Unlikely formal intersections

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Ax’s theorem about formal intersections

**Setting**

Let us fix:

- $G$, an algebraic group over $\mathbb{C}$;
- $\mathcal{A}$, a complex analytic subgroup of $G$;
- $V$, an irreducible algebraic subvariety of $G$ containing 1;
- $W$, an analytic subvariety of $\mathcal{A}$ and $V$.

**Ax’s Theorem (Amer. J. Math, 1972)**

If $W$ is Zariski dense in $V$, then there is $B$, a complex analytic subgroup of $G$ containing $V$ and $\mathcal{A}$ such that

$$\dim(B) \leq \dim(\mathcal{A}) + \dim(V) - \dim(W).$$
Assume that the intersection $\mathcal{W}$ is unlikely, i.e. for some $d > 0$ we have

$$\dim(\mathcal{W}) = \dim(\mathcal{A}) + \dim(\mathcal{V}) - \dim(\mathcal{G}) + d.$$

- Ax’s theorem: there is $B$ as above of codimension at least $d$.
- Inside $B$ the intersection is not unlikely anymore.
- Ax’s theorem looks similar to CIT. A differential version of Ax’s theorem (closely related to this one) implies weak CIT.

**Question**

Is there a direct proof of

$$\text{Ax’s theorem } \Rightarrow \text{ weak CIT}$$

not going through a (logical) compactness argument?
Formal version

Setting (\( C \) a field of characteristic 0)

- Let \( G \) be an algebraic group over \( C \);
- Let \( A \) be a formal subgroup of \( \hat{G} \);
- Let \( V \) be an irreducible algebraic subvariety of \( G \) containing 1;
- Let \( \mathcal{W} \) be a formal subvariety of \( A \) and \( \hat{V} \).

Ax’s Theorem, formal version

If \( \mathcal{W} \) is Zariski dense in \( V \), then there is \( B \), a formal subgroup of \( \hat{G} \) containing \( \hat{V} \) and \( A \) such that

\[
\dim(B) \leq \dim(A) + \dim(V) - \dim(\mathcal{W}).
\]

Question

Is the above theorem true for \( C \) of positive characteristic?
“Formal intersection Ax” implies “Ax-Schanuel”

- Let $x$ be an $n$-tuple of formal power series in several variables over $C$ without constant terms, linearly independent over $\mathbb{Q}$;
- Let $G = \mathbb{G}_a^n \times \mathbb{G}_m^n$;
- Let $V$ be the algebraic locus of $(x, \exp(x))$ over $C$;
- Let $\mathcal{A}$ be the graph of $\exp : \mathbb{G}_a^n \to \mathbb{G}_m^n$;
- Let $\mathcal{W}$ be the formal locus of $(x, \exp(x))$ over $C$.

Formal group $B$ given by Ax’s theorem coincides with $\hat{G}$ here, so

$$2n \leq \dim(\mathcal{A}) + \dim(V) - \dim(\mathcal{W}) = n + \text{trdeg}_C(x, \exp(x)) - \text{rk}(J_x),$$

$$\text{trdeg}_C(x, \exp(x)) \geq n + \text{rk}(J_x).$$

The same proof works for $A$ semi-abelian.
Let $A$ be a semi-abelian variety of dimension $n$. Similarly as above, Ax’s theorem easily implies the following.

**Theorem**

Assume that

- $Y \subseteq A$ is an algebraic subvariety;
- $X \subseteq \mathbb{G}_a^n$ is a maximal algebraic subvariety such that $\exp(X) \subseteq Y$;
- $Y' := \exp(X)^{\text{Zar}}$.

Then $Y'$ is an algebraic subgroup of $A$ and $X = \text{Lie}(Y')$. 

“Ax-Lindemann-Weierstrass”
Dense formal subvarieties (characteristic 0)

The above applications suggested me a more general statement which also looks better for positive characteristic generalizations.

**Setting**
- Let $A$ be a commutative algebraic group over $C$;
- Let $V$ be an algebraic variety over $C$ and $v \in V(C)$;
- Let $\mathcal{F} : \hat{V} \to \hat{A}$ be a “special” formal map;
- Let $\mathcal{W}$ be a formal subvariety of $\hat{V}$ such that $\mathcal{F}(\mathcal{W}) = 0$.

**Theorem (easily following from Ax’s proof)**
If $\mathcal{W}$ is Zariski dense in $V$, then there is $A$, a formal subgroup of $\hat{A}$ such that $\mathcal{F}(\hat{V}) \subseteq A$ and

$$\dim(A) \leq \dim(V) - \dim(\mathcal{W}).$$
Remarks

- A continuous map between Hausdorff spaces which is constant on a dense set is constant everywhere.
- The same principle applies to an algebraic map between algebraic varieties and the Zariski topology.
- In the Ax’s theorem situation the categories are mixed: a formal map is constant on a Zariski dense set. The theorem says that the above principle can be saved at the cost of quotienting out by a subgroup of a controlled dimension.
I call a formal map \( \mathcal{F} : \hat{V} \to \hat{G} \) “special” if it has certain properties of formal homomorphisms (even when \( V \) is not a group!)

**Definition**

\( \mathcal{F} \) is **special** if it takes invariant differential forms on \( G \) into algebraic differential forms on \( V \).

**Example**

\[
\exp^* \left( \frac{dX}{X} \right) = \frac{\exp(X) dX}{\exp(X)} = dX.
\]

Formalizations of algebraic maps are special.
Formal homomorphisms are special.
Assume that $\text{char}(C) = p > 0$. There is no exponential map anymore. But there are other interesting formal homomorphisms.

**Example**

- **Additive power series.**
  \[
  \mathcal{F} : \hat{G}_a \rightarrow \hat{G}_a, \quad \mathcal{F} = \sum c_i X^{p^i}.
  \]

- **Multiplicative power series.** For $\gamma = \sum a_i p^i \in \mathbb{Z}_p$
  \[
  \mathcal{F} : \hat{G}_m \rightarrow \hat{G}_m, \quad \mathcal{F} = X^\gamma.
  \]

  $\mathcal{F}$ corresponds to $\prod (X^{p^i} + 1)^{a_i} - 1$.

- A formal isomorphism between $\hat{G}_m$ and an ordinary elliptic curve (defined over $\mathbb{C}^\text{alg}$).
Towards positive characteristic Ax

Setting

Let us fix:

- $C$, a perfect field of characteristic $p > 0$;
- $A$, a commutative algebraic group over $C$;
- $V$, an algebraic variety over $C$ and $v \in V(C)$;
- $\mathcal{F} : \hat{V} \to \hat{A}$, a formal map;
- $\mathcal{W}$, a formal subvariety of $\hat{V}$ such that $\mathcal{F}(\mathcal{W}) = 0$.

I will describe a positive characteristic variant of Ax’s theorem. Unfortunately, I have to put extra assumptions on the formal map $\mathcal{F}$ and the algebraic group $A$. 
**Limit maps**

**Definition (positive characteristic)**

I call a formal map $\mathcal{F} : \hat{V} \to \hat{A}$ an *$A$-limit* if there is a sequence of rational maps $(f_m : V \to A)_m$ such that $f_m(v) = 0$ and $\mathcal{F}$ is the limit of $(f_m)_m$ in a certain strong sense, i.e.

$$f_{m+1} - f_m \in A((\mathcal{O}_V)_V^p)^{m+1}).$$

**Example**

- For $\mathcal{F} = \sum c_i X^{p^i}$, $\mathcal{F}$ is the limit of $(\sum_{i=0}^m c_i X^{p^i})_m$.
- For $\mathcal{F} = X^\gamma$, $\mathcal{F}$ is the limit of $(X^{\sum_{i=0}^m a_i p^i})_m$, where

$$\gamma = \sum a_i p^i \in \mathbb{Z}_p.$$
Questions about limit maps

1. Any $A$-limit map $\mathcal{F} : \hat{V} \to \hat{A}$ is special (in the proper sense involving higher differential forms).

2. The converse is true for $A$ affine.

3. More generally, the converse is true for $A$ such that

   $$\ker(H^1(K^p, A) \to H^1(K, A)) = 0,$$

   where $C \subseteq K$ is a finitely generated field extension.

4. Is the above map on cohomology always injective?

5. Formal homomorphisms are special. Are they $A$-limits?
**Integrable groups**

**Definition**

Let $D$ be a 1-dimensional algebraic group. I call $D$ **integrable** if for any $c \in \mathbb{C}$ there is an algebraic endomorphism $\varphi : D \to D$ such that $\varphi^*$ (the map induced on differential forms) is the multiplication by $c$.

**Example**

The following algebraic groups are integrable:

- Any 1-dimensional $D$ over $\mathbb{F}_p$.
- $\mathbb{G}_a$ over any $\mathbb{C}$. 
Main Theorem

Setting

Let us fix:

- $D$, an integrable algebraic group;
- $A := D^n$;
- $V$, an algebraic variety over $\mathbb{C}$ and $v \in V(\mathbb{C})$;
- $\mathcal{F} : \hat{V} \to \hat{A}$, an $A$-limit map;
- $W$, a formal subvariety of $\hat{V}$ such that $\mathcal{F}(W) = 0$.

Theorem (K.)

If $W$ is Zariski dense in $V$, then there is $A$, a formal subgroup of $\hat{A}$ such that $\mathcal{F}(\hat{V}) \subseteq A$ and

$$\dim(A) \leq \dim(V) - \dim(W).$$
For $D = \mathbb{G}_a$ or $D = \mathbb{G}_m$, we can replace “$A$-limit” with “special”.

In the characteristic 0 case any commutative algebraic group is isomorphic to $\mathbb{G}_a^n$ as a formal group, so our theorem may be thought of as a generalization of Ax’s theorem to the arbitrary characteristic case. However it is not satisfactory, since it does not fully answer the original question.
Application I: Additive transcendence

We assume that:

- $F$ is an additive power series over $\mathbb{F}_p$ which is transcendental over the ring of additive polynomials.
- $x_1, \ldots, x_n$ are power series over $\mathbb{F}_p$ without a constant term.

**Theorem (K.)**

*If $x_1, \ldots, x_n$ are linearly independent over the ring of additive polynomials, then*

$$\text{trdeg}_{\mathbb{F}_p}(x_1, F(x_1), \ldots, x_n, F(x_n)) \geq n + 1.$$
Application II: Multiplicative transcendence

We assume that:

- $\gamma \in \mathbb{Z}_p$ is transcendental over $\mathbb{Q}$.
- $F$ is the multiplicative power series corresponding to $\gamma$.
- $x_1, \ldots, x_n$ are power series over $\mathbb{F}_p$ with constant term 1.

**Theorem (K.)**

If $x_1, \ldots, x_n$ are multiplicatively independent, then

$$\text{trdeg}_{\mathbb{F}_p}(x_1, F(x_1), \ldots, x_n, F(x_n)) \geq n + 1.$$
Application III: Ax-Lindemann-Weierstrass for \( p > 0 \)

Let us fix

- \( D \), an integrable algebraic group;
- \( \gamma \), a formal endomorphism of \( D \) which is not algebraic;
- \( Y \), an algebraic subvariety of \( D^n \) containing 0;

**Theorem (K. proof to be checked)**

If \( X \) is an algebraic subvariety containing 0 and maximal such \( \gamma(\hat{X}) \subseteq \hat{Y} \) and \( Y' := \exp(X)^{\text{Zar}} \), then both \( X \) and \( Y' \) are algebraic subgroups of \( D^n \).