Smoothing transform and thin tails

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joint work with Gerold Alsmeyer *(University of Münster)*

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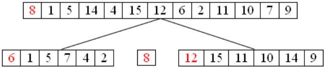

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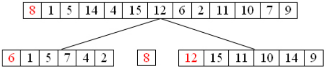

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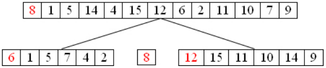

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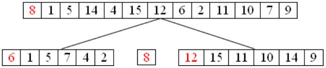

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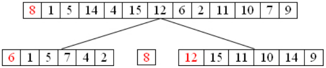

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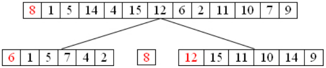

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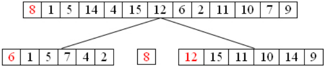

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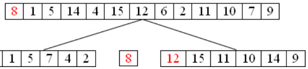

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$g(u) = 2u \log(u) + 2(1 - u) \log(1 - u) + 1$. When $n \to \infty$ we get

$X^{(qs)} \stackrel{d}{=} U X_1^{(qs)} + (1 - U) X_2^{(qs)} + g(U), \quad X_1^{(qs)}, X_2^{(qs)}$ independent of U.

Examples:

$$X \overset{d}{=} \lambda X + \xi_0, \quad X \overset{d}{=} e^{-T} X + \int_0^T e^{-s}\, \mathrm{d}\xi_s,$$

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Question 1: Given $(C, N, T_1, T_2, \ldots)$, when μ can be found? When μ is unique?

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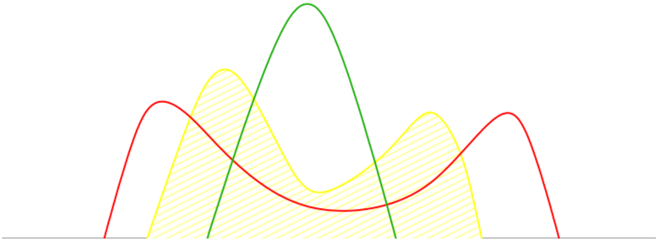

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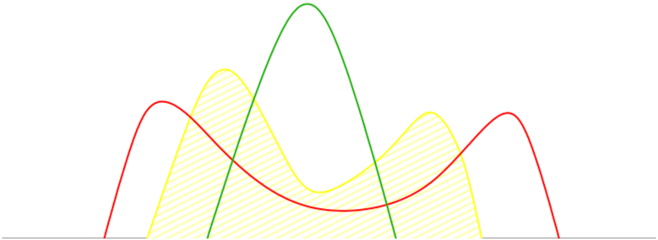

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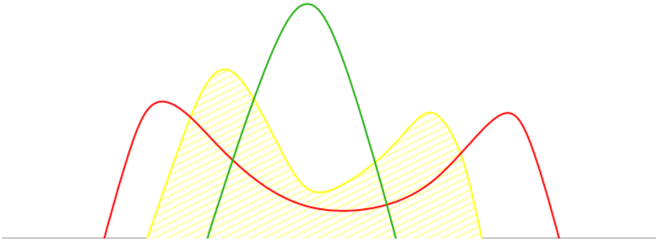

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Hence $(\mathcal{M}, \mathrm{d}_2) \subset L^2[0,1]$.

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Theorem (U. Rösler 1992)

Assume that

$$\mathbb{E}\left[\sum_{k=1}^{N} T_k^2\right] < 1, \quad \mathbb{E}[C] = 0, \quad 0 < \mathbb{E}\left[C^2\right] < \infty. \qquad (\spadesuit)$$

Then $\mathcal{S} \colon (\mathcal{M}, \mathrm{d}_2) \to (\mathcal{M}, \mathrm{d}_2)$ *is a contraction.*

$$\mathcal{S}(\eta) = \mathcal{L}\left(\sum_{k=1}^{N} T_k Y_k + C\right).$$

$$\mathrm{d}_2(\mu, \eta) = \inf\left\{ \sqrt{\mathbb{E}\left[(X - Y)^2\right]} \;\middle|\; X \sim \mu,\ Y \sim \eta \right\}.$$

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Proof.

For $\eta, \nu \in \mathcal{M}$, take (Y_k, Z_k) iid such that $\mathrm{d}(\eta,\nu) = \sqrt{\mathbb{E}\left[(Y_k - Z_k)^2\right]}$.

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$$= \mathbb{E}[(Y_1 - Z_1)^2]\mathbb{E}\left[\sum_{k=1}^{N} T_k^2\right] = \mathrm{d}_2(\eta, \nu)^2 \mathbb{E}\left[\sum_{k=1}^{N} T_k^2\right]$$

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Corollary

Assume (♠), then there exists a unique solution (in $\mathcal{M}$) of

$$X \stackrel{d}{=} \sum_{k=1}^{N} T_k X_k + C.$$

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Let $\Theta > 0$ then $\Theta \in \mathbb{D}_\Psi$ if, and only if $\exists\, \Phi\colon [0,\Theta] \to (0,+\infty)$, $\Phi(0) = 1$, $\Phi(\theta) \ge 1$ such that

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Proof.

For $Z_0 = 0$ we have

$$\mathbb{E}[\exp\{\theta Z_0\}] = 1 \le \Phi(\theta) \qquad \text{for } \theta \in [0, \Theta].$$

Consider $Z_n \stackrel{d}{=} \mathcal{S}^n(\delta_0)$.

Proof continued.

$Z_n \overset{d}{=} \mathcal{S}(Z_{n-1})$ with $Z_0 = 0$.

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On the other hand if $\Theta \in \mathbb{D}_\Psi$ then $\Phi(\theta) = \Psi(\theta)$ satisfies

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Rough idea.

For Φ increasing sufficiently fast

$$\mathbb{E}\left[\exp\{\theta C\}\Phi(\max T_k\theta)\prod_j \Phi(T_j\theta)\right] \approx \mathbb{E}\left[e^{\theta C}\mathbb{1}_{\{\max_k T_k = 1\}}\right]\Phi(\theta) \le \Phi(\theta).$$

Example

Suppose $A \stackrel{d}{=} B(\alpha, 1)$, and let $N = n \geq 1$ such that $\alpha < \frac{2}{n-1}$, $T_1 = T_2 = \ldots = T_n = A$ and take C independent of A with $\mathbb{E}[C] = 0$ and $\varphi(\theta) = \mathbb{E}[\exp\{\theta C\}]$.

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By computing the derivative $\frac{\mathrm{d}}{\mathrm{d}\theta}$ one gets

$$\Psi'(\theta) = \frac{\alpha\varphi(\theta)}{\theta} \Psi(\theta)^n + \left(\frac{\varphi'(\theta)}{\varphi(\theta)} - \frac{\alpha}{\theta} \right) \Psi(\theta)$$

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and so

$$\Psi^{n-1}(\theta) = \frac{\varphi(\theta)^{n-1}}{1 - \int_0^\theta (\varphi(s)^n - 1) \left(\frac{\theta}{s}\right)^{\alpha(n-1)+1} \alpha(n-1)s^{-1}\,\mathrm{d}s}$$

We see that $\mathbb{D}_\Psi$ is given by

$$\mathbb{D}_\Psi = \left\{ \theta : \int_0^\theta (\varphi(s)^n - 1) \left(\frac{\theta}{s}\right)^{\alpha(n-1)+1} \alpha(n-1)\, ds < 1 \right\} \quad (1)$$

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As for $s < \theta$, the function

$$\alpha \; \mapsto \; \alpha \left(\frac{\theta}{s}\right)^{\alpha(n-1)+1}$$

is increasing, the set $\mathbb{D}_\Psi$ in (1) gets smaller, while the probabilities in (2) get bigger with increasing α.

Corollary

There exists a unique (in $\mathcal{M}$) solution to

$$X^{(qs)} \stackrel{d}{=} UX_1^{(qs)} + (1-U)X_2^{(qs)} + g(U),$$

where $X_1^{(qs)}, X_2^{(qs)}$ are independent of U.

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Question 3: Can we say something more about $\mathbb{P}[X^{(qs)} > x]$ as $x \to \infty$? Or in general about $\mathbb{P}[X > x]$?

Recall that if

$$\mathbb{E}\left[\exp\{\theta C\}\prod_{k=1}^{N}\Phi(T_k\theta)\right] \leq \Phi(\theta)$$

then

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$$\log \mathbb{P}[X > x] \leq \inf_{\theta} \left\{\log \Phi(\theta) - \theta x\right\}.$$

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for some $p \geq 1$ *and for* $\gamma, D > 0$

$$\mathbb{P}[\max_{1 \leq k \leq N} T_k \in (1-\delta, 1]] \leq D\delta^\gamma.$$

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Then

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}[X > x]}{x \log x} \le -\frac{\gamma}{c^+},$$

where $c^+ = \|C^+\|_\infty < \infty$.

Proof.

We will show that $\Phi(\theta) = e^{\theta^p e^{b\theta}}$, with $b := c^+/\gamma$ satisfies

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$\|C^+\|_\infty = c^+$, $\| \max T_k\|_\infty = 1$. We need to know that these maxima are attained on the same set, that is

$$\mathbb{P}[C \approx c^+, \max T_k \approx 1] > 0.$$

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Theorem (G. Alsmeyer, P. D.)

As always, assume $\|N\|_\infty < \infty$*,*

$$\mathbb{E}\left[\sum_{k=1}^{N} T_k^2\right] < 1, \quad \mathbb{E}[C] = 0, \quad 0 < \mathbb{E}[C^2] < \infty, \quad \sum_{k=1}^{N} T_k^p \le 1.$$

for some $p \ge 1$*,* $q < c^+$ *for* $\gamma, d > 0$

$$d\delta^\gamma \le \mathbb{P}[\max_{1 \le k \le N} T_k < 1 - \delta,\ C > c^+ - \delta].$$

Then

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}[X > x]}{x \log x} \ \ge \ -\frac{\gamma}{c^+},$$

where $c^+ = \|C^+\|_\infty < \infty$*.*

Proof.

$$X \stackrel{d}{=} \sum_{|v|=n} L(v)X(v) + \sum_{k=0}^{n-1} \sum_{|v|=k} L(v)C(v)$$

$$\mathcal{T} := \{\text{first vertex } v \text{ s.t.} T_k(v) \leq 1-\delta \text{ or } C(v) \leq c := c^+ - \delta\}.$$

$$X \stackrel{d}{=} \sum_{v\in\mathcal{T}} L(v)X(v) + \sum_{v\prec\mathcal{T}} L(v)C(v).$$

If τ denotes the length of $\mathcal{T}$, then on the set $\{X(v) \geq 0,\ v \in \mathcal{T}\}$,

$$X \geq \sum_{v\prec\mathcal{T}} L(v)C(v) \geq \sum_{k=0}^{\tau-1} L(e_k)C(e_k) > \sum_{j=0}^{\tau-1} (1-\delta)^j c = \frac{1-(1-\delta)^\tau}{\delta} c.$$

Note that τ has geometric distribution with
$1-p = \mathbb{P}[\max_{1\leq k\leq N} T_k < 1-\delta,\ C > c^+ - \delta]$

$$\mathbb{P}\left[X > \frac{1-(1-\delta)^k}{\delta} c\right] \gtrsim \mathbb{P}[\tau = k]$$

Corollary

There exists a unique (in $\mathcal{M}$) solution to

$$X^{(qs)} \overset{d}{=} UX_1^{(qs)} + (1-U)X_2^{(qs)} + g(U),$$

where $X_1^{(qs)}, X_2^{(qs)}$ are independent of U.

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$$\mathbb{E}[\exp\{\theta X^{(qs)}\}] < \infty.$$

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Furthermore

$$\lim_{x\to\infty} \frac{\log \mathbb{P}[X^{(qs)} > x]}{x \log x} = -1.$$