Iterative $\partial$-varieties via group scheme actions

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Iterative $C$-derivations

$C \subseteq K$ is a field extension.

**Definition of iterative (Hasse-Schmidt) $C$-derivations**

$\partial = (\partial_n)_{n \in \mathbb{N}}$ is an iterative $C$-derivation if for each $n, m \in \mathbb{N}$:

1. $\partial_n : K \to K$ is additive and $\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y)$,
2. $\partial_n|_C = 0$,
3. $\partial_0 = \text{id}$,
4. $\partial_m \circ \partial_n = \binom{n+m}{m} \partial_{n+m}$ (iterativity).

- Let $\partial : K \to K[X]$, $\partial(x) = \sum \partial_n(x)X^n$. Then 1 (with 2) is equivalent to $\partial$ being a ring homomorphism ($C$-algebra map).
- 3 is equivalent to $\partial$ being a section of $K[X] \to K$. 

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- Let $\partial : K \rightarrow K[\!\![X]\!\!]$, $\partial(x) = \sum \partial_n(x)X^n$. Then 1 (with 2) is equivalent to $\partial$ being a ring homomorphism ($C$-algebra map).
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- Let \( \partial : K \to K[[X]] \), \( \partial(x) = \sum \partial_n(x)X^n \). Then 1 (with 2) is equivalent to \( \partial \) being a ring homomorphism (\( C \)-algebra map).
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The iterativity condition

Let \( c_K : K[[X]] \to K[[X_1, X_2]], c_K(F) = F(X_1 + X_2) \).

Since

\[
c\left( \sum_n a_n X^n \right) = \sum_{i,j} \binom{i+j}{i} a_{i+j} X_1^i X_2^j,
\]

\( \partial \) is iterative iff the following diagram is commutative:

\[
\begin{array}{ccc}
K & \xrightarrow{\partial} & K[[X]] \\
\downarrow{\partial} & & \downarrow{c_K} \\
K[[X]] & \xrightarrow{\partial^*} & K[[X_1, X_2]],
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Formal group and complete Hopf algebra

- Consider \( c : C[[X]] \to C[[X_1, X_2]], c(F) = F(X_1 + X_2). \)
- Since \( C[[X_1, X_2]] \cong C[[X]] \widehat{\otimes}_C C[[X]] \), the following diagram is commutative:

\[
\begin{array}{ccc}
C[[X]] & \xrightarrow{c} & C[[X]] \widehat{\otimes}_C C[[X]] \\
\downarrow c & & \downarrow \text{id} \widehat{\otimes} c \\
C[[X]] \widehat{\otimes}_C C[[X]] & \xrightarrow{c \widehat{\otimes} \text{id}} & C[[X]] \widehat{\otimes}_C C[[X]] \widehat{\otimes}_C C[[X]]
\end{array}
\]

- This diagram and two more things mean that \( C[[X]] \) becomes a complete Hopf algebra (a co-group object in the category of complete \( C \)-algebras) corresponding to the formal group \( \hat{G}_a \).
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This diagram and two more things mean that $C[X]$ becomes a complete Hopf algebra (a co-group object in the category of complete $C$-algebras) corresponding to the formal group $\hat{G}_a$. 
Complete Hopf algebra co-action

- Since $K[X] \cong K \widehat{\otimes}_C C[X]$, $\partial$ is an iterative $C$-derivation iff the following diagram is commutative:

\[
\begin{array}{ccc}
K & \xrightarrow{\partial} & K \widehat{\otimes}_C C[X] \\
\downarrow & & \downarrow \text{id} \otimes c \\
K \widehat{\otimes}_C C[X] & \xrightarrow{\partial \otimes \text{id}} & K \widehat{\otimes}_C C[X] \widehat{\otimes}_C C[X]
\end{array}
\]

- This means that $\partial$ is a co-action of the complete Hopf algebra $C[X]$ on $K$. Note that $\partial$ is a section of $K[X] \to K$ iff the following diagram is commutative ("1 · x = x" axiom):

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K & \xrightarrow{\partial} & K \widehat{\otimes}_C C[X] \\
\downarrow \text{id} & & \downarrow \text{mod } X \\
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Other formal group laws

- What happens if we take a 1-dimensional formal group (law) other than \(X_1 + X_2\)?
- If \(\text{char}(C) = 0\), then all 1-dimensional formal groups are isomorphic.
- If \(\text{char}(C) > 0\), then e.g. \(\hat{\mathbb{G}}_a \not\cong \hat{\mathbb{G}}_m\) and 1-dimensional formal groups are classified by \(\mathbb{N} \cup \{\infty\}\).
- Assume we take the iterative law \(X_1X_2 + X_1 + X_2\) corresponding to \(\hat{\mathbb{G}}_m\). What is the corresponding theory of existentially closed fields with \(\hat{\mathbb{G}}_m\)-iterative derivations?
- The iterative law \(X_1X_2 + X_1 + X_2\) (apparently) gives \((i \leq j)\):

\[
\partial_i \circ \partial_j = \sum_{k=j}^{i+j} \frac{k!}{(k-j)!(k-i)!(i+j-k)!} \partial_k.
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Kernel of the Frobenius map

- We prefer to have actual Hopf algebras (corresponding to group schemes rather than formal groups).
- Since $\mathbb{C}[X] \cong \varprojlim \mathbb{C}[X]/(X^n)$ we want a map $c_n$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbb{C}[X] & \xrightarrow{c} & \mathbb{C}[X_1, X_2] \\
\downarrow & & \downarrow \\
\mathbb{C}[X]/(X^n) & \xrightarrow{c_n} & \mathbb{C}[X_1, X_2]/(X_1^n, X_2^n)
\end{array}
$$

- We need $c(X^n) \in (X_1^n, X_2^n)\mathbb{C}[X_1, X_2]$. For $n > 1$, it happens if and only if $\text{char}(\mathbb{C}) = p > 0$ and $n = p^m$.
- $\mathbb{C}[X]/(X^p^m)$ is a Hopf algebra, so $\text{Spec}(\mathbb{C}[X]/(X^p^m))$ is a group scheme: $\ker(\text{Fr}^m : \mathbb{G}_a \to \mathbb{G}_a)$ \textit{(not an algebraic group!)}. 
We prefer to have actual Hopf algebras (corresponding to group schemes rather than formal groups).

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$C[X]/(X^{p^m})$ is a Hopf algebra, so $\text{Spec}(C[X]/(X^{p^m}))$ is a group scheme: $\ker(\text{Fr}^m : G_\alpha \to G_\alpha)$ (not an algebraic group!).
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$$
\begin{array}{ccc}
C[X] & \overset{c}{\longrightarrow} & C[X_1, X_2] \\
\downarrow & & \downarrow \\
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Iterative $C$-derivations in positive characteristic

- An iterative $C$-derivation on $K$ is a compatible sequence of co-actions $\partial_m$ of the Hopf algebras $C[X]/(X^{p^m})$ on $K$.

- “Compatible” here means that for $n < m$ the following diagram is commutative:

$$
\begin{array}{c}
K \xrightarrow{\partial_n} K \otimes_C C[X]/(X^{p^n}) \\
\downarrow \quad \downarrow \\
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\[
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\downarrow \hspace{1cm} \downarrow \\
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\end{array}
\]
Iterative $C$-schemes

The situation looks more natural when we go to the category of $C$-schemes. Hopf algebras correspond to group schemes and co-actions correspond to group scheme actions.

**Definition of the category of iterative $C$-schemes**

- Let $\alpha_n := \text{Spec}(C[X]/(X^{p^n}))$. It is a group scheme.
- An iterative $C$-scheme is a $C$-scheme $X$ together with a compatible sequence of group scheme actions

$$\partial_n : \alpha_n \times_C X \to X.$$

- An iterative $C$-morphism is an equivariant $C$-morphism.

If $X = \text{Spec}(R)$ then an iterative $C$-scheme structure on $X$ is the same as an iterative $C$-derivation on $R$. 
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If $X = \text{Spec}(R)$ then an iterative $C$-scheme structure on $X$ is the same as an iterative $C$-derivation on $R$. 
Iterative $\partial$-schemes

We fix a $C$-iterative field $(K, \partial)$ which is existentially closed and $\omega$-saturated. Then $S := \text{Spec}(K)$ is an iterative $C$-scheme. Assume $C = K^{p\infty}$.

- An (iterative) $\partial$-scheme is an iterative $C$-scheme $X$ with an iterative $C$-morphism $X \rightarrow S$.
- (Iterative) $\partial$-morphism is a $K$-morphism which is $C$-iterative.
- $\partial$-point of a $\partial$-scheme $X$ is a $K$-point $x$ such that the corresponding morphism $x : S \rightarrow X$ is a $\partial$-morphism.
- $X^\#$ is the set of all $\partial$-points.

If $X = \text{Spec}(R)$ then an iterative $\partial$-scheme structure on $X$ is the same as an iterative $C$-derivation on $R$ extending $\partial$. 
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We fix a $C$-iterative field $(K, \partial)$ which is existentially closed and $\omega$-saturated. Then $S := \text{Spec}(K)$ is an iterative $C$-scheme. Assume $C = K^{p^\infty}$.

An (iterative) $\partial$-scheme is an iterative $C$-scheme $X$ with an iterative $C$-morphism $X \to S$.

(Iterative) $\partial$-morphism is a $K$-morphism which is $C$-iterative.

$\partial$-point of a $\partial$-scheme $X$ is a $K$-point $x$ such that the corresponding morphism $x : S \to X$ is a $\partial$-morphism.

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$X^\#$ is “very” very thin and of finite $U$-rank.

Lemma

If a $\partial$-scheme $X$ has an open affine subvariety, then $X^\# \neq \emptyset$.

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An open sub-scheme $U \to X$ (even in the étale topology) has a unique $\partial$-structure such that $U \to X$ is a $\partial$-morphism.

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Trivializable $\partial$-schemes

**Definition**

- If $X_C$ is a $C$-scheme, then it has a $C$-iterative scheme structure given by the 0-derivation (trivial action).
- Then $X := X_C \times_C S$ has a $\partial$-structure coming from the 0-structure on $X_C$ and the $\partial$-structure on $S$. We call such a $\partial$-schemes trivial and $\partial$-isomorphic to them trivializable.

**Remark**

- If $X$ is a trivial $\partial$-scheme, then $X^\# = X_C(C)$.
- If $X$ is a trivializable $\partial$-variety, then $X^\#$ is $C$-internal.
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The automorphism functor

Let \( C \) be an arbitrary category with fiber products (e.g. category of \( C \)-schemes or category of sets). Let \( X \to Y \) be a morphism in \( C \).

**Automorphisms**

- It is possible to extend the group of automorphisms of \( X \) over \( Y \) to the following contravariant functor:

\[
A_{X/Y} : C_Y \to \text{Gps}, \quad A_{X/Y}(Z) = \text{Aut}_Z(X \times_Y Z).
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- \( \text{Aut}_Y(X) \) is a group object in \( C \) if \( A_{X/Y} \) is representable.

**Example**

In the category of sets \( \text{Aut}(X) \) indeed represents \( A_X \) – for any set \( Z \) a map \( f : Z \to \text{Aut}(X) \) corresponds to the \( Z \)-automorphism of \( X \times Z \) given by \( \bar{f}(x, z) = (f(z)(x), z) \).
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The first trivialization theorem

**Theorem (K., Pillay)**

If $X$ is a projective variety over $K$ and $D_1, D_2$ are iterative $\partial$-scheme structures on $X$, then $(X, D_1) \cong (X, D_2)$. In particular, if $X$ descents to $\mathbb{C}$, then $(X, D_1)$ is trivializable.

**Proof**

- By Matsumura/Oort, $G := \text{Aut}_K(X)$ is a group scheme whose connected component is an algebraic group.
- $D_1$ and $D_2$ give together a $\partial$-structure $D_{12}$ on $G$ such that $g \in G^\#$ iff $g$ is a $\partial$-isomorphism between $(X, D_1)$ and $(X, D_2)$.
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- How one finds $D_{12}$ (the $\partial$-structure on the automorphism group)?

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For a group object $\alpha$, if we have a group action $\partial$ of $\alpha$ on $Y$ with two liftings $D_1, D_2$ to group actions of $\alpha$ on $X$, then we get a group action of $\alpha$ on $G$ such that equivariant morphisms $Y \rightarrow G$ correspond to equivariant isomorphisms from $(X, D_1)$ to $(X, D_2)$.
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If $\text{char}(C) = 0$ and $X$ is a projective $\partial$-variety, then $X$ is trivializable (in particular $X$ descents to $C$).

Positive characteristic

We would like to prove the same thing. By the previous theorem, it would be enough to prove the descent part.
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**Theorem (K., Pillay)**

*If $X$ is a smooth projective $\partial$-variety whose canonical or anti-canonical divisor is ample, then $X$ is trivializable.*

**Idea of the proof**

- The (anti-)canonical sheaf $K_X$ inherits the $\partial$-structure.
- For each $n \in \mathbb{N}$, $K_X^n$ induces $f_n : X \to \mathbb{P}^N$ which is a $\partial$-morphism, where $\mathbb{P}^N$ has the trivial $\partial$-structure.
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Abelian varieties and projective curves

Moduli spaces

- Benoit has proved that $X$ has a $\partial$-structure if and only if for all $n$, $X$ descends to $K^{p^n}$ (note that $C = \cap_n K^{p^n}$).
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A smooth projective $\partial$-variety is trivializable if it is an abelian variety or a curve.

Question

Assume $X$ is a projective variety which descends to $K^{p^n}$ for each $n$. Does it descend to $C$? If Yes, we have the full result.
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