A NOTE ON A THEOREM OF AX

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1. Introduction

In [1], James Ax proved the following theorem:

Theorem 1.1. *Let* (K, ∂) *be a differential field of characteristic 0 with the field of constants* C*. For* $x_1, \ldots, x_n \in K, y_1, \ldots, y_n \in K^*$*, if*

$$\partial x_1 = \frac{\partial y_1}{y_1}, \ldots, \partial x_n = \frac{\partial y_n}{y_n}$$

and $\partial x_1, \ldots, \partial x_n$ *are* $\mathbb{Q}$*-linearly independent, then* $\operatorname{trdeg}_C(x, y) \geqslant n+1$.

The above theorem has some implications in mathematical logic. It was used by Boris Zilber [20] to prove *Weak CIT*, a weak version of *Conjecture on Intersection with Tori* (CIT) stated in [20]. CIT is a finiteness statement about intersections of subtori of a given torus with certain subvarieties of this torus. Weak CIT was crucial in the *bad field* construction [3].

During the Model Theory and Applications to Algebra and Analysis Semester in the Newton Institute in Cambridge (spring 2005), Boris Zilber suggested to look at possible generalizations of 1.1 to the positive characteristic case. Unfortunately, we have not fully succeeded as yet, therefore this note is concerned mostly with characteristic 0 generalizations of 1.1 and only a small discussion about the positive characteristic case is given.

Ax's theorem above may be thought of as a certain statement about a torus and a vector group. Ax himself proved theorems going way beyond the torus case in [2], but they were not phrased in the differential terms. Brownawell and Kubota [7] proved a version of 1.1 in the case of an elliptic curve and Jonathan Kirby in his thesis [12] proved a generalization of 1.1 to the case of a semi-abelian variety and a vector group. Finally, Bertrand [4] proved a further generalization to the case of an algebraic group with no vectorial quotients and a vector group.

*Supported by the Polish grant MEN no. N201 032 32/2231.

There is a common feature in all of these statements – there is no clear way how to transfer them to positive characteristic. In our statement, the main character is not an algebraic group, but a *formal map* (which may be though of as an *analytic map*) between algebraic groups. In this interpretation Kirby's and Bertrand's statements are about the exponential map on the Lie algebra of an algebraic group. The reason their statements do not transfer to positive characteristic becomes clear – there are no exponential maps in positive characteristic! The statement we get includes all the known statements of this form and is meaningful in the positive characteristic too. It is stated in terms of differential equations related to a formal map between (the formalizations of) algebraic groups. We do not claim the proofs here are very original. The main point – a usage of differential forms is almost the same as in [2] or in [12]. However, as the referee has pointed out, Theorem 6.12 is the first result in this area to depend on finer information than the algebraic/transcendental dichotomy. The theorem considers raising to a power α of degree over the field of constants greater than some n, and this degree is used essentially in the proof.

It is worth mentioning that Daniel Bertrand and Anand Pillay [5] have proved a generalization of 1.1 going into a different direction – they do not assume that a given algebraic group is defined over constants. We do not go into this direction here. It should be also mentioned that in [1] and [12], they deal with several commuting derivations. We do not think it would be difficult to transfer the main results of this paper to the multi-derivation case, but we stick with one derivation for simplicity of the presentation.

The paper is organized as follows. In Section 2, we recall several facts about differential forms. In Section 3, we relate logarithmic derivatives with invariant differential forms. In Section 4, we define the differential equation of a formal map. In Section 5, we define nowhere algebraic formal maps and prove an Ax-like statement about them (5.5), which is the main result of this paper. In Section 6, we show how 5.5 specializes to the previously known statements. In Section 7, we discuss an alternative formulation of 5.5 in terms of linear dependence. In Section 8, we give a brief discussion of the positive characteristic case.

I would like to thank the referee for the very helpful report.

2. Invariant Differential Forms

All the rings are of characteristic 0 unless we explicitly state that the characteristic is positive. Throughout the paper C is an algebraically closed field, $C \subseteq K$ is a field extension, R is a C-algebra and V is an algebraic variety over C.

We will briefly recall the definition of Kähler differentials (see [9] and [10]). The *module of Kähler differentials* of R over C is the R-module $\Omega_{R/C}$ given together with the derivation $\mathrm{d} : R \to \Omega_{R/C}$ having the following universal property – for any R-module M and any C-derivation $\partial : R \to M$, there is a unique R-module map $\partial^* : \Omega_{R/C} \to M$ such that $\partial = \partial^* \circ \mathrm{d}$ (see [9]). We will denote $\Omega_{R/C}$ just by Ω_R. If V is affine, Ω_V is defined as $\Omega_{C[V]}$. If V is arbitrary, Ω_V is defined using an affine open cover of V (see [10]). Note that $\Omega_\bullet$ is a functor, which is covariant on C-algebras and contravariant on C-varieties. For a subvariety $W \subseteq V$ and $\omega \in \Omega_V$, $\omega|_W$ denotes the image of ω by the map $\Omega_V \to \Omega_W$ induced by the inclusion map $W \subseteq V$. On any commutative algebraic group A, we have Ω_A^{inv}, the C-subspace of Ω_A consisting of A-invariant forms. All results of this section about invariant forms generalize to the case of a non-commutative algebraic group, where invariant means left-and-right invariant.

If V is affine and $x \in V(K)$ we have the following commutative diagram:

$$\begin{array}{ccc} C[V] & \xrightarrow{x} & K \\ \downarrow{\scriptstyle \mathrm{d}} & & \downarrow{\scriptstyle \mathrm{d}} \\ \Omega_V & \xrightarrow{\Omega_x} & \Omega_K \end{array}$$

For any $\omega \in \Omega_V$, we define:

$$\omega_K : V(K) \to \Omega_K, \quad \omega_K(x) = \Omega_x(\omega).$$

If V is arbitrary, we define the map $\omega_K : V(K) \to \Omega_K$ using an open affine cover of V (see [18] and [13]).

Example 2.1. Consider

$$\omega := \mathrm{d}X \in \Omega_{C[X]} = \Omega_{\mathbb{A}^1}.$$

We clearly have $\omega_K(x) = \mathrm{d}x$. Thus $\omega_K^{-1}(0) = C$. Hence $\omega_K^{-1}(0)$ is not even a constructible subset of K.

Lemma 2.2. *For any $\omega \in \Omega_V$, the set $\omega_K^{-1}(0)$ is $\mathrm{Aut}(K/C)$-invariant.*

Proof. Take $x \in \omega_K^{-1}(0)$. We can assume V is affine and $R = C[V]$. Then x corresponds to a C-algebra map $x : R \to K$. Take any $\sigma \in$

$G(K/C)$. Then $\sigma(x)$ is just the composition of σ with x. Since $\Omega_{\sigma(x)} = \Omega_\sigma \circ \Omega_x$, the result follows. □

Question 2.3. What is the definable nature of $\omega_K^{-1}(0)$? In particular:

(1) Is $\omega_K^{-1}(0)$ definable in the structure $(K, +, \cdot, C)$?
(2) Is $\omega_K^{-1}(0)$ of the form $f^{-1}(W(C))$ for a certain defined over C morphism $f : V \to W$?

We will define one more map using $\omega \in \Omega_V$. Let TV denote the tangent bundle of V. There is a natural bijection:

$$TV(K) \quad \longleftrightarrow \quad V(K[X]/(X^2)).$$

We will denote the ring $K[X]/(X^2)$ by $K[\varepsilon]$, it is called the *ring of dual numbers*. Assume that V is affine and $R = C[V]$. Any point $x \in TV(K)$ corresponds to a C-algebra morphism $x : R \to K[\varepsilon]$. So, x is of the form $x_1 + x_2\varepsilon$, where x_1, x_2 are functions from R to K. It is easy to see that x_1 is a ring homomorphism and x_2 is a derivation from R to K, where the R-algebra structure on K is given by x_1. Therefore, x induces a map $x^* : \Omega_R \to K$. We define an algebraic map

$$\omega_K^* : TV(K) \to K, \quad \omega_K^*(x) = x^*(\omega).$$

This gives us also a K-definable morphism $\omega^* : V \to T^*V$, which is the well-known section of the cotangent bundle often used to *define* differential forms. Hence ω induces two maps on $V(K)$ which are quite different and should not be confused.

It will be useful to emphasize the relation between Ω_V and TV in functorial terms. The argument above implies the existence of a natural bijection

$$\mathrm{Hom}_C(R, K[\varepsilon]) \longleftrightarrow \mathrm{Hom}_C(S\Omega_R, K),$$

where $S\Omega_R$ is the symmetric algebra on Ω_R. Therefore the functor of the ring of dual numbers is right-adjoint to the functor of the symmetric algebra on differential forms and for an affine algebraic variety V, we have $C[TV] = S\Omega_V$. In this interpretation Ω_V (being a subspace of $S\Omega_V$) corresponds to certain regular functions on V – those functions which are C-linear on fibers (similarly as $C[X] = S(KX)$). For an algebraic group A, the space Ω_A^{inv} corresponds to those fiber-wise linear functions which can be obtained by translations from C-linear functionals on T_0A. Hence Ω_A^{inv} corresponds exactly to T_0^*A – the dual space to the Lie algebra of A.

Let us quote a result of Rosenlicht [18] (generalized later by Kolchin [13]).

Proposition 2.4. *If $\omega \in \Omega_A^{\mathrm{inv}}$, then $\omega_K : A(K) \to \Omega_K$ is a group homomorphism.*

The next proposition is crucial for an application of the theory of invariant forms to the proof of our main result. It is related to 5.4 from [12]. This is also the main obstacle for a positive characteristic generalization of the results here.

Proposition 2.5. *Suppose* $\omega \in \Omega_A^{\text{inv}}$ *and* $x \in \ker(\omega_K)$*. Then there is a connected algebraic subgroup* $H \leqslant A$ *defined over* C *such that* $x \in H(K) + A(C)$ *and* $\omega|_H = 0$*.*

Proof. For any field extension $K \subseteq M$, the induced map $\Omega_K \to \Omega_M$ is an embedding – it follows e.g. from the description of differentials given in 16.4 of [9]. For any C-algebra map $y : R \to K$ (a "K-point") and the corresponding map $\tilde{y} : R \to M$ (the corresponding "M-point"), we have the following commutative diagram:

$$\begin{array}{ccc} \Omega_R & \xrightarrow{\Omega_y} & \Omega_K \\ & \searrow^{\Omega_{\tilde{y}}} & \downarrow \\ & & \Omega_M \end{array}$$

Therefore, we can always replace K with M in this proof. Thus without loss K is algebraically closed and $|C|^+$-saturated.

Let O_x denote the orbit of x under $\mathrm{Aut}(K/C)$ and V be the locus of x over C. Take $c \in V(C)$, which exists since C is algebraically closed. By a theorem of Chevalley (see Chapter II Section 7 in [8]), the group generated by $V(K) - c$ is of the form $H(K)$, where H is a connected algebraic subgroup of A defined over C. By the $|C|^+$-saturation of K, O_x is Zariski dense in V. Therefore, $O_x - c$ generates $H(K)$ as well. By 2.2, ω_K vanishes on O_x. By 2.4, ω_K vanishes on $H(K)$. Clearly, $x \in H(K) + A(C)$. We need the following:

Claim
Assume V is smooth and irreducible, $\omega \in \Omega_V$ and $\omega_K = 0$. Then $\omega = 0$.

Proof of Claim. We can assume again that V is affine and $R = C[V]$. Since R embeds in K over C (saturation again), we can also assume that $R \subset K$ and it is enough to show that the induced map $\Omega_R \to \Omega_K$ is an embedding. Since we know it for extensions of fields, it is enough to show that $\Omega_R \to \Omega_{R_0}$ is an embedding, where R_0 is the fraction field of R. By 16.9 in [9], $\Omega_{R_0} \cong R_0 \otimes_R \Omega_R$. Hence it is enough to show that $\Omega_R \to R_0 \otimes_R \Omega_R$ is an embedding. It exactly means that Ω_R should be torsion-free.
Take a non-zero $\omega \in \Omega_R$ and $I = \{f \in R | f\omega = 0\}$. Since $\omega \neq 0$, I

is proper. Take a maximal ideal $P \triangleleft R$ containing I. By 16.22 in [9], Ω_R is locally free (here we finally use the smoothness assumption). In particular, the R_P-module $\Omega_R \otimes_R R_P$ is free. But in R_P we localize only by elements which do not annihilate ω, hence $\tilde{\omega}$, the image of ω in $\Omega_R \otimes_R R_P$, is non-zero. Let $\tilde{I} = \{f \in R | f\tilde{\omega} = 0\}$. Clearly $I \subseteq \tilde{I}$. Since $\Omega_R \otimes_R R_P$ is free, it is torsion-free, thus $\tilde{I} = 0$. Hence $I = 0$ and Ω_R is torsion-free as well. □

Since H is smooth and connected, using Claim 2 we get $\omega|_H = 0$. □

Corollary 2.6. *For each $\omega \in \Omega_A^{\mathrm{inv}}$, there is an algebraic subgroup $H \leqslant A$ defined over C such that $\ker(\omega_K) = H(K) + A(C)$ and $\omega|_H = 0$.*

Proof. For each $x \in \ker(\omega_K)$, take H_x given by 2.5. Since each H_x is connected, the group H generated by all the H_x's is algebraic (by a theorem of Chevalley as in the proof of 2.5) and gives the result. □

Remark 2.7. By the above corollary, we can see that Question 2.3(2) has an affirmative answer for an invariant form – one should take the projection morphism $f : A \to A/H$.

3. Logarithmic Derivative

From now on (K, ∂) is a differential field and C is its field of constants. Most of the facts concerning logarithmic derivatives presented in this section can be found in [15] or [16].
The derivation ∂ induces a ring homomorphism:

$$K \ni x \mapsto x + \partial(x)\varepsilon \in K[\varepsilon],$$

which we will also denote by ∂ (recall from Section 2 that $K[\varepsilon] = K[X]/(X^2)$). Therefore ∂ also induces a map

$$\partial_V : V(K) \to V(K[\varepsilon]) = TV(K).$$

In the next lemma we will show that this map is a natural transformation. For a morphism $\phi : V \to W$ between algebraic varieties let $\phi' : TV \to TW$ denote the induced morphism on the tangent spaces. On the level of K-points, ϕ' is just ϕ applied to $V(K[\varepsilon])$.

Lemma 3.1. *For any morphism $\phi : V \to W$ between algebraic varieties over C, the following diagram is commutative:*

$$\begin{array}{ccc} TV(K) & \xrightarrow{\phi'} & TW(K) \\ \uparrow{\scriptstyle \partial_V} & & \uparrow{\scriptstyle \partial_W} \\ V(K) & \xrightarrow{\phi} & W(K) \end{array}$$

Proof. It is enough to notice that ϕ is a natural transformation between the functors of rational points, so the following diagram is commutative:

$$\begin{array}{ccc} V(K[\varepsilon]) & \xrightarrow{\phi_{K[\varepsilon]}} & W(K[\varepsilon]) \\ \uparrow{\scriptstyle \partial_V} & & \uparrow{\scriptstyle \partial_W} \\ V(K) & \xrightarrow{\phi_K} & B(K) \end{array}$$

□

The map ∂_V is usually not algebraic except the important case of the 0-section $0_V : V(K) \to TV(K)$ coming from the 0-derivation on K. If $V \subseteq \mathbb{A}^n$ is affine, then $TV \subseteq \mathbb{A}^{2n}$ is affine as well, $\partial_{\mathbb{A}^n} = (\mathrm{id}, \partial^{\times n})$ and $\partial_V = \partial_{\mathbb{A}^n}|_V$. Hence the map ∂_V may be thought of as an application of ∂ to the K-points of V.

Let A be a commutative algebraic group defined over C. Then TA is a commutative algebraic group as well and the projection map $TA \to A$ is a group homomorphism. Hence T_0A, the fiber over 0, is a commutative algebraic group too and we have an exact sequence

$$0 \to T_0A \to TA \to A \to 0.$$

Since the 0-section is also an algebraic group homomorphism, this sequence splits and we have another projection map $p_A : TA \to T_0A$.

Definition 3.2. The *logarithmic derivative on* A is defined as follows:

$$l\partial_A : A(K) \to T_0A(K), \quad l\partial_A = p_A \circ \partial_A.$$

Example 3.3. The tangent spaces at identity are naturally identified with K below.

(1) $l\partial_{\mathbb{G}_a}(x) = \partial x$.
(2) $l\partial_{\mathbb{G}_m}(x) = \frac{\partial x}{x}$.
(3) $l\partial_E(x, y) = \frac{\partial x}{y}$, where E is the elliptic curve given by $y^2 = f(x)$, for a cubic polynomial f.

Proposition 3.4. *Let A, B be commutative algebraic groups over C and $\phi : A \to B$ an algebraic group homomorphism. Then:*

(1) $\ker(l\partial_A) = A(C)$.
(2) *The following diagram is commutative (naturality of the logarithmic derivative):*

$$\begin{array}{ccc} T_0A(K) & \xrightarrow{\phi'} & T_0B(K) \\ \uparrow{\scriptstyle l\partial_A} & & \uparrow{\scriptstyle l\partial_B} \\ A(K) & \xrightarrow{\phi} & B(K) \end{array}$$

Proof. (1) is exactly 2.2(iii) from [16]. For (2) it is enough to use 3.1 and notice that the following diagram is commutative:

$$\begin{array}{ccc}
T_0A(K) & \xrightarrow{\phi'} & T_0B(K) \\
{\scriptstyle p_A}\uparrow & & {\scriptstyle p_B}\uparrow \\
TA(K) & \xrightarrow{\phi'} & TB(K)
\end{array}$$

which is easy to see after decomposing TA into $T_0A \times A$. □

We add now differential forms into the picture. Let us fix $\omega \in \Omega_V$. In a presence of the derivation ∂, the two maps ω_K and ω_K^* are related to each other by the diagram below.

Lemma 3.5. *The following diagram is commutative*

$$\begin{array}{ccc}
V(K) & \xrightarrow{\partial_V} & TV(K) \\
{\scriptstyle \omega_K}\downarrow & & \downarrow{\scriptstyle \omega_K^*} \\
\Omega_K & \xrightarrow{\partial^*} & K
\end{array}$$

Proof. For this proof one needs only to unravel the definitions and use the adjointness between functors $R \mapsto R[\varepsilon]$ and $R \mapsto S\Omega_R$ (see Section 2). Let us assume that V is affine and $R = C[V]$. To simplify the diagrams we assume the following notation: for C-algebras R, S, T let $[R, S]$ denote $\mathrm{Hom}_C(R, S)$ and for $f \in [S, T]$ and $r \in R$, let $f_\bullet : [R, S] \to [R, T]$ be the map obtained by the composition and $\mathrm{ev}_r : [R, S] \to S$ the map obtained by the evaluation. We have to show that the following diagram is commutative († denotes the adjointness map):

$$\begin{array}{ccc}
[R, K] & \xrightarrow{\partial_\bullet} & [R, K[\varepsilon]] \\
{\scriptstyle \Omega}\downarrow & & \downarrow{\scriptstyle \dagger} \\
[\Omega_R, \Omega_K] & \xrightarrow{\partial^*_\bullet} & [\Omega_R, K] \\
{\scriptstyle \mathrm{ev}_\omega}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_\omega} \\
\Omega_K & \xrightarrow{\partial^*} & K
\end{array}$$

The lower diagram is clearly commutative. The upper one is commutative by the naturality of † and the observation that $\partial^\dagger = \partial^*$. □

As we have already noticed, we can identify Ω_A^{inv} with the dual space to T_0A. Using this identification, we give a description of the logarithmic derivative in terms of invariant forms (see also [15]). We denote by the same symbol the map $TA(K) \to K$ induced by a differential form and the restriction of this map to $T_0A(K)$.

Lemma 3.6. *Let A be a commutative algebraic group over C. Choose $\bar{\omega}$, a basis of Ω_A^{inv}. The following diagram is commutative:*

$$\begin{array}{ccc} A(K) & \xrightarrow{l\partial_A} & T_0A(K) \\ \downarrow{\scriptstyle \bar{\omega}_K} & & \downarrow{\scriptstyle \bar{\omega}_K^*} \\ \Omega_K^{\times n} & \xrightarrow{(\partial^*)^{\times n}} & K^n \end{array}$$

Proof. This time we need to show that the following diagram is commutative:

$$\begin{array}{ccccc} A(K) & \xrightarrow{\partial_A} & TA(K) & \xrightarrow{p_A} & T_0A(K) \\ \downarrow{\scriptstyle \bar{\omega}_K} & & \downarrow{\scriptstyle \bar{\omega}_K^*} & \swarrow{\scriptstyle \bar{\omega}_K^*} & \\ \Omega_K^{\times n} & \xrightarrow{(\partial^*)^{\times n}} & K^n & & \end{array}$$

The left-hand side diagram is commutative by 3.5. Commutativity of the right-hand side diagram is given by the invariance of $\bar{\omega}$ – the function $\bar{\omega}_K^*$ is determined by its values on $T_0A(K)$. □

4. The differential equation of a formal map

Let us assume first that $C = \mathbb{C}$. For each algebraic group G over $\mathbb{C}$, the set of its $\mathbb{C}$-rational points $G(\mathbb{C})$ is a complex Lie group. Any local analytic homomorphism $\phi : A(\mathbb{C}) \to B(\mathbb{C})$ induces a linear map $\phi' : T_0A(\mathbb{C}) \to T_0B(\mathbb{C})$. By tensoring, we also get a linear map (denoted by the same symbol) $\phi' : T_0A(K) \to T_0B(K)$.

Definition 4.1. The *differential equation of* ϕ is

$$l\partial_B(y) = \phi'(l\partial_A(x)).$$

Example 4.2. The tangent spaces in (1) and (2) below are naturally identified with K.

(1) For $\exp : \mathbb{G}_a(\mathbb{C}) \to \mathbb{G}_m(\mathbb{C})$ its differential equation is

$$\frac{\partial y}{y} = \partial x,$$

since exp induces the identity map on the tangent spaces. It is the same differential equation as the one in 1.1.

(2) Let $\alpha \in \mathbb{C}$. Consider the following local analytic map

$$\mathbb{G}_m(\mathbb{C}) \ni x \mapsto x^\alpha \in \mathbb{G}_m(\mathbb{C}).$$

Its differential equation is:

$$\frac{\partial y}{y} = \alpha \frac{\partial x}{x}.$$

(3) If ϕ is an algebraic map, we get from 3.4(2) that (x, y) is a solution of the differential equation of ϕ if and only if $y - f(x) \in B(C)$. Therefore, the solution set of the differential equation of ϕ is the graph of ϕ up to constants.

By (3) above, we can intuitively think about the solution set of the differential equation of a local analytic homomorphism ϕ as (up to constants) the graph of a possibly non-existing function $\phi : A(K) \to B(K)$.

If $C \neq \mathbb{C}$, we need to replace the notion of local analytic with the notion of *formal*.

Definition 4.3 (Bochner [6]). An n-dimensional *formal group* (law) over C is a tuple of power series $F \in C[\![X, Y]\!]^{\times n}$ ($|X| = |Y| = |Z| = n$) satisfying:

- $F(0, X) = F(X, 0) = X$,
- $F(X, F(Y, Z)) = F(F(X, Y), Z)$.

A formal group is *commutative* if it moreover satisfies:

- $F(X, Y) = F(Y, X)$.

A *morphism* from an n-dimensional formal group G into an m-dimensional formal group F is a tuple of power series $f \in C[\![X]\!]^{\times m}$ such that:

- $F(f(X), f(Y)) = f(G(X, Y))$.

There is a well-known formalization functor (see pages 5 and 13 in [14]) from the category of algebraic groups to the category of formal groups. Briefly, for an algebraic group G, let $R := \mathcal{O}_{G,1}$ be the local ring at identity. The group multiplication from G induces a comultiplication on R, i.e. a map

$$\mu : R \to R \otimes_C R.$$

Then μ induces a comultiplication on the completion of R, i.e. a map

$$\hat{\mu} : \hat{R} \to \hat{R} \hat{\otimes}_C \hat{R}.$$

Since G is smooth, $\hat{R} \cong C[\![X]\!]$ ($|X| = \dim G$), so $\hat{R} \hat{\otimes}_C \hat{R} \cong C[\![X_1, X_2]\!]$. The *formalization of* G, denoted $\hat{G}$, is defined as $\hat{\mu}(X) \in C[\![X_1, X_2]\!]$. One can check that any algebraic group homomorphism $f : A \to B$ functorially induces a formal map $\hat{f} : \hat{A} \to \hat{B}$. But there are formal maps between $\hat{A}$ and $\hat{B}$ which are not algebraic (as the exponential map), i.e. not being the formalization of an algebraic homomorphism.

Abusing the language a bit, we will sometimes say for a formal map $\phi : \hat{A} \to \hat{B}$ that ϕ is a formal map between algebraic groups A and B.

Example 4.4. The formalization of $\mathbb{G}_a$ is $X+Y$ and the formalization of $\mathbb{G}_m$ is $X+Y+XY$. Note that the axioms of formal groups correspond to the fact that 0 is the neutral element, so XY is *not* a formal group.

It is easy to see that formal maps between algebraic groups still induce linear maps on their tangent spaces. If $C = \mathbb{C}$, then the category of formal groups is equivalent to the category of local complex Lie groups and this equivalence commutes with the linear maps induced by formal maps on tangent spaces. Hence the following definition generalizes 4.1:

Definition 4.5. Let A, B be commutative algebraic groups over C and ϕ a formal map between A and B. The *differential equation of* ϕ is

$$l\partial_B(y) = \phi'(l\partial_A(x)).$$

By 4.2(3) again, we can think of the set of solutions of the differential equation of ϕ as (up to constants) the graph of a possibly non-existing function $\phi : A(K) \to B(K)$. Note that the category of formal groups is not concrete, so morphisms can not be understood as functions (unless K is a metric field and a series may converge). However, the solution set of the differential equation of ϕ may be thought of as the "blurred graph" of ϕ (blurred by constants). There may be a relation between this "blurred graph" and the blurred (pseudo-)exponentiation from Section 8 of [12].
Since for two commutative algebraic groups A, B we have the following bijection:

$$\mathrm{Hom}_{\mathbf{Formal}}(A, B) \ni f \mapsto f' \in \mathrm{Hom}_{\mathbf{Linear}}(T_0A, T_0B),$$

we could have forgotten about formal or even analytic maps and just talk about the induced linear maps on tangent spaces. However we find the language of formal maps more natural as e.g. the original theorem of Ax should say something about the exponential map and not about the identification (a trivial one) of the tangent spaces it induces. This formalism is also useful for finding positive characteristic analogues of Ax's theorem.

5. Ax theorem for nowhere algebraic formal maps

Recall the statement of Ax's theorem from the introduction:

Theorem 5.1 (Ax)**.** *If* $(\partial x_i = \frac{\partial y_i}{y_i})_{1\leqslant i\leqslant n}$ *and* $(\partial x_i)_{1\leqslant i\leqslant n}$ *are* $\mathbb{Q}$*-linearly independent, then* $\mathrm{trdeg}_C(x, y) \geqslant n+1$.

Note the following scheme in the above statement:

A good differential equation + linear independence $\Longrightarrow$ large trdeg.

The differential equation there is actually the differential equation of the exponential map (see 4.2(1)). In Theorem 5.7 from [12] and Proposition 1.b from [4], the differential equations were those of the map $\exp : \mathrm{Lie}(A) \to A$ for A semi-abelian (Kirby) or without $\mathbb{G}_a$-quotients (Bertrand). Thus our statement should identify those formal maps which yield differential equations giving large transcendence degree. Note that algebraic maps are not good – if ϕ is algebraic, then any $(x, \phi(x))$ is a solution to the differential equation of ϕ (see 4.2(3)) and the transcendence degree is always smaller that the dimension of the domain. However even certain non-algebraic formal maps are not good as the example below shows.

Example 5.2. For the differential equation $\frac{\partial z}{z} = \sqrt{2}\frac{\partial y}{y}$ take:

- y_1, y_2 such that $\frac{\partial y_1}{y_1} = 1, \frac{\partial y_2}{y_2} = \sqrt{2} + 1,$
- $z_1 = y_1^{-1} y_2,\ z_2 = y_1 y_2.$

Then $\frac{\partial y_1}{y_1}, \frac{\partial y_2}{y_2}$ are $\mathbb{Q}$-linearly independent, $\mathrm{trdeg}_C(y_1, y_2, z_1, z_2) \leqslant 2$ and:

- $\frac{\partial z_1}{z_1} = \sqrt{2} + 1 - 1 = \sqrt{2}\frac{\partial y_1}{y_1}, \quad \frac{\partial z_2}{z_2} = \sqrt{2}\frac{\partial y_2}{y_2}.$

We see that our map should be "very far" from being algebraic. Below is a definition which works (to some extent) for our purposes, but it is perhaps not very beautiful.

Definition 5.3. A formal map ϕ between algebraic groups A, B is *nowhere algebraic* if there are no:

- algebraic subgroups $A_0 \leqslant A, B_0 < B$,
- algebraic epimorphism $\psi : A_0 \twoheadrightarrow B/B_0$,
- $v \in T_0 A_0(C)$ such that $\psi'(v) = \bar{\phi}'(v) \neq 0$, where $\bar{\phi}$ is the following composition:

$$A_0 \longrightarrow A \xrightarrow{\phi} B \longrightarrow B/B_0.$$

At first, it seemed to me it would be more natural just to demand in the definition above that ϕ and ψ should not coincide on A_0, so "nowhere" would refer to the algebraic structure (rather than formal) then. But the formal map $X^{\sqrt{2}}$ satisfies this condition and still does not yield an Ax-like statement as was shown in 5.2. Therefore a "good" formal map should not coincide with an algebraic one even in the formal sense (i.e. on the tangent bundle) as in the definition above.

Before the statement of the main result, we need to phrase the linear dependence condition in more geometric terms.

Lemma 5.4. *For $a_1, \dots, a_n \in K^*$, $\frac{\partial a_1}{a_1}, \dots, \frac{\partial a_n}{a_n}$ are linearly dependent over $\mathbb{Q}$ if and only if $(a_1, \dots, a_n)$ belongs to a constant coset of a proper subtorus of $\mathbb{G}_m^n$.*

Proof. Note that $\frac{\partial x}{x} = l\partial_{\mathbb{G}_m}(x)$. Let $a = (a_1, \dots, a_n)$.
$\Longrightarrow$ Take $(k_1, \dots, k_n) \in \mathbb{Z}^n \setminus \{0\}$ such that $\sum k_i l\partial_{\mathbb{G}_m}(a_i) = 0$ and define

$$f : \mathbb{G}_m^n \to \mathbb{G}_m, \quad f(x_1, \dots, x_n) = \prod_{i=1}^{n} x_i^{k_i}.$$

Since $l\partial_{\mathbb{G}_m}(f(a)) = 0$, $f(a) \in C$. Let $T := \ker(f)$ and take $c \in \mathbb{G}_m^n(C)$ such that $f(c) = f(a)$. Then $f(c^{-1}a) = 1$, so $a \in cT(K)$.
$\Longleftarrow$ If $a \in T(K) + \mathbb{G}_m^n(C)$ for a proper subtorus $T < \mathbb{G}_m^n$, then (by 3.4(2)) $l\partial_{\mathbb{G}_m^n}(a) \in T_0T(K)$. Since $T_0T(K)$ is a $\mathbb{Q}$-definable proper linear subspace of $T_0\mathbb{G}_m^n(K)$, the tuple $l\partial_{\mathbb{G}_m^n}(a)$ is $\mathbb{Q}$-linearly dependent. $\square$

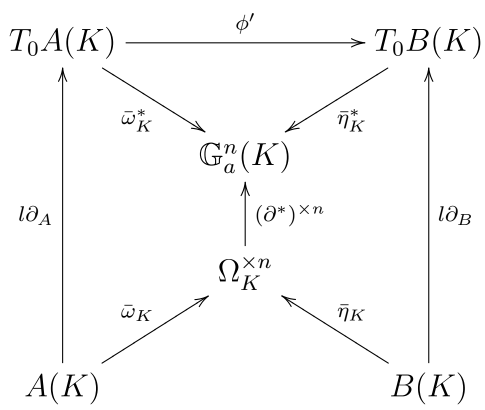
We are ready now to state and prove the main result of this paper.

Theorem 5.5. *Let A, B be commutative algebraic groups over C of dimension n and $\phi : \hat{A} \to \hat{B}$ be a nowhere algebraic formal map. If $(a, b) \in (A \times B)(K)$ satisfies the differential equation of ϕ and* $\operatorname{trdeg}_C(a, b) \leqslant n$, *then there is a proper algebraic subgroup $B_0 < B$ defined over C such that $b \in B_0(K) + B(C)$.*

Proof. From the description of invariant differential forms given in Section 2, we know that the choice of a basis of Ω_A^{inv} corresponds exactly to a choice of a linear isomorphism between T_0A and $\mathbb{G}_a^n$. Take $\bar{\omega} = (\omega_1, \dots, \omega_n)$, an arbitrary basis of Ω_A^{inv} and the basis $\bar{\eta} = (\eta_1, \dots, \eta_n)$ of Ω_B^{inv} such that the following diagram is commutative:

$$\begin{array}{ccc} T_0A(K) & \xrightarrow{\phi'} & T_0B(K) \\ & {\scriptstyle \bar{\omega}_K^*} \searrow \quad \swarrow {\scriptstyle \bar{\eta}_K^*} & \\ & K^n & \end{array}$$

From 3.6 we get a commutative diagram:



Chasing this diagram, we get:

$$\partial^*(\omega_1(a) - \eta_1(b)) = 0, \ldots, \partial^*(\omega_n(a) - \eta_n(b)) = 0.$$

Let $g := (a, b)$, $G := A \times B$ and $\pi_A : G \to A, \pi_B : G \to B$ be the projection maps. For $i \leqslant n$ let us define:

$$\xi_i := \pi_A^*(\omega_i) - \pi_B^*(\eta_i) \in \Omega_G^{\text{inv}}.$$

We clearly have:

$$\partial^*(\xi_1(g)) = 0, \ldots, \partial^*(\xi_n(g)) = 0.$$

In this paragraph we proceed exactly as in [12], so we are going to be brief. Since $\operatorname{trdeg}_C(g) \leqslant n$, we have $\dim_{C(g)} \Omega_{C(g)} \leqslant n$. Since

$$\xi_1(g), \ldots, \xi_n(g) \in \ker(\partial^* : \Omega_K \to K) \cap \Omega_{C(g)}$$

($\Omega_{C(g)}$ is naturally embedded into Ω_K here), we get that $\xi_1(g), \ldots, \xi_n(g)$ are linearly dependent over K. Since each ξ_i is invariant on the commutative group G, it is closed. Therefore each $\xi_i(g)$ is closed, hence $\xi_1(g), \ldots, \xi_n(g)$ are linearly dependent over C by a Lie derivative argument (see [2] or [12]). Take a non-zero tuple $(c_1, \ldots, c_n) \in C^n$ such that $c_1\xi_1(g) + \ldots + c_n\xi_n(g) = 0$ and define:

$$\xi := c_1\xi_1 + \ldots + c_n\xi_n \in \Omega_G^{\text{inv}}.$$

We have:

$$\xi|_A \neq 0, \quad \xi|_B \neq 0, \quad \xi(g) = 0.$$

By 2.5, there is an algebraic subgroup $H < G$ over C such that $g \in H(K) + G(C)$ and $\xi|_H = 0$.

Let us define $H^A := \pi_A(H)$, $H_A := H \cap (A \times 0)$ and similarly for H^B, H_B. Clearly:

$$g \in (H^A \times H^B)(K) + G(C).$$

Thus we are done, if H^B is proper. Since $\xi|_B \neq 0$, we have $H_B \neq H$. Hence we can also assume that $b \notin H_B(K) + B(C)$. We will show that $H^B = B$ contradicts the assumption that ϕ is nowhere algebraic.

Assume $H^B = B$. The algebraic subgroup $H < G$ induces an algebraic epimorphism $\psi : H^A \to \bar{B} := B/H_B$ such that $\psi(a) = \bar{b} := b + H_B$. By 3.4(2), we have:

$$\psi'(l\partial_A(a)) = l\partial_{\bar{B}}(\bar{b}), \quad \bar{\phi}'(l\partial_A(a)) = l\partial_{\bar{B}}(\bar{b}),$$

where $\bar{\phi}$ is the composition of ϕ with the formalization of the projection map $B \to \bar{B}$. Hence $\bar{\phi}'$ coincides with ψ' on $l\partial_A(a)$. Since $b \notin H_B(K) + B(C)$, by 3.4 we get $l_\partial \bar{B}(\bar{b}) \neq 0$, which contradicts the assumption that ϕ is nowhere algebraic. □

The statement of the above theorem is non-symmetric, since the notion of a nowhere algebraic formal map is non-symmetric. But in many cases (including all the previously known ones), we get a symmetric statement, i.e proper algebraic subgroups of both A and B. It is discussed in details in the next section.

6. Specializations of the main theorem and a torus case

In this section we will show how Theorem 5.5 generalizes some of the known Ax-like statements. We will put these statements into more general contexts and still show how 5.5 specializes to these contexts. We will also discuss another Ax-like statement regarding certain formal automorphisms of tori which does not fit to the scope of 5.5.

Definition 6.1. We call two algebraic groups G and H *essentially different* if any algebraic subgroup $T < G \times H$ is isogenous to a group of the form $G_0 \times H_0$ for G_0 (resp. H_0) an algebraic subgroup of G (resp. H).

Example 6.2. Essentially different algebraic groups:

1. $G = \mathbb{G}_a^n$, H is a semi-abelian variety;
2. $G = \mathbb{G}_m^n \times \mathbb{G}_a^n$, H is an abelian variety;
3. (generalization of (2) by Poincaré's Reducibility Theorem) $G = G_1 \times \cdots \times G_n$, $H = H_1 \times \cdots \times H_m$, where each G_i, H_j is a simple algebraic group, and G_i is not isogenous to H_j;
4. $G = \mathbb{G}_m^n$, H is an extension of an abelian variety by a vector group.

Proposition 6.3. *If A, B are essentially different and $\phi : \hat{A} \to \hat{B}$ is an isomorphism, then ϕ and ϕ^{-1} are nowhere algebraic.*

Proof. By symmetry, it is enough to deal with ϕ. Assume ϕ is not nowhere algebraic and take algebraic subgroups $A_1 \leqslant A, B_1 < B$ with infinite index $[B : B_1]$ and an algebraic epimorphism $\psi : A_0 \twoheadrightarrow B/B_1$. Then the preimage in $A \times B$ of the graph of ψ is not isogenous to a group of the form $A_0 \times B_0$. □

Theorem 6.4. *Assume A, B are essentially different commutative algebraic groups of dimension n and ϕ is a formal isomorphism between them. Let $(a, b) \in (A \times B)(K)$ satisfy the differential equation of ϕ and $\mathrm{trdeg}_C(a, b) \leqslant n$. Then there are proper algebraic subgroups $A_0 < A, B_0 < B$ such that $a \in A_0(K) + A(C)$ and $b \in B_0(K) + B(C)$.*

Proof. By 6.3, ϕ and ϕ^{-1} are nowhere algebraic. By 5.5, we get $B_0 < B$ with the required properties. Clearly, (b, a) satisfies the differential equation of ϕ^{-1}. By 5.5 again, we get $A_0 < A$ with the required properties. □

Remark 6.5. If A is a semi-abelian variety, then A and $\mathbb{G}_a^n$ are essentially different. Hence 6.4 generalizes 5.7 from [12]. Note that the choice of an arbitrary basis of the space of invariant forms in [12] corresponds exactly to the choice of an arbitrary formal isomorphism here.

Definition 6.6. We say that two algebraic groups G and H *have no common quotients* if for any normal algebraic subgroups $G_0 \lhd G, H_0 \lhd H$; $G/G_0 \cong H/H_0$ implies G/G_0 is finite.

Fact 6.7. *If G and H are essentially different, then they have no common subquotients, i.e. for any algebraic subgroups $G_1 \leqslant G$, $H_1 \leqslant H$, G_1 and H_1 have no common quotients.*

Proof. Assume G_1, H_1 have a common infinite algebraic quotient Q. Then there are algebraic epimorphisms $\phi : G_1 \to Q, \psi : H_1 \to Q$. It is easy to see that the preimage of the diagonal by $\phi \times \psi$ is not isogenous to a group of the form $H_0 \times G_0$. □

Example 6.8.
(1) Let $G = \mathbb{G}_a^n$, $H = E(A)$, where A is a semi-abelian variety and $E(A)$ is its maximal vectorial non-split extension. Then G and H have no common quotients but are not essentially different.
(2) If A and B are abelian varieties, then by Poincaré's Reducibility Theorem they are essentially different if and only if they have no common quotients.

Fact 6.9. *Assume A, B are commutative algebraic groups and for any algebraic subgroup $A_1 \leqslant A$, A_1 and B have no common quotients. Let ϕ be a formal isomorphism between A and B. Then ϕ is nowhere algebraic.*

Proof. Obvious from the definitions. □

Theorem 6.10. *If A, B, ϕ are as in 6.9, $(a, b) \in (A \times B)(K)$ satisfies the differential equation of ϕ and $\mathrm{trdeg}_C(a, b) \leqslant \dim(A)$, then there is a proper algebraic subgroup $B_0 < B$ such that $b \in B_0(K) + B(C)$.*

Proof. Clear by 6.9 and 5.5. □

Remark 6.11. Note that if $A = \mathbb{G}_a^n$ and we are in the situation of 6.10, then we also get a proper subgroup $A_0 < A$ defined over C such that $a \in A_0(K) + A(C)$. This is because algebraic subgroups over C of $\mathbb{G}_a^n$ are in 1-to-1 correspondence (modulo constants) with linear subspaces defined over C of $T_0\mathbb{G}_a^n$ and this correspondence is given by the logarithmic derivative map, which is here just the n-th cartesian power of ∂. Therefore, 6.10 includes Bertrand's Proposition 1.b from [4] – an Ax-like statement for $\exp : \mathrm{Lie}(A) \to A$, where A has no $\mathbb{G}_a$-quotients.

Unfortunately, it seems like the argument given in the proof of 5.5 alone is not enough to prove an Ax-like statement for another class of formal maps – raising to irrational powers. The proof of the theorem below combines an argument from 5.5 with some arguments which seem to be specific to the case of torus – rigidity and decomposition into the product of 1-dimensional subgroups (see also 6.15). It is also easy to see that raising to an irrational power is still *not* nowhere algebraic.

Theorem 6.12. *If $\alpha \in C$, $[\mathbb{Q}(\alpha) : \mathbb{Q}] > n$, $y_1, z_1, \ldots, y_n, z_n \in K^*$ satisfy*

$$\frac{\partial z_1}{z_1} = \alpha \frac{\partial y_1}{y_1}, \ldots, \frac{\partial z_n}{z_n} = \alpha \frac{\partial y_n}{y_n}$$

and $\frac{\partial z_1}{z_1}, \ldots, \frac{\partial z_n}{z_n}$ are $\mathbb{Q}$-linearly independent, then

$$\mathrm{trdeg}_C(y_1, z_1, \ldots, y_n, z_n) \geqslant n + 1.$$

Proof. Let $g := (y_1, \ldots, y_n, z_1, \ldots, z_n)$ and assume $\mathrm{trdeg}_C(g) \leqslant n$. If we proceed as in the proof of 5.5, we get $(a_1, \ldots, a_n) \in C^n \setminus \{0\}$ such that for

$$\xi := \sum_{i=1}^{n} a_i \left(\alpha \frac{\mathrm{d}X_i}{X_i} - \frac{\mathrm{d}Y_i}{Y_i} \right),$$

we have $\xi_K(g) = 0$. Let T be a proper subtorus of $\mathbb{G}_m^{2n}$ such that $\ker(\xi_K) = T(K) + \mathbb{G}_m^{2n}(C)$ (see 2.6).
For $l \leqslant 2n$, let $\pi_l : \mathbb{G}_m^{2n} \to \mathbb{G}_m^l$ denote the projection on the first l coordinates. If $\pi_n(T) \neq \mathbb{G}_m^n$ we are done as in the proof of 5.5. Assume $\pi_n(T) = \mathbb{G}_m^n$. We will reach a contradiction.
Let $\dim T = n + k$, $0 \leqslant k < n$. After permuting some coordinates,

we may assume that $\pi_{n+k}(T) = \mathbb{G}_m^{n+k}$. It means there is a matrix $B = (b_{ij}) \in M_{n-k,n+k}(\mathbb{Z})$ and non-zero integers $l_1, \ldots, l_{n-k}$ such that T is given by the following equations:

$$(\dagger) \qquad X_{n+k+j}^{l_j} = \prod_{i=1}^{n+k} X_i^{b_{ij}}, \quad 0 < j \leqslant n-k.$$

By 2.4, there is a matrix $A \in M_{n-k,n+k}(\mathbb{Q})$ (we divide the i-th row of B by l_i) such that for any $x = (x_1, \ldots, x_{2n}) \in T(K)$ we have

$$(*) \qquad A\left(\frac{\mathrm{d}x_1}{x_1}, \ldots, \frac{\mathrm{d}x_{n+k}}{x_{n+k}}\right) = \left(\frac{\mathrm{d}x_{n+k+1}}{x_{n+k+1}}, \ldots, \frac{\mathrm{d}x_{2n}}{x_{2n}}\right).$$

On the other hand (since $\xi_K(x) = 0$) we have:

$$(**) \qquad \alpha\left(a_1\frac{\mathrm{d}x_1}{x_1} + \cdots + a_n\frac{\mathrm{d}x_n}{x_n}\right) = a_1\frac{\mathrm{d}x_{n+1}}{x_{n+1}} + \cdots + a_n\frac{\mathrm{d}x_{2n}}{x_{2n}}.$$

Since we are not going to use the derivation ∂ anymore, we can assume as in the proof of 2.5 that K is sufficiently saturated. Hence we can and will take x generic over C in $T(K)$. Since $\pi_{n+k}(T) = \mathbb{G}_m^{n+k}$, the elements $x_1, \ldots, x_{n+k}$ are algebraically independent over C. Therefore

$$(***) \qquad \dim_C\left(\frac{\mathrm{d}x_1}{x_1}, \ldots, \frac{\mathrm{d}x_{n+k}}{x_{n+k}}\right) = n+k.$$

By $(*), (**)$ and $(***)$ we get (A^T is A transposed):

$$A^T(a_{k+1}, \ldots, a_n) = (\alpha a_1, \ldots, \alpha a_n, -a_1, \ldots, -a_k).$$

If we decompose A into a block form $A = [A_1 A_2 A_3]$, where $A_1 \in M_{n-k,k}(\mathbb{Q}), A_2 \in M_{n-k,n-k}(\mathbb{Q})$ and $A_3 \in M_{n-k,k}(\mathbb{Q})$, then we have:

$$A_2^T(a_{k+1}, \ldots, a_n) = \alpha(a_{k+1}, \ldots, a_n).$$

If $(a_{k+1}, \ldots, a_n) \neq 0$, then α is an eigenvalue of A_2^T. Therefore, α is a zero of the characteristic polynomial of A_2^T, which has rational coefficients and degree $n-k \leqslant n$, a contradiction. Hence $(a_{k+1}, \ldots, a_n) = 0$. Take $v \in K \setminus C$. Then $(0, \ldots, 0, v) \in \ker(\xi_K)$. Therefore, there is $(c_1, \ldots, c_{2n}) \in \mathbb{G}_m^{2n}(C)$ such that $(c_1, \ldots, c_{2n-1}, vc_{2n}) \in T(K)$. By the description of T given in $(\dagger)$, we see that v is algebraic over C, a contradiction, since C is algebraically closed. □

Note that Example 5.2 shows that the statement of 6.12 can not be improved.

Remark 6.13. A statement similar to 6.12 appears as 5.8 in [12]. The constant $\alpha \in C$ there is arbitrary, but after assuming $\mathrm{trdeg}_C(y,z) \leqslant n$ the conclusion there is weaker:

$$(y,z) \in T(K) + \mathbb{G}_m^n(C) \text{ for a proper subtorus } T < \mathbb{G}_m^{2n}.$$

This condition is equivalent to $\mathbb{Q}$-dependence of $\frac{\partial y_1}{y_1}, \frac{\partial z_1}{z_1}, \ldots, \frac{\partial y_n}{y_n}, \frac{\partial z_n}{z_n}$ (see 5.4).

Question 6.14. What is a general definition of a "good" formal map for which:

- one can prove 5.5,
- it includes the nowhere algebraic case and formal maps from 6.12?

Is it the condition (much nicer and weaker than 5.3) that for ϕ, a formal map, $\ker(\phi' - \psi') = 0$ for any algebraic ψ? I am not able to prove 5.5 for such formal maps as yet. Formal maps from 6.12 are such. Note that the formal map from Example 5.2 is not like this: for

$$\psi : \mathbb{G}_m^2 \to \mathbb{G}_m^2, \quad \psi(x,y) = (x^{-1}y, xy)$$

we have $(1, \sqrt{2}+1) \in \ker(\phi' - \psi')$.

Let E be an elliptic curve, $\alpha \in C$ and ϕ_α a formal automorphism of E such that ϕ'_α is the multiplication by α on T_0E. By similar arguments as in the proof of 6.12 one can show:

Theorem 6.15. *Let $(y,z) \in E^{2n}(K)$ satisfy the differential equation of $\phi_\alpha^{\times n}$ and $\mathrm{trdeg}_C(y,z) \geqslant n+1$. Then:*

- *If E has no complex multiplication and $[\mathbb{Q}(\alpha) : \mathbb{Q}] > n$, then there are proper algebraic subgroups $E_1, E_2 < E^n$ such that $y \in E_1(K) + E^n(C)$ and $z \in E_2(K) + E^n(C)$.*
- *If E has a complex multiplication τ and $[\mathbb{Q}(\alpha,\tau) : \mathbb{Q}(\tau)] > n$, then there are proper algebraic subgroups $E_1, E_2 < E^n$ such that $y \in E_1(K) + E^n(C)$ and $z \in E_2(K) + E^n(C)$.*

One can guess from 6.12 and 6.15 that the field $\mathbb{Q}$ plays a different role for a torus and an elliptic curve without a complex multiplication, than for an elliptic curve with a complex multiplication (see also [7] or [11]). This relation will be made clearer in the next section.

Analyzing Poizat's proof of Weak CIT (see 3.2 in [17]) one can see that there are three crucial ingredients in it: Ax's theorem 1.1, rigidity of tori (countably many algebraic subgroups) and the following property:

$$(*) \qquad \mathrm{trdeg}_C(a_1,\ldots,a_n) \leqslant \dim_C(\partial a_1,\ldots,\partial a_n).$$

Since Ax's theorem also holds when torus is replaced by any semi-abelian variety, Weak CIT holds in such a context as well, which was proved by Kirby, see 5.15 in [12].
Let us concentrate on the property $(*)$. Note that for a given $k \in \mathbb{N}$, the condition

$$\dim_C(\partial a_1, \dots, \partial a_n) \leqslant k$$

is definable in $(a_1, \dots, a_n)$ and implies the *non-definable* condition

$$\operatorname{trdeg}_C(a_1, \dots, a_n) \leqslant k.$$

For a commutative algebraic group A let us formulate the following condition:

$$(*_A) \qquad \forall a \in A(K) \quad \operatorname{trdeg}_C(a) \leqslant \dim_C(l\partial_A(a)).$$

Then $(*)$ is just $(*_{\mathbb{G}_a^n})$.
One could wonder for which algebraic groups A, the property $(*_A)$ holds. E.g. if it held for a torus, it would give another proof of Weak CIT using 6.12. Similarly for elliptic curves using 6.15. Unfortunately $(*_A)$ does not hold neither for A being a torus nor for A being a power of an elliptic curve. Quite ironically, a counterexample is given by the corresponding Ax-like statement 6.12 or 6.15. Let us focus on the torus case. Take $\alpha \in C$ non-algebraic and $a, b \in K^* \setminus C^*$ such that $\frac{\partial a}{a} = \alpha \frac{\partial b}{b}$. Clearly $\dim_C l\partial_{\mathbb{G}_m^2}(a, b) = 1$. But by 6.12, $\operatorname{trdeg}_C(a, b) = 2$.
Therefore if for an algebraic group A the property $(*_A)$ holds, then A should not have "very non-algebraic" formal endomorphisms, since we get counterexamples to $(*_A)$ along the "blurred graph" (see comments at the end of Section 4) of such a formal map. Note that vector groups are exactly like that – any formal endomorphism of a vector group is algebraic. However we should have in mind we are in the characteristic 0 situation here (note the last sentence of this paper).

7. Schanuel Conditions

The original theorem of Ax (1.1) is stated in terms of Schanuel-like inequalities. By 5.4, the linear condition there is equivalent to a condition about subtori. The latter condition is easier to deal with if one goes from torus to more complicated algebraic groups. The problem is that in 1.1 the base field for linear dependence is clear (being just $\mathbb{Q}$), but for algebraic groups other than tori it gets more complicated. Jonathan Kirby has proved results into this direction about products of elliptic curves (see Propositions 1. and 2. in [11]). In this section, we will give a general result in terms of Schanuel-like inequalities, which includes the Ax's theorem and the statements from [11].

Definition 7.1. For an algebraic group G defined over K, we define its *absolute field of definition*:

$$\mathbb{Q}_G := \mathbb{Q}(\bigcup\{\text{field of definition of } H | H, \text{ a connected algebraic subgroup of } G\}).$$

Example 7.2. Absolute fields of definitions:

(1) $G = \mathbb{G}_a^n$, $n > 1$: $\mathbb{Q}_G = K$;
(2) $G = \mathbb{G}_m^n$: $\mathbb{Q}_G = \mathbb{Q}$;
(3) E is an elliptic curve over $\mathbb{Q}$, $G = E^n$, $n > 1$:
$$\mathbb{Q}_G = \mathrm{End}_{\mathbb{Q}}(E) := \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}(\tau),$$
where τ is a complex multiplication on E (if any);
(4) More generally, if A is a simple commutative algebraic group, then for $n > 1$:
$$\mathbb{Q}_{A^n} = \mathrm{End}_{\mathbb{Q}}(E);$$
(5) G is a semi-abelian variety defined over $\mathbb{Q}$: $\mathbb{Q}_G \subseteq \mathbb{Q}^{\mathrm{acl}}$.

The next theorem is a version of 5.5 stated in terms of linear dependence.

Theorem 7.3. *Let A, B, ϕ, a, b, n be as in 5.5. If* $\dim_{\mathbb{Q}_B \cap C} l\partial_B(b) = n$ *then* $\mathrm{trdeg}_C(a, b) \geqslant n + 1$.

Proof. If $\mathrm{trdeg}_C(a, b) \leqslant n$, we get by 5.5 a proper algebraic subgroup $B_0 < B$ defined over C, $b_0 \in B_0(K)$ and $c \in B(C)$ such that $b = b_0 + c$. Since B_0 is also defined over $\mathbb{Q}_B$, it is defined over $\mathbb{Q}_B \cap C$. By 3.4 we get:

$$\dim_{\mathbb{Q}_B \cap C} l\partial_B(b) = \dim_{\mathbb{Q}_B \cap C} l\partial_B(b_0) = \dim_{\mathbb{Q}_B \cap C} l\partial_{B_0}(b_0) \leqslant \dim B_0 < n.$$

□

Remark 7.4. We do not know if 5.5 can be obtained from 7.3. It is unclear if an analogue of 5.4 holds, i.e. for $b \in B(K)$, if

$$\dim_{\mathbb{Q}_B \cap C} l\partial_B(b) < n$$

implies that b belongs to a constant coset of a proper algebraic subgroup of B. It is quite clear that $\mathbb{G}_m$ in 5.4 may be replaced by any simple commutative algebraic group. It is worth noticing that for $B = \mathbb{G}_a$, we get the $(*)$ property considered at the end of the previous section.

Now we state a version of 6.4.

Theorem 7.5. *Let A, B, ϕ, a, b, n be as in 6.4 and assume A, B, ϕ' are defined over $\mathbb{Q}$. If*

$$\max[\dim_{\mathbb{Q}_A \cap C} l\partial_B(b), \dim_{\mathbb{Q}_B \cap C} l\partial_B(b)] = n,$$

then $\mathrm{trdeg}_C(a, b) \geqslant n + 1$.

Proof. Using 6.4 we get (as in the proof of 7.3):

$$\dim_{\mathbb{Q}_A \cap C} l\partial_A(a) < n, \ \dim_{\mathbb{Q}_B \cap C} l\partial_B(b) < n.$$

Since ϕ' is defined over $\mathbb{Q}$ and $l\partial_B(b) = \phi'(l\partial_A(a))$, we get

$$\dim_{\mathbb{Q}_A \cap C} l\partial_B(b) = \dim_{\mathbb{Q}_A \cap C} l\partial_A(a) < n.$$

□

Example 7.6. (1) If ϕ is a formal isomorphism between $\mathbb{G}_a^n$ and a semi-abelian variety A and (v, a) satisfies the differential equation of ϕ, then we get:

$$\dim_{\mathbb{Q}_A}(l\partial_A(a)) = n \implies \operatorname{trdeg}_C(v, a) \geqslant n + 1.$$

(2) Assume A, ϕ and (v, a) are as in (1).
- If $A = E^n$ for an elliptic curve E, we get:

$$\dim_{\mathbb{Q}(\tau)}(l\partial_A(a)) = n \implies \operatorname{trdeg}_C(v, a) \geqslant n + 1,$$

  where τ is a complex multiplication in E (if any). This is Proposition 1. from [11].
- More generally, if E above is any simple semi-abelian variety, then we get:

$$\dim_{\mathrm{End}_{\mathbb{Q}}(E)}(l\partial_A(a)) = n \implies \operatorname{trdeg}_C(v, a) \geqslant n + 1.$$

- Yet more generally, if $A = A_1^{l_1} \times \cdots \times A_k^{l_k}$, where A_i are pairwise non-isogenous simple semi-abelian varieties we get a condition involving all $\mathrm{End}_{\mathbb{Q}}(A_i)$ which is a bit cumbersome to write down. If A_i are 1-dimensional, we get Proposition 2. from [11].

(3) If ϕ is a formal isomorphism between $\mathbb{G}_m^n$ and an abelian variety A (or even an extension of an abelian variety by a vector group) and (a, b) satisfies the differential equation of ϕ, then we get:

$$\dim_{\mathbb{Q}}(l\partial_{\mathbb{G}_m^n}(a)) = n \implies \operatorname{trdeg}_C(a, b) \geqslant n + 1.$$

8. Positive characteristic

In this section we assume $\mathrm{char}(K) = p > 0$. If ∂ is a derivation on K with the field of constants C, then $K^p \subseteq C$, so K is algebraic over C and any Ax-like statement is meaningless. To get a more reasonable field of constants we replace the derivation ∂ with a *Hasse-Schmidt derivation* $D = (D_i)_{i<\omega}$, i.e.: each $D_i : K \to K$ is additive, D_0 is the identity map,

$$D_i(xy) = \sum_{n+m=i} D_n(x) D_m(y),$$

$$D_i \circ D_j = \binom{i+j}{i} D_{i+j}.$$

Let C be the constant field of *all* D_i. K is usually not algebraic over C. To proceed as in Section 2, we need to:

- Replace the tangent bundle $TV(K) = V(K[X]/X^2)$ with the *bundle of arcs*:
$$\operatorname{Arc} V(K) = V(K[\![X]\!]).$$
- For A, a commutative algebraic group, replace $T_0 A$ with U_A, which denotes the fiber of $\operatorname{Arc} A \to A$ over 0.
- It is more difficult to find a good replacement of differential forms. It turns out that Vojta's Hasse-Schmidt differential forms [19] work well. However 2.5 remains the main problem.

To proceed as in Sections 3 and 4 we need the following:

Definition 8.1. $lD_A : A(K) \to U_A(K)$ is the *HS-logarithmic derivative* – the composition of $D_A : A(K) \to \operatorname{Arc} A(K)$ and the projection map $\operatorname{Arc} A(K) \to U_A(K)$.

Example 8.2.

$$lD_{\mathbb{G}_a}(x) = (D_i(x))_{i \geqslant 1}, \quad lD_{\mathbb{G}_m}(x) = \left(\frac{D_i(x)}{x}\right)_{i \geqslant 1}.$$

Remark 8.3. Any formal map $\phi : A \to B$ induces an algebraic homomorphism $U_\phi : U_A \to U_B$.

Definition 8.4. The *HS-differential equation of* ϕ is

$$U_\phi(lD_A(x)) = lD_B(y).$$

One can now state and try to prove a version of 5.5 in this context. This is work in progress.

The main difference between the characteristic 0 case and the positive characteristic case is that for A, B commutative algebraic groups of the same dimension U_A need not be isomorphic to U_B if characteristic is positive. In characteristic 0 case, all commutative formal groups of dimension n are isomorphic to $\widehat{\mathbb{G}}_a^n$. This is no longer true in positive characteristic even for 1-dimensional algebraic groups, e.g. $pU_{\mathbb{G}_a} = 0$ and $pU_{\mathbb{G}_m} \neq 0$, so $U_{\mathbb{G}_a} \ncong U_{\mathbb{G}_m}$ (note that $A \cong B$ implies $U_A \cong U_B$). Actually, the situation in dimension 1 has a nice description: to each 1-dimensional formal group F one associates its height $\operatorname{ht}(F) \in \mathbb{N} \cup \{\infty\}$, which is the height of the p-th power map on F. It is well-known that

$F \cong G$ if and only if $\mathrm{ht}(F) = \mathrm{ht}(G)$. Let E_o be an ordinary elliptic curve and E_s a supersingular elliptic curve. Then we have:

$$\mathrm{ht}(\mathbb{G}_a) = \infty,\ \mathrm{ht}(\mathbb{G}_m) = 1,\ \mathrm{ht}(E_o) = 1,\ \mathrm{ht}(E_s) = 2.$$

And for each $n \in \mathbb{N}$, there is a formal group F such that $\mathrm{ht}(F) = n$. Hence there are formal algebraic groups in dimension 1 which are not algebraic, so Riemann's Existence Theorem does not hold in this context.

But there are still formal isomorphisms between E_o and $\mathbb{G}_m$. They are the main candidates for a positive characteristic version of 5.5.

We finish with a remark that unlike in the characteristic 0 case, there are formal endomorphisms of $\mathbb{G}_a$ which are not algebraic e.g. $\sum_{n=0}^{\infty} X^{p^n}$ (thanks to Thomas Scanlon for pointing out this example to me).

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