What do we know about $\mathbb{F}_p((t))$?

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In the year 1965 Ax and Kochen generated much interest in the model theory of valued fields through their proof of a correct version of Artin's Conjecture about non-trivial zeros of forms over the p-adic numbers. Since then one of the best known open problems in model theoretic algebra is whether the elementary theory of the field $\mathbb{F}_p((t))$ of formal Laurent series over the field with p elements is decidable. Although this field looks so similar to the field $\mathbb{Q}_p$ of p-adic numbers and the theory of the latter has been shown by Ax, Kochen and Ershov to be decidable, several excellent model theorists tried in vein to solve this problem. But this is not due to a lack of knowledge in model theory. In contrast to $\mathbb{Q}_p$, the field $\mathbb{F}_p((t))$ is a valued field of positive characteristic, and we simp ly do not know enough about the structure of such valued fields. For this reason, model theoretical questions have stimulated new research in a classical area of algebra: valuation theory.

One way to prove the decidability is to find a complete recursive axiom system for the elementary theory of $\mathbb{F}_p((t))$. A complete recursive axiom system for $\mathbb{Q}_p$ is well known since the work of Ax, Kochen and Ershov. An adaptation of their axiom system to the case of $\mathbb{F}_p((t))$ is: "henselian defectless valued field of characteristic p with value group a $\mathbb{Z}$-group and residue field $\mathbb{F}_p$".

I showed in 1989 that this axiom system is not complete (cf. [3]). In positive characteristic, the existence of additive polynomials lets us formulate elementary properties that are independent of the "classical ones" that we know from $\mathbb{Q}_p$. A polynomial f over an infinite field K is called additive if $f(a+b) = f(a) + f(b)$ for all a, b in K. If the characteristic of K is 0, then the only additive polynomials are of the form cX with c in K. But if the characteristic is $p > 0$, then for i nstance, X^p and the Artin-Schreier-polynomial $X^p - X$ are additive. I will explain the particular role that additive polynomials play in valuation theory in positive characteristic (cf. [4]).

Following an idea of Lou van den Dries, attempts have been made to understand the model theory of valued fields as valued modules over rings of additive polynomials ([5]), but many questions have remained open. It seems that forgetting about multiplication and replacing it by the application of the Frobenius instead does not make matters easier. In fact, this "reduct" may already encode all the problems we have in dealing with non-perfect valued fields.

In 1998 I developed an elementary axiom scheme which holds in maximal valued fields of positive characteristic and describes valuation theoretic properties of additive polynomials on such fields (cf. [3]). This axiom scheme is independent of the above axioms. In 1999, in joint work with Lou van den Dries ([2]), I developed a particularly simple version of this axiom scheme for $\mathbb{F}_p((t))$, using the fact that this field is locally compact.

Further, I will describe the technical problems that make the elementary theory of $\mathbb{F}_p((t))$ so hard to handle, in contrast to all other theories of valued fields for which a good model theory is known.

Here is at least one positive result: Denef and Schoutens showed that if resolution of singularities in positive characteristic holds, then the existential elementary theory of $\mathbb{F}_p((t))$ is decidable ([1]).

References

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