Iterative ∂ -varieties via group scheme actions

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$C \subseteq K$ is a field extension.

Definition of iterative (Hasse-Schmidt) *C*-derivations

 $\partial = (\partial_n)_{n \in \mathbb{N}}$ is an iterative *C*-derivation if for each $n, m \in \mathbb{N}$

- ① $\partial_n : K \to K$ is additive and $\partial_n(xy) = \sum_{i+j=n} \partial_i(x) \partial_j(y)$,
- $\partial_0 = id$,
- - Let $\partial: K \to K[\![X]\!]$, $\partial(x) = \sum \partial_n(x) X^n$. Then 1 (with 2) is equivalent to ∂ being a ring homomorphism (*C*-algebra map).
 - 3 is equivalent to ∂ being a section of $K[X] \to K$.



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- **1** $\partial_n : K \to K$ is additive and $\partial_n(xy) = \sum_{i+j=n} \partial_i(x) \partial_j(y)$,
- $\partial_n|_C=0,$
- $\partial_0 = id$,
- $\bullet \ \partial_m \circ \partial_n = \binom{n+m}{m} \partial_{n+m} \text{ (iterativity)}.$
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The iterativity condition

Let
$$c_K : K[X] \to K[X_1, X_2], c_K(F) = F(X_1 + X_2).$$

Since

$$c(\sum_{n} a_n X^n) = \sum_{i,j} {i+j \choose i} a_{i+j} X_1^i X_2^j,$$

 ∂ is iterative iff the following diagram is commutative:

where
$$\partial_*(\sum a_n X^n) = \sum \partial_j(a_i) X_1^i X_2^j$$
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Formal group and complete Hopf algebra

- Consider $c: C[X] \to C[X_1, X_2], c(F) = F(X_1 + X_2).$
- Since $C[X_1, X_2] \cong C[X] \widehat{\otimes}_C C[X]$, the following diagram is commutative:

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 This diagram and two more things mean that C[X] becomes a complete Hopf algebra (a co-group object in the category of complete C-algebras) corresponding to the formal group G

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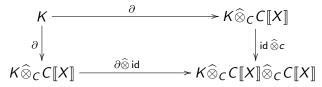
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Complete Hopf algebra co-action

• Since $K[X] \cong K \widehat{\otimes}_C C[X]$, ∂ is an iterative C-derivation iff the following diagram is commutative:

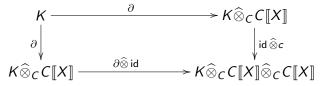


• This means that ∂ is a co-action of the complete Hopf algebra C[X] on K. Note that ∂ is a section of $K[X] \to K$ iff the following diagram is commutative (" $1 \cdot x = x$ " axiom):

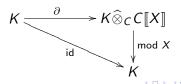


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- What happens if we take a 1-dimensional formal group (law) other that $X_1 + X_2$?
- If char(C) = 0, then all 1-dimensional formal groups are isomorphic.
- If char(C) > 0, then e.g. $\widehat{\mathbb{G}}_a \ncong \widehat{\mathbb{G}}_m$ and 1-dimensional formal groups are classified by $\mathbb{N} \cup \{\infty\}$.
- Assume we take the iterative law $X_1X_2 + X_1 + X_2$ corresponding to $\widehat{\mathbb{G}}_m$. What is the corresponding theory of existentially closed fields with $\widehat{\mathbb{G}}_m$ -iterative derivations?
- The iterative law $X_1X_2 + X_1 + X_2$ (apparently) gives $(i \leq j)$:

$$\partial_j \circ \partial_j = \sum_{k=j}^{i+j} \frac{k!}{(k-j)!(k-i)!(i+j-k)!} \partial_k.$$

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- We prefer to have actual Hopf algebras (corresponding to group schemes rather than formal groups).
- Since $C[X] \cong \varprojlim C[X]/(X^n)$ we want a map c_n such that the following diagram is commutative:

$$C[X] \xrightarrow{c} C[X_1, X_2]$$

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$$C[X]/(X^n) \xrightarrow{c_n} C[X_1, X_2]/(X_1^n, X_2^n)$$

- We need $c(X^n) \in (X_1^n, X_2^n) C[X_1, X_2]$. For n > 1, it happens if and only if char(C) = p > 0 and $n = p^m$.
- $C[X]/(X^{p^m})$ is a Hopf algebra, so $Spec(C[X]/(X^{p^m}))$ is a group scheme: $ker(Fr^m : \mathbb{G}_a \to \mathbb{G}_a)$ (not an algebraic group!).

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Iterativity again

Iterative C-derivations in positive characteristic

- An iterative C-derivation on K is a compatible sequence of co-actions ∂_m of the Hopf algebras $C[X]/(X^{p^m})$ on K.
- "Compatible" here means that for n < m the following diagram is commutative:

$$\begin{array}{ccc}
K & \xrightarrow{\partial_n} & K \otimes_C C[X]/(X^{p^n}) \\
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The situation looks more natural when we go to the category of *C*-schemes. Hopf algebras correspond to group schemes and co-actions correspond to group scheme actions.

Definition of the category of iterative ${\mathcal C}$ -schemes

- Let $\alpha_n := \operatorname{Spec}(C[X]/(X^{p^n}))$. It is a group scheme.
- An iterative C-scheme is a C-scheme X together with a compatible sequence of group scheme actions

$$\partial_n : \alpha_n \times_C X \to X$$
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• An iterative *C*-morphism is an equivariant *C*-morphism.



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We fix a C-iterative field (K, ∂) which is existentially closed and ω -saturated. Then $S := \operatorname{Spec}(K)$ is an iterative C-scheme. Assume $C = K^{p^{\infty}}$.

Iterative ∂-schemes

- An (iterative) ∂ -scheme is an iterative C-scheme X with an iterative C-morphism $X \to S$.
- (Iterative) ∂ -morphism is a K-morphism which is C-iterative.
- ∂ -point of a ∂ -scheme X is a K-point x such that the corresponding morphism $x:S\to X$ is a ∂ -morphism.
- X^{\sharp} is the set of all ∂ -points.



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∂ -points exist

Iterative differential equations

- If $X \subseteq \mathbb{A}^n$, then X^{\sharp} is given by an infinite system of iterative differential equations (so is type-definable in $(K, +, \cdot, \partial)$).
- It plays the same role as $(X, f)(K, \sigma)$ from Scanlon's talk.
- X^{\sharp} is "very" very thin and of finite U-rank.

Lemma

If a ∂ -scheme X has an open affine subvariety, then $X^{\sharp} \neq \emptyset$.

Proof

- An open sub-scheme $U \to X$ (even in the étale topology) has a unique ∂ -structure such that $U \to X$ is a ∂ -morphism.
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∂-points exist

Iterative differential equations

- If $X \subseteq \mathbb{A}^n$, then X^{\sharp} is given by an infinite system of iterative differential equations (so is type-definable in $(K, +, \cdot, \partial)$).
- It plays the same role as $(X, f)(K, \sigma)$ from Scanlon's talk.
- X^{\sharp} is "very" very thin and of finite *U*-rank.

Lemma

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Definition

- If X_C is a C-scheme, then it has a C-iterative scheme structure given by the 0-derivation (trivial action).
- Then X := X_C ×_C S has a ∂-structure coming from the 0-structure on X_C and the ∂-structure on S. We call such a ∂-schemes trivial and ∂-isomorphic to them trivializable.

- If X is a trivial ∂ -scheme, then $X^{\sharp} = X_{\mathcal{C}}(\mathcal{C})$.
- If X is a trivializable ∂ -variety, then X^{\sharp} is C-internal.

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Let \mathcal{C} be an arbitrary category with fiber products (e.g. category of C-schemes or category of sets). Let $X \to Y$ be a morphism in \mathcal{C} .

Automorphisms

It is possible to extend the group of automorphisms of X over
 Y to the following contravariant functor:

$$A_{X/Y}: \mathcal{C}_Y \to \mathbf{Gps}, \quad A_{X/Y}(Z) = \mathrm{Aut}_Z(X \times_Y Z)$$

• Aut $_Y(X)$ is a group object in $\mathcal C$ if $A_{X/Y}$ is representable.

Example

In the category of sets $\operatorname{Aut}(X)$ indeed represents A_X – for any set Z a map $f:Z\to\operatorname{Aut}(X)$ corresponds to the Z-automorphism of $X\times Z$ given by $\overline{f}(x,z)=(f(z)(x),z)$.

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Theorem (K., Pillay)

If X is a projective variety over K and D_1, D_2 are iterative ∂ -scheme structures on X, then $(X, D_1) \cong (X, D_2)$. In particular, if X descents to C, then (X, D_1) is trivializable.

- By Matsumura/Oort, $G := Aut_K(X)$ is a group scheme whose connected component is an algebraic group.
- D_1 and D_2 give together a ∂ -structure D_{12} on G such that $g \in G^{\sharp}$ iff g is a ∂ -isomorphism between (X, D_1) and (X, D_2) .
- Since G^0 is an algebraic variety and is open in G, we know that G^{\sharp} is non-empty.

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Finding ∂ -structure on G continued

Idea of the proof (category of sets)

• Seems to be very simple – in the category of sets if we have $D_1, D_2 : \alpha \to \operatorname{Aut}(X)$ extending given $\partial : \alpha \to \operatorname{Aut}(Y)$, then for $\phi \in \operatorname{Aut}_Y(X), g \in \alpha$, we define:

$$D_{12}(g)(\phi) := D_1(g)^{-1} \circ \phi \circ D_2(g).$$

- However Aut_Y(X) does not represent A_{X/Y}, e.g. because it has no map to Y! Note that a group over Y is not a group! It is a function G → Y such that each fiber is a group.
 E.g. A_{X/Y} is represented by the disjoint union of the automorphism groups of the fibers of X → Y.
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Theorem (K., Pillay)

If X is a smooth projective ∂ -variety whose canonical or anti-canonical divisor is ample, then X is trivializable.

- The (anti-)canonical sheaf K_X inherits the ∂ -structure.
- For each $n \in \mathbb{N}$, K_X^n induces $f_n : X \to \mathbb{P}^N$ which is a ∂ -morphism, where \mathbb{P}^N has the trivial ∂ -structure.
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Moduli spaces

- Benoit has proved that X has a ∂ -structure if and only if for all n, X descents to K^{p^n} (note that $C = \cap_n K^{p^n}$).
- Hence, if a ∂ -variety X belongs to a C-definable family with a fine moduli space, then X descents to C.

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A smooth projective ∂ -variety is trivializable if it is an abelian variety or a curve.

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