Iterative ∂-varieties via group scheme actions

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Iterative C-derivations

$C \subseteq K$ is a field extension.

Definition of iterative (Hasse-Schmidt) C-derivations

$\partial = (\partial_n)_{n \in \mathbb{N}}$ is an *iterative C-derivation* if for each $n, m \in \mathbb{N}$:

1. $\partial_n : K \to K$ is additive and $\partial_n(xy) = \sum_{i+j=n} \partial_i(x)\partial_j(y)$,
2. $\partial_n|_C = 0$,
3. $\partial_0 = \mathrm{id}$,
4. $\partial_m \circ \partial_n = \binom{n+m}{m}\partial_{n+m}$ (*iterativity*).

- Let $\partial : K \to K[\![X]\!]$, $\partial(x) = \sum \partial_n(x)X^n$. Then 1 (with 2) is equivalent to ∂ being a ring homomorphism (C-algebra map).
- 3 is equivalent to ∂ being a section of $K[\![X]\!] \to K$.

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The iterativity condition

Let $c_K : K[\![X]\!] \to K[\![X_1, X_2]\!], c_K(F) = F(X_1 + X_2)$.

Since

$$c(\sum_n a_n X^n) = \sum_{i,j} \binom{i+j}{i} a_{i+j} X_1^i X_2^j,$$

∂ is iterative iff the following diagram is commutative:

$$\begin{array}{ccc} K & \xrightarrow{\ \partial\ } & K[\![X]\!] \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle c_K} \\ K[\![X]\!] & \xrightarrow{\ \partial_*\ } & K[\![X_1, X_2]\!], \end{array}$$

where $\partial_*(\sum a_n X^n) = \sum \partial_j(a_i) X_1^i X_2^j$.

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Formal group and complete Hopf algebra

- Consider $c : C[\![X]\!] \to C[\![X_1, X_2]\!], c(F) = F(X_1 + X_2)$.
- Since $C[\![X_1, X_2]\!] \cong C[\![X]\!]\widehat{\otimes}_C C[\![X]\!]$, the following diagram is commutative:

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- This diagram and two more things mean that $C[\![X]\!]$ becomes a complete Hopf algebra (a co-group object in the category of complete C-algebras) corresponding to the formal group $\widehat{\mathbb{G}}_a$.

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Complete Hopf algebra co-action

- Since $K[\![X]\!] \cong K \widehat{\otimes}_C C[\![X]\!]$, ∂ is an iterative C-derivation iff the following diagram is commutative:

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- This means that ∂ is a co-action of the complete Hopf algebra $C[\![X]\!]$ on K. Note that ∂ is a section of $K[\![X]\!] \to K$ iff the following diagram is commutative ("$1 \cdot x = x$" axiom):

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Other formal group laws

- What happens if we take a 1-dimensional formal group (law) other that $X_1 + X_2$?
- If $\mathrm{char}(C) = 0$, then all 1-dimensional formal groups are isomorphic.
- If $\mathrm{char}(C) > 0$, then e.g. $\widehat{\mathbb{G}}_a \not\cong \widehat{\mathbb{G}}_m$ and 1-dimensional formal groups are classified by $\mathbb{N} \cup \{\infty\}$.
- Assume we take the iterative law $X_1X_2 + X_1 + X_2$ corresponding to $\widehat{\mathbb{G}}_m$. What is the corresponding theory of existentially closed fields with $\widehat{\mathbb{G}}_m$-iterative derivations?
- The iterative law $X_1X_2 + X_1 + X_2$ (apparently) gives ($i \leqslant j$):

$$\partial_i \circ \partial_j = \sum_{k=j}^{i+j} \frac{k!}{(k-j)!(k-i)!(i+j-k)!} \partial_k .$$

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Kernel of the Frobenius map

- We prefer to have actual Hopf algebras (corresponding to group schemes rather than formal groups).
- Since $C[\![X]\!] \cong \varprojlim C[X]/(X^n)$ we want a map c_n such that the following diagram is commutative:

$$\begin{array}{ccc} C[\![X]\!] & \xrightarrow{\quad c \quad} & C[\![X_1, X_2]\!] \\ \downarrow & & \downarrow \\ C[X]/(X^n) & \xrightarrow{\quad c_n \quad} & C[X_1, X_2]/(X_1^n, X_2^n) \end{array}$$

- We need $c(X^n) \in (X_1^n, X_2^n)C[\![X_1, X_2]\!]$. For $n > 1$, it happens if and only if $\mathrm{char}(C) = p > 0$ and $n = p^m$.
- $C[X]/(X^{p^m})$ is a Hopf algebra, so $\mathrm{Spec}(C[X]/(X^{p^m}))$ is a group scheme: $\ker(\mathrm{Fr}^m : \mathbb{G}_a \to \mathbb{G}_a)$ (not an algebraic group!).

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Iterativity again

Iterative C-derivations in positive characteristic

- An iterative C-derivation on K is a compatible sequence of co-actions ∂_m of the Hopf algebras $C[X]/(X^{p^m})$ on K.
- "Compatible" here means that for $n < m$ the following diagram is commutative:

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Iterative C-schemes

The situation looks more natural when we go to the category of C-schemes. Hopf algebras correspond to group schemes and co-actions correspond to group scheme actions.

Definition of the category of iterative C-schemes

- Let $\alpha_n := \mathrm{Spec}(C[X]/(X^{p^n}))$. It is a group scheme.
- An iterative C-scheme is a C-scheme X together with a compatible sequence of group scheme actions

$$\partial_n : \alpha_n \times_C X \to X.$$

- An iterative C-morphism is an equivariant C-morphism.

If $X = \mathrm{Spec}(R)$ then an iterative C-scheme structure on X is the same as an iterative C-derivation on R.

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Iterative ∂-schemes

We fix a C-iterative field (K, ∂) which is existentially closed and ω-saturated. Then $S := \mathrm{Spec}(K)$ is an iterative C-scheme. Assume $C = K^{p^\infty}$.

Iterative ∂-schemes

- An (iterative) ∂-scheme is an iterative C-scheme X with an iterative C-morphism $X \to S$.
- (Iterative) ∂-morphism is a K-morphism which is C-iterative.
- ∂-point of a ∂-scheme X is a K-point x such that the corresponding morphism $x : S \to X$ is a ∂-morphism.
- $X^\sharp$ is the set of all ∂-points.

If $X = \mathrm{Spec}(R)$ then an iterative ∂-scheme structure on X is the same as an iterative C-derivation on R extending ∂.

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- ∂-point of a ∂-scheme X is a K-point x such that the corresponding morphism $x : S \to X$ is a ∂-morphism.
- $X^\sharp$ is the set of all ∂-points.

If $X = \mathrm{Spec}(R)$ then an iterative ∂-scheme structure on X is the same as an iterative C-derivation on R extending ∂.

∂-points exist

Iterative differential equations

- If $X \subseteq \mathbb{A}^n$, then $X^\sharp$ is given by an infinite system of iterative differential equations (so is type-definable in $(K, +, \cdot, \partial)$).
- It plays the same role as $(X, f)(K, \sigma)$ from Scanlon's talk.
- $X^\sharp$ is "very" very thin and of finite U-rank.

Lemma

If a ∂-scheme X has an open affine subvariety, then $X^\sharp \neq \emptyset$.

Proof

- An open sub-scheme $U \longrightarrow X$ (even in the étale topology) has a unique ∂-structure such that $U \longrightarrow X$ is a ∂-morphism.
- (K, ∂) is existentially closed and ω-saturated, so $X^\sharp \supseteq U^\sharp \neq \emptyset$.

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Trivializable ∂-schemes

Definition

- If X_C is a C-scheme, then it has a C-iterative scheme structure given by the 0-derivation (trivial action).
- Then $X := X_C \times_C S$ has a ∂-structure coming from the 0-structure on X_C and the ∂-structure on S. We call such a ∂-schemes trivial and ∂-isomorphic to them trivializable.

Remark

- If X is a trivial ∂-scheme, then $X^\sharp = X_C(C)$.
- If X is a trivializable ∂-variety, then $X^\sharp$ is C-internal.

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The automorphism functor

Let $\mathcal{C}$ be an arbitrary category with fiber products (e.g. category of C-schemes or category of sets). Let $X \to Y$ be a morphism in $\mathcal{C}$.

Automorphisms

- It is possible to extend the group of automorphisms of X over Y to the following contravariant functor:

$$A_{X/Y} : \mathcal{C}_Y \to \mathbf{Gps}, \quad A_{X/Y}(Z) = \mathrm{Aut}_Z(X \times_Y Z).$$

- $\mathrm{Aut}_Y(X)$ is a group object in $\mathcal{C}$ if $A_{X/Y}$ is representable.

Example

In the category of sets $\mathrm{Aut}(X)$ indeed represents A_X – for any set Z a map $f : Z \to \mathrm{Aut}(X)$ corresponds to the Z-automorphism of $X \times Z$ given by $\bar{f}(x, z) = (f(z)(x), z)$.

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The first trivialization theorem

Theorem (K., Pillay)

If X is a projective variety over K and D_1, D_2 are iterative ∂-scheme structures on X, then $(X, D_1) \cong (X, D_2)$.
In particular, if X descents to C, then (X, D_1) is trivializable.

Proof

- By Matsumura/Oort, $G := \mathrm{Aut}_K(X)$ is a group scheme whose connected component is an algebraic group.
- D_1 and D_2 give together a ∂-structure D_{12} on G such that $g \in G^\sharp$ iff g is a ∂-isomorphism between (X, D_1) and (X, D_2).
- Since G^0 is an algebraic variety and is open in G, we know that $G^\sharp$ is non-empty.

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Finding ∂-structure on G

- How one finds D_{12} (the ∂-structure on the automorphism group)?
- Assume $\mathcal{C}$ is an arbitrary category with fiber products and $X \to Y$ is a morphism such that $A_{X/Y}$ is representable by G.
- In our case $Y = S = \mathrm{Spec}(K)$.

Lemma

For a group object α, if we have a group action ∂ of α on Y with two liftings D_1, D_2 to group actions of α on X, then we get a group action of α on G such that equivariant morphisms $Y \to G$ correspond to equivariant isomorphisms from (X, D_1) to (X, D_2).

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- Seems to be very simple – in the category of sets if we have $D_1, D_2 : \alpha \to \mathrm{Aut}(X)$ extending given $\partial : \alpha \to \mathrm{Aut}(Y)$, then for $\phi \in \mathrm{Aut}_Y(X), g \in \alpha$, we define:

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- However $\mathrm{Aut}_Y(X)$ does not represent $A_{X/Y}$, e.g. because it has no map to Y! Note that a group over Y is not a group! It is a function $G \to Y$ such that each fiber is a group. E.g. $A_{X/Y}$ is represented by the disjoint union of the automorphism groups of the fibers of $X \to Y$.
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A theorem of Buium

Theorem (Buium)

If $\operatorname{char}(C) = 0$ *and* X *is a projective* ∂*-variety, then* X *is trivializable (in particular* X *descents to* C*).*

Positive characteristic

We would like to prove the same thing. By the previous theorem, it would be enough to prove the descent part.

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The second trivialization theorem

Theorem (K., Pillay)

If X is a smooth projective ∂-variety whose canonical or anti-canonical divisor is ample, then X is trivializable.

Idea of the proof

- The (anti-)canonical sheaf K_X inherits the ∂-structure.
- For each $n \in \mathbb{N}$, K_X^n induces $f_n : X \to \mathbb{P}^N$ which is a ∂-morphism, where $\mathbb{P}^N$ has the trivial ∂-structure.
- Ampleness says that f_n is an embedding for some n.
- This trivializes the ∂-structure on X.

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Abelian varieties and projective curves

Moduli spaces

- Benoit has proved that X has a ∂-structure if and only if for all n, X descents to K^{p^n} (note that $C = \cap_n K^{p^n}$).
- Hence, if a ∂-variety X belongs to a C-definable family with a fine moduli space, then X descents to C.

Theorem

A smooth projective ∂-variety is trivializable if it is an abelian variety or a curve.

Question

Assume X is a projective variety which descents to K^{p^n} for each n. Does it descent to C? If Yes, we have the full result.

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