ii) $\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.
- Hence $\mathcal{F} = \psi(P) \subseteq \mathcal{G}$ is an uncountable Borel family of pairwise isomorphic groups such that the groups $\{ \varphi(G) \mid G \in \mathcal{F} \}$ are pairwise incomparable with respect to $\preceq$.
- This completes the proof of the Main Theorem.

Towards a proof of the Main Lemma ...

Notation

- From now on, let V be the actual set-theoretic universe.
- Let $\mathbb{P}$ be a forcing notion.

Definition

- *The relation R on the Polish space X is $\boldsymbol{\Sigma}^1_n$ if $R(\bar{v})$ has the form*

$$(\exists x_1 \in X_1)(\forall x_2 \in X_2) \cdots B(x_1, x_2, \cdots, \bar{v}),$$

  where $X_1, \cdots, X_n$ are Polish spaces and $B(\bar{x}, \bar{v})$ is a Borel relation.
- *In this case, $R^{V^{\mathbb{P}}}$ denotes the relation obtained by applying the definition of R within the generic extension $V^{\mathbb{P}}$.*
- *R is absolute for $V^{\mathbb{P}}$ if $R^{V^{\mathbb{P}}} \cap V = R$.*

Shoenfield Absoluteness

Theorem (Shoenfield)

If $R \in V$ is a $\mathbf{\Sigma}^1_2$ relation, then R is absolute for every *generic extension $V^{\mathbb{P}}$.*

An Application

If $\preceq$ is a countable Borel quasi-order on the Polish space X, then $\preceq^{V^{\mathbb{P}}}$ is a countable Borel quasi-order on $X^{V^{\mathbb{P}}}$.

Proof.

Let $\mathrm{Perf}(X)$ be the Polish space of nonempty perfect subsets of X. Then $\preceq$ is countable if and only if

$$(\forall x \in X)\,(\forall P \in \mathrm{Perf}(X))\,(\exists y \in X)\;[\; y \in P \wedge y \not\preceq x \;].$$

$\square$

Martin-Solovay Absoluteness

Theorem (Martin-Solovay)

Let $\kappa \in V$ be a Ramsey cardinal. If $R \in V$ is a $\boldsymbol{\Sigma}^1_3$ relation and $|\mathbb{P}| < \kappa$, then R is absolute for $V^{\mathbb{P}}$.

An Application (LC)

$\leq_c$ is a countable $\boldsymbol{\Sigma}^1_2$ quasi-order on $2^{\mathbb{N}}$.

Proof.

If $\mathbb{P}$ is the poset of finite functions $p : \omega \to \omega_1$, then for all $x \in 2^{\mathbb{N}} \cap V$,

$$V^{\mathbb{P}} \vDash (\, \exists f \in (2^{\mathbb{N}})^{\mathbb{N}}\,)(\, \forall z \in 2^{\mathbb{N}}\,)[\, z \in L[\,x\,] \Longrightarrow (\, \exists n\,)\ f(n) = z\,].$$

By Martin-Solovay, this $\boldsymbol{\Sigma}^1_3(x)$ statement also holds in V. □

Virtual equivalence classes

Definition (Kanovei après Hjorth après Harrington)

Let E be a Borel equivalence relation on the Polish space X and let $\mathbb{P}$ be a forcing notion. Then a $\mathbb{P}$-name τ is a virtual E-class if:

- $\Vdash_{\mathbb{P}} \ \tau \in X^{V^{\mathbb{P}}}$
- $\Vdash_{\mathbb{P}\times\mathbb{P}} \ \tau_{left} \ E^{V^{\mathbb{P}\times\mathbb{P}}} \ \tau_{right}$

Here τ_{left}, τ_{right} are the $(\mathbb{P}\times\mathbb{P})$-names such that if $G \times H$ is $(\mathbb{P}\times\mathbb{P})$-generic, then $\tau_{left}[G \times H] = \tau[G]$ and $\tau_{right}[G \times H] = \tau[H]$.

Virtual equivalence classes

Example

- Let $E = E_{cntble}$ and let $\mathbb{P}$ consist of all finite injective partial functions $p : \mathbb{N} \to 2^{\mathbb{N}}$.
- If G is $\mathbb{P}$-generic, then $g = \bigcup G$ is a bijection between $\mathbb{N}$ and $2^{\mathbb{N}} \cap V$.
- Hence if τ is the canonical $\mathbb{P}$-name such that $\tau[G] = g$, then τ is a virtual E_{cntble}-class.

A reminder ...

Main Lemma

Suppose that X is a Polish space and that $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ is any Borel map. Then at least one of the following must hold:

(a) *There exists $x \in X$ such that for all $r \in 2^{\mathbb{N}}$, there exists $z \in \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \mathrm{range}(z)$ such that $\theta(z) = x$.*

(b) *For each countable Borel quasi-order $\preccurlyeq$ on X, there exists a perfect subset $P \subseteq \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that*

   (i) *$y \; E_{cntble} \; z$ for all $y, z \in P$; and*

   (ii) *$\theta(y), \theta(z)$ are incomparable with respect to $\preccurlyeq$ for all $y \neq z \in P$.*

Moreover, if (LC) holds, then the conclusion also holds with respect to the quasi-order $\leq_c$ of relative constructibility.

Towards a proof of the Main Lemma ...

- Let $\theta : \mathrm{Inj}(\mathbb{N}, 2^{\mathbb{N}}) \to X$ be any Borel map.
- Let $\preceq$ be either a countable Borel quasi-order on X or else the relative constructibility relation $\leq_c$.

Notation

- $x \perp y \iff x, y$ are $\preceq$-incomparable.
- $x \,||\, y \iff x, y$ are $\preceq$-comparable.

- Let $\mathbb{P}$ consist of all finite injective partial functions $p : \mathbb{N} \to 2^{\mathbb{N}}$ and let τ be the corresponding virtual E_{cntble}-class.

The Fundamental Dichotomy

Are $\theta(\tau_{left}), \theta(\tau_{right})$ comparable with respect to $\preceq^{V^{\mathbb{P}\times\mathbb{P}}}$?

Case 1: $(\exists p_0 \in \mathbb{P})\ \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

Claim

There exists $p_1 \leq p_0$ such that $\langle p_1, p_1 \rangle \Vdash \theta(\tau_{\text{left}}) = \theta(\tau_{\text{right}})$.

Proof.

- Suppose not and let $\mathbb{Q}$ collapse $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set.
- Working in $V^{\mathbb{Q}}$, there exists a perfect subset $P \subseteq \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ such that $\theta(P)$ is an uncountable Borel set of pairwise $\preceq$-comparable elements.
- Let $Z \subseteq \theta(P)$ be a perfect subset.
- By Kuratowski-Ulam, both $A = \{ (x, y) \in Z \times Z \mid x \preceq y \}$ and $B = \{ (x, y) \in Z \times Z \mid y \preceq x \}$ are meager subsets of $Z \times Z$.
- Since $Z \times Z = A \cup B$, this contradicts the Baire Category Theorem.

□

Case 1: $(\exists p_0 \in \mathbb{P})\ \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

Working in V and assuming that $X = [0,1]$, we can inductively define conditions

$$p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_n \geq \cdots$$

and closed intervals $I_n \subseteq [0,1]$ with rational endpoints

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that the following conditions hold:

- $|I_n| = 2^{-(n-1)}$
- $p_n \Vdash \theta(\tau) \in I_n$.

Still working in V, let

$$\bigcap_{n \geq 1} I_n = \{x\}.$$

Case 1: $(\exists p_0 \in \mathbb{P}) \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

Claim

$p_1 \Vdash \theta(\tau) = x$.

Proof.

- Otherwise, there exists $q \leq p_1$ and $n \geq 1$ such that $q \Vdash \theta(\tau) \notin I_n$.
- But then $\langle q, p_n \rangle \leq \langle p_1, p_1 \rangle$ satisfies

$$\langle q, p_n \rangle \Vdash \theta(\tau_{\text{left}}) \notin I_n \text{ and } \theta(\tau_{\text{right}}) \in I_n,$$

which is a contradiction.

□

Case 1: $(\exists p_0 \in \mathbb{P})\ \langle p_0, p_0 \rangle \Vdash \theta(\tau_{\text{left}}) \mid\mid \theta(\tau_{\text{right}})$.

- Let $G \subseteq \mathbb{P}$ be V-generic with $p_1 \in G$.
- Then $V[G] \vDash \theta(\tau[G]) = x$.
- Hence for each $r \in 2^{\mathbb{N}} \cap V$,

$$V[G] \vDash (\exists z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}}))(\exists n \in \mathbb{N})\ [z(n) = r \text{ and } \theta(z) = x].$$

- By Shoenfield Absoluteness, this $\boldsymbol{\Sigma}^1_1$ property of the reals $r, x \in 2^{\mathbb{N}} \cap V$ must also hold in V.
- Thus, in V, for all $r \in 2^{\mathbb{N}}$, there exists $z \in \text{Inj}(\mathbb{N}, 2^{\mathbb{N}})$ with $r \in \text{range}(z)$ such that $\theta(z) = x$.

Case 2: $(\forall p \in \mathbb{P})\ \langle p, p \rangle \not\Vdash \theta(\tau_{\text{left}}) \,||\, \theta(\tau_{\text{right}})$.

- Once again, let $\mathbb{Q}$ collapse $\mathcal{P}(\mathbb{P} \times \mathbb{P})$ to a countable set.
- Then $V^{\mathbb{Q}}$ satisfies the following statement:

$$(\exists P \in \text{Perf}(\text{Inj}(\mathbb{N}, 2^{\mathbb{N}})))\,(\forall x)\,(\forall y)$$
$$[\,(x, y \in P \wedge x \neq y) \Longrightarrow (x\, E_{cntble}\, y \wedge \theta(x) \perp \theta(y))\,].$$

- Applying either Shoenfield or Martin-Solovay Absoluteness, this statement also holds in V.
- This completes the proof of the Main Lemma.

An open problem

Definition

$\mathcal{G}^*_{fg}$ *is the space of f.g. groups with underlying set* $\mathbb{N}$.

Theorem

There does not exist a Borel map $\psi : \mathcal{G}^*_{fg} \to \mathbb{N}^{<\mathbb{N}}$ *such that*

- $\psi(G)$ *generates* G*; and*
- *if* $G \cong H$*, then* $\mathrm{Cay}(G, \psi(G)) \cong \mathrm{Cay}(H, \psi(H))$.

Conjecture

- Suppose that $\psi : \mathcal{G}^*_{fg} \to \mathbb{N}^{<\mathbb{N}}$ is any Borel map such that $\psi(G)$ generates G.
- Then what ???

The End