Some model theory of partial differential equations

joint work (in progress) with James Freitag and Omar León Sánchez

Piotr Kowalski

Instytut Matematyczny, Uniwersytet Wrocławski

1 September 2026

Workshop on Model Theory and Applications II, Chania

DCF_0

- All fields considered in this talk have characteristic 0.
- The model companion of the theory of (ordinary, i.e. with one derivation) differential fields is denoted by DCF_0.
- The existence of DCF_0 was shown by Robinson and nice axioms (in one variable) were given by Blum.
- DCF_0 is ω-stable of Morley rank ω.
- For $a \in \mathbb{U}$, where $(\mathbb{U}, \partial) \models \mathrm{DCF}_0$ is a monster model, and $K \subset \mathbb{U}$ let us denote

$$K\langle a\rangle := K(a, \partial(a), \partial^2(a), \ldots), \qquad \dim_{\mathrm{alg}}(\mathrm{tp}(a/K)) := \mathrm{trdeg}_K(K\langle a\rangle).$$

- For a 1-type p in DCF_0, we have

$$\mathrm{U}(p) \leqslant \mathrm{RM}(p) \leqslant \dim_{\mathrm{alg}}(p).$$

Ranks in DCF_0

- Take now $a = (a_1, \ldots, a_n) \in \mathbb{U}^n$ and for any $f : \mathbb{U} \to \mathbb{U}$ let
$$f(a) := (f(a_1), \ldots, f(a_n)).$$
- There are $d, h \in \mathbb{N}$ such that for all sufficiently large $t \in \mathbb{N}$ we have
$$\mathrm{trdeg}_K K\left(a, \partial(a), \ldots, \partial^t(a)\right) = d(t+1) + h \qquad \text{(Kolchin polynomial)}.$$
- Let us define
$$\dim_{\mathrm{alg}}(\mathrm{tp}(a/K)) := \omega d + h.$$
- For an arbitrary type p in DCF_0, we still have
$$\mathrm{U}(p) \leqslant \mathrm{RM}(p) \leqslant \dim_{\mathrm{alg}}(p).$$
- It is also true that "one rank is finite iff all ranks are finite" or more precisely
$$\omega d \leqslant \mathrm{U}(p) \leqslant \mathrm{RM}(p) \leqslant \dim_{\mathrm{alg}}(p) < \omega(d+1). \qquad \text{(Pong)}$$

Kolchin polynomial in $\mathrm{DCF}_{0,m}$

- McGrail showed that the model companion of differential fields with m commuting derivations exists and has Morley rank ω^m.
- This model companion is denoted by $\mathrm{DCF}_{0,m}$.
- Let

$$(\mathbb{U};\partial_1,\ldots,\partial_m)\models \mathrm{DCF}_{0,m},\quad K\subset\mathbb{U},\quad a=(a_1,\ldots,a_n)\in\mathbb{U}^n,\quad p(x)=\mathrm{tp}(a/K).$$

- There is again the following Kolchin polynomial

$$w_p(T)=\sum_{i=0}^{\tau}\alpha_i\binom{T+i}{i},\qquad \alpha_i\in\mathbb{Z},\qquad \alpha_\tau\neq 0$$

such that for all sufficiently large $t\in\mathbb{N}$, we have

$$w_p(t)=\mathrm{trdeg}_K K\left(\left\{\left(\partial_1^{i_1}\circ\ldots\circ\partial_m^{i_m}\right)(a)\mid i_1+\cdots+i_m\leqslant t\right\}\right).$$

McGrail's claim about ranks in $\mathrm{DCF}_{0,m}$

- McGrail claimed the following for a type p in $\mathrm{DCF}_{0,m}$

$$\omega^\tau \alpha_\tau \leqslant \mathrm{U}(p) \leqslant \mathrm{RM}(p) < \omega^\tau(\alpha_\tau + 1),$$

  where τ and α_τ come from the Kolchin polynomial above.
- McGrail's claim implies that "one rank finite iff all ranks finite", i.e. that the following are equivalent for a type p in $\mathrm{DCF}_{0,m}$:
  - $\mathrm{RM}(p)$ is finite,
  - $\mathrm{U}(p)$ is finite,
  - $\mathrm{trdeg}_K(K\langle a\rangle)$ is finite.
- People noticed that the inequality

$$\omega^\tau \alpha_\tau \leqslant \mathrm{U}(p)$$

  is problematic.

Süer's counterexample: the heat equation

- Sonat Süer gave the following counterexample to the problematic inequality.
- Assume $m = 2$ and set

$$H = \left\{x \in \mathbb{U} \mid \partial_1(x) = \partial_2^2(x)\right\}.$$

- Sonat showed that $\mathrm{U}(H) = \omega$.
- For a generic $a \in H$, we have

$$\partial_1^i\left(\partial_2^j(a)\right) = \partial_2^{2i+j}(a), \text{ thus} \qquad w(t) = 2t + 1.$$

- Hence $\tau = 1$ and $\alpha_\tau = 2$, so the problematic inequality gives

$$\omega + \omega = \omega^\tau \alpha_\tau \leqslant \mathrm{U}(H) = \omega, \qquad \text{a contradiction.}$$

- The lower bound fails.
- The example has U-rank ω, so it does not settle the finite-rank claim.

Omar's suggestion: the equation $\partial_1(x) = x^3 + c$

- In his "*Zilber dichotomy for* $\mathrm{DCF}_{0,m}$" paper, Omar asked whether the set defined by the following differential equation

$$\partial_1(x) = x^3 + c, \qquad c \text{ is generic}$$

  has finite U-rank in $\mathrm{DCF}_{0,2}$.
- We showed that this set is strongly minimal in $\mathrm{DCF}_{0,2}$ (and much more).
- Thus, the finite-rank claim is also false.
- Apparently, it is still unknown whether there is a type p in $\mathrm{DCF}_{0,2}$ such that $\mathrm{U}(p) = 1$ and $\mathrm{RM}(p) = \omega$.
- Independently, Sonat together with AI-Claude showed strong minimality in $\mathrm{DCF}_{0,m}$ of the set above. Sonat asked me to include the following disclaimer: "*I read the paper only twice and I have been away from this sort of stuff for many years. Thus it is quite possible that there are errors in it and the final proof may collapse.*"
- The degree 3 in this example is very non-accidental. More on this later.

Main question

- For the rest of the talk, we assume for simplicity that $m = 2$, but everything remains valid for any finite $m \geqslant 2$.
- Let
$$(\mathbb{U}; \partial_1, \partial_2) \models \mathrm{DCF}_{0,2}$$
be a monster model.
- Take $Q \in \mathbb{U}[X, Y]$ and the corresponding order-one definable set
$$S = \{x \in \mathbb{U} \mid Q(x, \partial_1(x)) = 0\}$$
such that S is strongly minimal in the reduct $(\mathbb{U}; \partial_1) \models \mathrm{DCF}_0$.

Main question

When does S remain strongly minimal after adding ∂_2?

First result

Our first result includes the definable set given by the original equation

$$\partial_1(x) = x^3 + c.$$

Theorem 1

Let $P \in \mathbb{U}[X]$ be such that

- $\deg(P) \geqslant 3$;
- *the constant term of P is (∂_1, ∂_2)-differentially transcendental over the other terms.*

Then, the set

$$\{x \in \mathbb{U} \mid \partial_1(x) = P(x)\}$$

is strongly minimal in $\mathrm{DCF}_{0,2}$.

Main result

- Let $k_0 \subset \mathbb{U}$ be a small differential subfield and we set

$$Q(X,Y) = q_0 + Q_0(X,Y), \qquad Q_0 \in k_0[X,Y], \qquad k := \left(k_0\langle q_0\rangle_{\partial_1}\right)^{\mathrm{alg}}.$$

- Assume that Q is irreducible, $Q_Y \neq 0$, and Q defines the plane curve C.
- Let (a_1,a_2) denote the generic point of C over k, so $k(C) = k(a_1,a_2)$, and let $\widetilde{\partial}$ be a derivation on $k(C)$ extending $\partial_1|k$ and such that $\widetilde{\partial}(a_1) = a_2$. We put:

$$L_C : k(C) \to k(C), \qquad L_C(f) := \widetilde{\partial}(f) + \frac{Q_X(a_1,a_2)}{Q_Y(a_1,a_2)} f, \qquad F := Q_Y(a_1,a_2)^{-1}.$$

Theorem 2

Assume that

- $F, L_C(F), L_C^2(F), \ldots$ *are linearly independent over* k *("geometric triviality");*
- q_0 *is* (∂_1,∂_2)*-differentially transcendental over* k_0 *(genericity).*

Then the set $S := \{x \in \mathbb{U} \mid Q(x,\partial_1(x)) = 0\}$ *is strongly minimal in* $\mathrm{DCF}_{0,2}$.

Theorem 2 is a vast generalization of Theorem 1

- In Theorem 1, we have the following definable set

$$\{x \in \mathbb{U} \mid \partial_1(x) = P(x)\},$$

  where
  - $\deg(P) \geqslant 3$;
  - the constant term of P is differentially transcendental over the other terms.
- It fits to the set-up of Theorem 2 after taking

$$Q := Y - P(X), \qquad q_0 = -P(0), \qquad k_0 = \text{field generated by non-constant terms of } P.$$

- It is clear that both the transcendence conditions coincide.
- It can be shown that in this set-up we have the following ($Q_Y = 1$ here):

$$1, L_C(1), L_C^2(1), \ldots \text{ are linearly independent over } k \qquad \Longleftrightarrow \qquad \deg(P) \geqslant 3.$$

- If $\deg(P) \leqslant 2$, then the set $\{x \in \mathbb{U} \mid \partial_1(x) = P(x)\}$ is non-orthogonal (in DCF_0) to $\partial_1^{-1}(0)$, so it cannot be geometrically trivial! More on it later.

Rough sketch of the proof of Theorem 2

1. Enough to show the following (an exercise using quantifier elimination in $\mathrm{DCF}_{0,2}$):

$$\forall\ k \subseteq K \prec \mathbb{U}\ \forall\ a \in S \setminus S(K) \qquad a \textit{ is } \partial_2\textit{-differentially transcendental over } K.$$

2. Suppose not and linearize the obtained ∂_2-algebraic relation into an equation

$$L(U) = b_1 + b_2 c.$$

3. Prove a general descent theorem: under linear independence of

$$b_2, L(b_2), L^2(b_2), \ldots$$

such an equation forces c and U to descend to the base field.

4. Apply descent to the highest new ∂_2-derivative of q_0.
5. Contradict linear independence of

$$b_2, L(b_2), L^2(b_2), \ldots.$$

The descent setting

- Let $K_0 \subseteq K$ be ordinary differential fields such that K_0 is differentially algebraically closed in K.
- Let V be an algebraic variety over K_0 and let D_0 be a derivation on $K_0(V)$ extending that of K_0. Extend D_0 to derivation D on $K(V) = \mathrm{Frac}(K \otimes_{K_0} K_0(V))$.
- Fix $\alpha \in K_0(V)$ and define the connection

$$L : K(V) \longrightarrow K(V), \qquad L(f) = D(f) + \alpha f.$$

Theorem 3 (Descent)

Assume that for some $b_2 \in K_0(V)$, the sequence $b_2, L(b_2), L^2(b_2), \ldots$ is linearly independent over K_0. IF for some $U \in K(V), c \in K, b_1 \in K_0(V)$, we have

$$L(U) = b_1 + b_2 c,$$

THEN

$$U \in K_0(V) \qquad \textit{and} \qquad c \in K_0.$$

Application of Theorem 3 to Theorem 2

- Assume Theorem 2 does not hold and take $a_1 := a, a_2 := \partial_1(a), K$ $(a, K$ are as in "rough sketch") and the smallest $r > 0$ such that

$$\partial_2^r(a_1) \in K\left(\partial_2^{<r}(a_1)\right)^{\mathrm{alg}}.$$

- Let us define

$$C := \mathrm{locus}_K\left(a_1, a_2\right), \qquad V := \mathrm{locus}_K\left(\partial_2^{<r}(a_1), \partial_2^{<r}(a_2)\right), \qquad D := \partial_1|_{K(V)},$$

  and set K_0 as the ∂_1-algebraic differential closure of $k(\partial_2^{<r}(q_0))$ in K.
- After applying ∂_2^r to the equation $Q(a_1, a_2) = 0$ and hitting both sides of the obtained equation with the normalized trace of $K(V)(\partial_2^r(a_1))/K(V)$, we get the linearized equation as in Theorem 3 for $c := \partial_2^r(q_0) \in K$.
- Theorem 3 implies that $c \in K_0$ contradicting the genericity of q_0.

Preservation of strong minimality implies triviality

- Let $(\mathbb{U}, \partial) \models \mathrm{DCF}_0$ and $C_\partial := \partial^{-1}(0)$ be the field of constants.
- Suppose that S is an order one strongly mininmal definable set in DCF_0.
- By Hrushovski-Sokolović and Buium, S is non-orthogonal to C_∂ or S is geometrically trivial.
- If S is non-orthogonal to C_∂, then S has Morley rank ω in $\mathrm{DCF}_{0,2}$.
- Therefore, we get (for S as above):

$$S \textit{ is strongly minimal in } \mathrm{DCF}_{0,2} \quad \Longrightarrow \quad S \textit{ is geometrically trivial in } \mathrm{DCF}_0.$$

- Hence, Theorem 2 provides a new criterion for triviality in DCF_0.
- It looks like a strange route to get triviality (checking strong minimality in an expansion), but maybe not?

Criterions of Hrushovski-Itai

- If C is a smooth algebraic curve over C_∂ and ω is a rational 1-form on C, then Hrushovski–Itai define the following strongly minimal set

$$\Xi(C,\omega) := \{a \in C(\mathbb{U}) \mid \omega(a)(\partial_C(a)) = 1\}.$$

- Hrushovski-Itai showed the following.

$$\Xi(C,\omega) \not\perp C_\delta \quad \Longleftrightarrow \quad \omega\text{: pullback of an invariant form on 1-dim. algebraic group.}$$

- For $C = \mathbb{P}^1$, writing

$$\omega = f(x)\,dx, \qquad\qquad f \in \mathbb{U}(X)$$

nonorthogonality of $\Xi(\mathbb{P}^1,\omega)$ to C_δ occurs if and only if either
  - f has no simple poles, or
  - all poles of f are simple and all nonzero residues have rational ratios.

Comparison to Hrushovski-Itai ("sanity check")

- Take now $(\mathbb{U}; \partial_1, \partial_2) \models \mathrm{DCF}_{0,2}$ and consider the situation from Theorem 1

$$\partial_1(x) = P(x), \qquad P(X) = p_0 + p_1 X + \cdots + p_d X^d \in \mathbb{U}[X], \qquad p_d \neq 0.$$

- Then we have

$$\Xi\left(\mathbb{P}^1, 1/P\,dx\right) = \{x \in \mathbb{U} \mid \partial_1(x) = P(x)\}.$$

- If p_0 is (∂_1, ∂_2)-differentially transcendental over the field generated by the other coefficients, then we have:
  1. P is square-free;
  2. the residues of $1/P(x)$ are $1/P'(\alpha_i)$ at the roots α_i of P;
  3. for $d \geqslant 3$, these residues do not all have rational ratios.
- This is consistent with the Hrushovski-Itai criterion for geometric triviality.

Possible future directions

- Show the following in the differential context

  strongly minimal, trivial, generic $\Longrightarrow$ strongly minimal with extra derivation.

- Find a general model-theoretic context such that the following holds

  strongly minimal and ??? $\Longrightarrow$ strongly minimal in a reasonable expansion.

- An example of such a context is given by the work of Wei Li and Thomas Scanlon who consider the differential-difference case. Wei announced at a conference in Singapore the following.
  "*if X is a strongly minimal set relative to $\mathrm{DCF}_{0,n}$ and the forking geometry on X is trivial, then X remains minimal when regarded as a definable set in $\mathrm{DCFA}_{0,n}$.*"
- Looks like no genericity assumption is needed in the differential-difference case!

The method of ansatz (or differential constraints)

The method of ansatz can be briefly described as follows.

- Start from a PDE of the form $F = 0$, which you want to solve.
- Add a constraint $G = 0$, which is an additional PDE.
- Substitute some consequences of the constraint equation $G = 0$ into the original equation $F = 0$ to obtain a system of ODE's.
- Solve these ODE's.

A useful ansatz should have at least three properties:

1. **Compatibility:** the enlarged system must have solutions.
2. **Reduction:** the enlarged system should have lower functional dimension than the original PDE.
3. **Computability:** the reduced equations should actually be easier to solve.

Example: an ansatz for the heat equation (separation of variables)

- Consider again the heat equation written now in the following form

$$u_t = \kappa u_{xx}, \qquad \kappa > 0 \text{ is the thermal diffusivity of the material,}$$

and we want to find solutions which are smooth functions $u = u(x,t)$.

- Add the following differential constraint

$$u\, u_{xt} - u_x u_t = 0 \qquad \text{implying the local factorization} \qquad u(x,t) = X(x)T(t).$$

- Substitution into the heat equation gives

$$XT_t = \kappa X_{xx} T_t \qquad \Longrightarrow \qquad \frac{T_t}{\kappa T} = \frac{X_{xx}}{X} = \lambda.$$

- Hence, the heat equation reduces to the two ordinary differential equations

$$\boxed{X'' - \lambda X = 0,} \qquad \boxed{T' - \kappa\lambda T = 0.}$$

Question of Clarkson and Mansfield and strong minimality

Open Question 2.9 in Clarkson, Mansfield "*Open Problems in Symmetry Analysis*"

How does one determine a priori which differential constraints are useful and which ansätze will yield computable solutions?

- We want to solve $F = 0$, so we add a constraint $G = 0$.
- If the set given by $G = 0$ is strongly minimal, then $G = 0$ is not a useful ansatz.
- Hence Theorem 2 provides a partial negative answer to Question 2.9:

We obtain specific criteria for no useful ordinary ansätze.

Initial arguments

- I think that I make many more mathematical mistakes than (a reasonable) AI.
- It is hard to separate "what was done by AI and what was not" here. I give some explanations below.
- The story starts with Omar's talk in Wrocław in January, where I heard first about these kind of problems.
- I started to work on the equation $\partial_1(x) = x^3 + c$ and I started to use AI to handle computations: first just Gemini from the Google search and then GPT "Think Deeper" from Microsoft Copilot provided by my university.
- Apparently this usage was on the "Wolfram Mathematica level".
- The first arguments regarding the equation $\partial_1(x) = x^3 + c$ were inductive on r, which is the ∂_2-order contradicting strong minimality.
- Then, I started to ask "global questions" (or, perhaps, "better questions") to AI and it gave a surprisingly good argument in the case of $r = 2$.
- Checking this argument, a pattern emerged for a general inductive argument (and a better ordinary descent statement!) in the case of the equation $\partial_1(x) = x^3 + c$.

Final arguments

- Looking at the ordinary statement, it seemed possible to replace $X^3 + c$ with a generic enough polynomial P of degree at least 3.
- I asked this question to GPT 5.6 Sol Pro (via a friend with access) and Pro gave the general descent criterium on fields of rational functions and with "$b_2 = 1$".
- We generalized this criterium to fields of the form $K(V)$ and any b_2: this is the descent Theorem 3 discussed before.
- However, I made a mistake and for some time I was getting results contradicting Hrushovski-Itai criterions (these criterions were my "sanity check").
- I could not find the mistake, so I asked the university provided AI for help.
- AI found a mistake (at the very beginning of the argument!), which was easy to fix and my life was relatively sane again.
- AI help with slides for this talk was not that great for me.
- But I still talk to AI a lot...