Vaught's conjecture: classifying countable models of weakly quasi-o-minimal theories

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T–complete theory in a countable language with infinite models.

$I(T, \aleph_0)$ –the number of countable models

Vaught's conjecture

$I(T, \aleph_0) \in \{1, 3, 4, ...\} \cup \{\aleph_0, 2^{\aleph_0}\}$ (independently of CH).

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Theorem (Steel 1976)

Every consistent (with the theory of colored trees) $L_{\omega_1\omega}$-sentence has either countably many or perfectly many countable models.

Classification problem

Assuming $I(T, \aleph_0) < 2^{\aleph_0}$ find a reasonable system of invariants that describes countable models up to $\cong$.

$I(T, \aleph_0) = 2^{\aleph_0}$ if

- T is a theory of linearly ordered structure with built-in Skolem functions (Shelah 1978)
- T has a definable infinite, discrete linear order (by Steel's methods,
  or T. *Vaught's conjecture for theories of discretely ordered structures.* https://arxiv.org/pdf/2212.13605.pdf

Theorem (Mayer 1988)

If T is o-minimal then $I(T, \aleph_0) \in \{3^m \cdot 6^n \mid m, n \in \omega\} \cup \{2^{\aleph_0}\}$.

- Strongly minimal T (Marsh 1966)
- Uncountably categorical T (Morley 1967)
- Theories of colored orders (Rubin 1974)
- Theories of one unary operation (Miller 1981)
- Stable theories with Skolem functions (Lascar 1981)
- $\aleph_0$-stable T (Shelah 1984)
- o-minimal T (Mayer 1988)
- Weakly minimal T (Saffe, Buechler, Newelski 1990)
- Superstable of finite U-rank T (Buechler 2008)
- Varieties (Hart, Starchenko, Valeriote 1994)
- Binary, weakly quasi-o-minimal T (Moconja, T. 2020)
- Weakly o-minimal of finite convexity rank (Kulpeshov 2020)

"Reasonable" open cases of VC:

- T binary (every formula is T-equivalent to a boolean combination of formulas in at most 2 free variables). Subcases: T binary stable; T binary, dp-minimal, ordered;
- T stable, one-based;
- T has Skolem functions;
- T weakly quasi-o-minimal.

Definition

Let $T = Th(\mathfrak{C}, <, , ...)$.

- T is o-minimal (with respect to $<$) if every definable $D \subset \mathfrak{C}$ is a boolean combination of intervals;
- T is weakly o-minimal (with respect to $<$) if every definable $D \subset \mathfrak{C}$ has finitely many convex components;
- T is quasi o-minimal (with respect to $<$) if every definable $D \subset \mathfrak{C}$ is a boolean combination of unary 0-definable sets and intervals (example $(\mathbb{Z}, +, <, 0)$);
- T is weakly quasi o-minimal (with respect to $<$) if every definable $D \subset \mathfrak{C}$ is a boolean combination of unary 0-definable sets and convex sets.

The main result

Theorem

Vaught's conjecture holds for weakly quasi-o-minimal satisfying:

(R) Every relatively definable equivalence relation on the locus of a complete 1-type is relatively 0-definable.

The following wqom theories satisfy (R):

- binary;
- quasi o-minimal;
- finite convexity rank and $I(T, \aleph_0) < 2^{\aleph_0}$;
- rosy and $I(T, \aleph_0) < 2^{\aleph_0}$.

The main result follows from:

Theorem

Let T be weakly quasi-o-minimal.

- If $I(\aleph_0, T) < 2^{\aleph_0}$ and T satisfies (R), then T is 1-trivial.
- If T is 1-trivial and $I(\aleph_0, T) < 2^{\aleph_0}$ then T is binary.

Weakly o-minimal types

Moconja, T. *Does weak quasi o-minimality behave better than weak o-minimality?* https://arxiv.org/pdf/1811.05228.pdf

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Definition

$p \in S_n(A)$ is a **weakly o-minimal type** if there is a relatively A-definable linear order $<_p$ on $p(\mathfrak{C})$ such that every relativrly definable subset of $p(\mathfrak{C})$ has finitely many convex components

$\mathbf{p} = (p, <_p)$ is called a **weakly o-minimal pair** over A.

For E a convex equivalence relation on $(X, <)$ define:

$$x <_E y \text{ iff } [x]_E < [y]_E \vee ([x]_E = [y]_E \wedge y < x)$$

For $\vec{E} = (E_1, E_2, ..., E_n)$ decreasing convex equivalences, define $<_{\vec{E}}$.

Theorem

Let $\mathbf{p} = (p, <)$ be a weakly o-minimal pair over A.

1. Any relatively A-definable linear order on $p(\mathfrak{C})$, $<_p$, is of the form $<_p = <_{\vec{E}}$ for some A-definable $\vec{E}$;
   $(p, <_p)$ is a weakly o-minimal pair.
2. T is weakly quasi o-minimal with respect to some (equivalently all) 0-definable order iff every $p \in S_1(T)$ is a weakly o-minimal type.
3. (Weak monotonicity theorem)
   If $(D, <_D)$ is a A-definable linear order and $f : p(\mathfrak{C}) \to D$ is relatively A-definable, then f is *weakly monotone*, i.e
   $f : (p(\mathfrak{C}), <_{\vec{E}}) \to D$ is increasing for some definable $\vec{E}$.

- Let $\mathbf{p} = (p, <_p)$ be a weakly o-minimal pair over A. Then p has exactly two A-invariant (equivalently non-forking) globalizations: the left one
  $$\mathbf{p}_l = \{\phi(x) \mid \phi(\mathfrak{C}) \cap p(\mathfrak{C}) \text{ is left-eventual in } p(\mathfrak{C})\}$$
  and the right one $\mathbf{p}_r$.
  - $a \models p$ is **right p-generic over** B, $B \lhd^{\mathbf{p}} a$, if $a \models \mathbf{p}_r{\restriction}AB$.
  - $B \lhd^{\mathbf{p}} a_1 <_p a_2$ implies $B \lhd^{\mathbf{p}} a_2$.
- Relatively A-definable equivalence relations on the locus of a wom type $p(\mathfrak{C})$ are convex and $\subseteq$-comparable.
- $x \not\perp y\,(A)$ is an equivalence relation on realizations of wom-types over A.
- $\not\perp^w$ and $\not\perp^f$ are equivalence relations on the wom-types over A.

Shifts

Shifts are redefined quasi-successors (generalized successor) from:

Alibek, Baizhanov, Zambarnaya. *Discrete order on a definable set and the number of models* (2014)

Motivated by: $I(T, \aleph_0) = 2^{\aleph_0}$ if T has a definable infinite, discrete linear order:

Conjecture

If T has a definable shift then $I(T, \aleph_0) = 2^{\aleph_0}$.

Let $(D, <)$ be a linear order.

- A convex set $S \subset D$ is a semi-interval if $a = \min S$ exists; we write S_a instead of S.
- $\mathcal{S} = (S_a \mid a \in D)$ is a monotone family of semi-intervals if $a < b$ implies $\sup S_a \leqslant \sup S_b$.

Let $(D, <)$ be a A-definable linear order and let $\mathcal{S} = (S_a \mid a \in D)$ be a monotone, A-definable family of semi-intervals of $(D, <)$.

- Recursively define $S_x^1 = S_x$, $S_x^{n+1} = \bigcup_{y \in S_x^n} S_y$.
- $S_x^1 \subseteq \S_x^2 \subseteq S_x^3 \subseteq ....$ are semi-intervals with minimum x
- Say that S_a is a $\mathcal{S}$-shift if $S_a^1 \subset S_a^2 \subset S_a^3 \subset ....$

Theorem

If T is weakly quasi-o-minimal with a definable shift (i.e. there is a monotone family of semi-intervals of $(\mathfrak{C}, <)$ that contains a shift) then $I(T, \aleph_0) = 2^{\aleph_0}$.

Corollary

If T is weakly quasi-o-minimal with $I(T, \aleph_0) < 2^{\aleph_0}$ and $a, b \in \mathfrak{C}$ then

$$a \perp b \, (dcl^{eq}(aA) \cap dcl^{eq}(bA))$$

Trivial types

trivial (stable) theory $\approx$ every pairwise independent set is independent

Definition

Let $\mathfrak{p} \in S_n(\mathfrak{C})$ be A-invariant and $p = \mathfrak{p} \restriction A$.

(a) $\mathfrak{p}$ is **trivial over** A if whenever $I = (a_i \mid i \in \omega)$ is a sequence of realizations of p such that $(a_i, a_j) \models \mathfrak{p}^2 \restriction A$ for all $i < j \in \omega$, then I is Morley over A.

(b) $\mathfrak{p}$ is **order-trivial over** A if whenever $I = (a_i \mid i \in \omega)$ is a sequence of realizations of p such that $(a_i, a_{i+1}) \models \mathfrak{p}^2 \restriction A$, then I is Morley over A.

Question

If $\mathfrak{p}$ is order-trivial over A, must there exist a definable partial order over some $B \supset A$ such that Morley sequences in $\mathfrak{p}$ over B are strictly increasing?

Fact

For a weakly o-minimal pair $\mathbf{p}$ over A tfae:

- $\mathbf{p}_r$ is trivial over A;
- $\mathbf{p}_r$ is order-trivial over A;
- $\mathbf{p}_l$ is trivial over A;
- $\mathbf{p}_l$ is order-trivial over A.

In that case, we say p is trivial.

Let $\mathbf{p} = (p, <_p)$ and $\mathbf{q} = (q, <_q)$ be wom pairs over A.

- Define: $\mathbf{p}$ and $\mathbf{q}$ are **directly non-orthogonal**, $\delta(\mathbf{p}, \mathbf{q})$, if for all $a \models p$ and $b \models q$:
  a is left $\mathbf{p}$-generic over b iff b is right $\mathbf{q}$-generic over a.
- δ is an equivalence relation with 2 classes on each $\not\perp^w$-class of wom pairs over A.
- Fix one δ-class $\mathcal{F}$; then $\lhd^{\mathcal{F}}$ is a well defined strict ordering ... A pairwise independent set is one that is totally ordered by $\lhd^{\mathcal{F}}$.

Proposition

- Triviality is preserved under $\not\perp^w$ of wom types.
- Pairwise independence implies independence for realizations of trivial $\not\perp^w$ types over A: if $a_0 \lhd^{\mathfrak{F}} a_1 \lhd^{\mathfrak{F}} a_2 \lhd^{\mathfrak{F}} ....$, then $\bar{a}_{<n} \lhd^{\mathfrak{F}} a_n$

Theorem

Assume $I(T, \aleph_0) < 2^{\aleph_0}$. Let A be finite and let $p \in S_n(A)$ be a trivial weakly o-minimal type.

- p is convex: there is an A-definable $(D, <)$ with $p(\mathfrak{C})$ a convex subset of D.
- p is simple: $x \not\perp y\,(A)$ is a relatively A-definable equivalence relation on $p(\mathfrak{C})$

A wqom theory is 1-trivial if every $p \in S_1(A)$ is trivial for all $A \subset \mathfrak{C}^{eq}$.

Proposition

If $I(\aleph_0, T) < 2^{\aleph_0}$ and T satisfies (R), then T is 1-trivial.

The main result follows from:

Theorem

If T is 1-trivial and $I(\aleph_0, T) < 2^{\aleph_0}$ then T is binary. In particular Vaught's conjecture holds for 1-trivial theories.

Theorem (Moconja, T. 2020)

T–countable, binary, weakly quasi-o-minimal theory.

- $I(\aleph_0, T) = 2^{\aleph_0}$ iff at least one of the following holds:
  - (a) T is not small;
  - (b) there is a non-convex type $p \in S_1(T)$;
  - (c) there is a non-simple type $p \in S_1(T)$;
  - (d) there are infinitely many $\not\perp^w$-classes of non-isolated types in $S_1(T)$;
  - (e) there is a non-isolated forking extension of some $p \in S_1(T)$ over an 1-element domain.
- $I(\aleph_0, T) = \aleph_0$ iff none of the above holds and there are infinitely many $\not\perp^f$-classes of non-isolated types in $S_1(T)$;
- If none of the above holds, then:

$$I(\aleph_0, T) = \Pi_{i \in w_T \setminus u}(|\alpha_i^{\mathfrak{F}}| + 2) \cdot \Pi_{j \in u}(|\alpha_j^{\mathfrak{F}}|^2 + 3|\alpha_j^{\mathfrak{F}}| + 2)$$

THANK YOU!!!