Model theory and differential equations

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Baby Steps Beyond the Horizon
Będlewo, September 2, 2024

Plan of the talk

1. Introduction to general concepts of model theory.
2. Differentially closed fields and strong minimality.
3. Examples: Painlevé and Schwarz differential equations.

What is model theory

- Model theory is a branch of logic. It was initiated by Tarski in 1930s.
- Model theory reached its current form mostly thanks to groundbreaking ideas and results of Saharon Shelah (mainly in 1970s) and Ehud Hrushovski (from 1980s till present).
- Currently model theory has connections with and applications to: diophantine geometry, algebraic geometry, algebraic dynamics, differential equations, combinatorics, ...

What is model theory about

- Analyzing definable properties of structures, where the terms "definable" and "structure" have a precise meaning coming from the first-order logic.
- The "first-order" assumption above may be relaxed sometimes but we will not get into that.
- In general, we have some fixed language L and then: L-formulas, L-sentences, L-theories, L-structures, and models of L-theories.
- I will just give some examples (next slide).

Model theory of fields

- Language: $L_r = \{+, \cdot, -, 0, 1\}$ (the language of rings).
- L_r-formulas, for example:
  - $\exists y \; x + x = y \cdot y$
  - $\forall x \; \exists y \; x = y \cdot y$
- L_r-sentences are L_r-formulas where all variables are quantified. For example: $\exists x \; x \cdot x = -1$
- L_r-theories: sets consisting of L_r-sentences. Examples: the theory of commutative rings with 1, the theory of fields, the theory of algebraically closed fields.
- L_r-structures: sets M together with two specified functions $+^M, \cdot^M : M \times M \to M$, one specified function $-^M : M \to M$, and two specified elements $0^M, 1^M$.
- Models of L_r-theories. For example: the models of the theory of fields are exactly those L_r-structures which are fields.

Existentially closed models

Definition

Let M be a model of T. We say that M is an existentially closed model of T, if for any quantifier free L_M-formula $\chi(x)$ (x is a tuple of variables) and any extension $M \subseteq N$ of models of T, we have:

$$\text{“}\exists x \chi(x) \text{ is true in } N\text{”} \quad \text{implies} \quad \text{“}\exists x\ \chi(x) \text{ is true in } M\text{”}.$$

Intuitively, all solvable in an extension of M “systems of (in)equations” (parameters from M) can be already solved in M.

Example (Hilbert's Nullstellensatz)

The class of existentially closed fields (that is: existentially closed models of the theory of fields) coincides with the class of algebraically closed fields.

Inductive theories and model companion

Definition

A theory T is inductive, if for each chain of models of T, its union is also a model of T.

Theorem

Assume that T is inductive and M is a model of T. Then, there is an extension $M \subseteq N$ of models of T such that N is an existentially closed model of T.

The proof is similar to the construction of an algebraic closure of a field: add solutions "one by one" and take the unions of chains.

Definition

For an inductive L-theory T, we call an L-theory T^* a model companion of T if the class of models of T^* coincides with the class of existentially closed models of T.

Model companions and non-companionable theories

1. The theory of pure sets (empty language) has a model companion, which is the theory of infinite sets.
2. The theory of linear orders has a model companion, which is the theory of dense linear orders without endpoints.
3. The theory of fields has a model companion, which is the theory of algebraically closed fields.
4. The theory of fields with an automorphism has a model companion, which is called ACFA.
5. The theory of commutative groups has a model companion: the theory of commutative divisible groups having infinitely many elements of order p for every prime p.
6. The theory of groups has no model companion.
7. The theory of commutative rings has no model companion.

Model theory of differential fields and DCF_0

- Language of differential rings: $L_{r,\partial} := L_r \cup \{\partial\}$.
- The following $L_{r,\partial}$-sentence expresses the Leibniz rule:

$$\forall x \forall y \quad \partial(x \cdot y) = \partial(x) \cdot y + x \cdot \partial(y).$$

- DF_0 is the $L_{r,\partial}$-theory of differential fields of characteristic 0, that is the theory of fields of characteristic 0 with an extra map ∂ which is additive and satisfies the Leibniz rule.
- DCF_0 is the model companion of DF_0 (A. Robinson).
- Blum gave the following axioms of DCF_0: if F has order greater than H, then there is x s.t. $F(x) = 0$ and $H(x) \neq 0$.
- There are no natural examples of differentially closed fields. This is not unusual, e.g. there is only one algebraically closed field "in nature": $\mathbb{C}$. The differential fields of meromorphic functions are "not so far" from being differentially closed.

Strongly minimal differential equations

We give a general model-theoretic concept in the special case of differential equations $F(y) = 0$ in one variable over $(\mathbb{C}(t), \frac{\mathrm{d}}{\mathrm{d}t})$.

Definition

We say that $F(y) = 0$ (as above) is strongly minimal, if for any differentially closed $(K, \partial) \supseteq (\mathbb{C}(t), \partial_t)$ the set $\{a \in K | F(a) = 0\}$ is infinite and for any differential equation $H(y) = 0$ over K:
the set $\{a \in K | F(a) = 0 \wedge H(a) = 0\}$ is finite or
the set $\{a \in K | F(a) = 0 \wedge H(a) \neq 0\}$ is finite.

- This notion makes sense for any language L (here: $L = L_{r,\partial}$), any L-theory (here: DCF_0) and any L-formula ("equation") in any number of variables (here: $F(y) = 0$).
- For the theory of algebraically closed fields, the strongly minimal formulas are those defining algebraic curves.

Strong minimal theories

There are the following three main strongly minimal theories (that is: the formula "$x = x$" is strongly minimal there).

1. The theory of algebraically closed fields.
2. The theory of infinite vector spaces over a fixed field F (the language $(+, -, 0, \cdot_\lambda)_{\lambda \in F}$).
3. The theory of infinite pure sets (the empty language).

Zilber's trichotomy conjecture and DCF_0

- Zilber conjectured that any strongly minimal theory is "closely related" to one of the three from the previous slide (algebraically closed fields, vector spaces, pure sets).
- Hrushovski gave a counterexample to Zilber's conjecture.
- However, Zilber's trichotomy conjecture still holds inside many structures, like differentially closed fields. Therefore, a strongly minimal differential equation fits into one of the following three types (we write y' for $\partial(y)$):
  1. "algebraically closed field like", for example $y' = 0$;
  2. "vector space like" or *modular*, for example *Picard-Painlevé VI*: $y'' = \frac{1}{2}(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t})(y')^2 + (\frac{1}{t-y} + \frac{1}{1-t} - \frac{1}{t})y' + \frac{y(y-1)}{2t(t-1)(y-t)}$;
  3. "pure set like" or *geometrically trivial*, examples later (those are the most interesting ones!).

Classical functions and irreducible equations

Let $D \subseteq \mathbb{C}$ be open and connected and $\mathcal{F}(D)$ be the differential field of meromorphic functions on D.

Definition (of classical functions, Umemura)

- Any $f \in \mathbb{C}(t)$ is classical.
- If $f_1, .., f_n \in \mathcal{F}(D)$ are classical and $f \in \mathcal{F}(D)$ is in the algebraic closure of the differential field generated by $\mathbb{C}(t)(f_1, \dots, f_n)$, then f is classical.
- If f' is classical or f'/f is classical, then f is classical. (Actually, more than that, but it is too technical.)

Definition (Painlevé)

An equation $F(y) = 0$ with coefficients from $\mathbb{C}(t)$ is irreducible (w.r.t. classical functions) if it has no classical solutions.

Strong minimality and irreducibility

The following result relates strong minimality and irreducibility.

Theorem (Nagloo-Pillay)

Let $F(y) = 0$ be a strongly minimal differential equation over $\mathbb{C}(t)$ which is modular or geometrically trivial. Then, for any $f \in \mathcal{F}(D) \setminus \mathbb{C}(t)^{\mathrm{alg}}$ such that $F(f) = 0$, f is not classical.

Its consequence is the following.

Criterium (Nagloo-Pillay)

If $F(y) = 0$ is a strongly minimal differential equation over $\mathbb{C}(t)$ of order at least two and having no algebraic (over $\mathbb{C}(t)$) solutions, then $F(y) = 0$ is irreducible with respect to classical functions.

Examples on next slides.

Painlevé differential equations

The six families of Painlevé differential equations $(\alpha, \beta, \gamma, \delta \in \mathbb{C})$.

$$\begin{aligned}
P_I: \quad & y'' = 6y^2 + t \\
P_{II}(\alpha): \quad & y'' = 2y^3 + ty + \alpha \\
P_{III}(\alpha,\beta,\gamma,\delta): \quad & y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y} \\
P_{IV}(\alpha,\beta): \quad & y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y} \\
P_V(\alpha,\beta,\gamma,\delta): \quad & y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} \\
& + \delta\frac{y(y+1)}{y-1} \\
P_{VI}(\alpha,\beta,\gamma,\delta): \quad & y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y+1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\
& + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)
\end{aligned}$$

Irreducibility of Painlevé differential equations

Theorem (Nagloo-Pillay, strongly inspired by "Japanese school")

Let $F(y) = 0$ be one of the Painlevé differential equations from the previous slide such that the (possible) coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are algebraically independent over $\mathbb{Q}$. Then, the differential equation $F(y) = 0$ is strongly minimal and geometrically trivial.

Corollary

Since it is known that there are no algebraic (over $\mathbb{C}(t)$) solutions to the differential equations as above, we get that these Painlevé differential equations are irreducible, so their solutions cannot be classical functions.

Schwarzian equations

- Let us define a Schwarzian equation $S_R(y) = 0$, where:

$$S_R(y) := \left(\frac{y''}{y'}\right)' - \frac{1}{2}\left(\frac{y''}{y'}\right)^2 + (y')^2 R(y)$$

  for some $R \in \mathbb{C}(y)$.
- If we take

$$R = R_j := \frac{y^2 - 1968y + 2654208}{y^2(y-1728)^2},$$

  then we get the differential equation of the classical j-function.
- More generally, there are certain automorphic functions h giving an appropriate $R_h \in \mathbb{C}(y)$ such that $S_{R_h}(h) = 0$.

Irreducibility of Schwarzian differential equations

Assertion of Painlevé (1895)

Differential equations $S_{R_h}(y) = 0$ (as in the previous slide) are irreducible.

The following result (*Annals of Mathematics*, 2020) implies Painlevé's claim from 1895.

Theorem (Casale, Freitag, Nagloo)

Those differential equations $S_{R_h}(y) = 0$ are strongly minimal and geometrically trivial.