Model theory of Galois actions
(joint work with Özlem Beyarslan)

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We fix a finitely generated (marked) group:

\[ \mathcal{G} = \langle \rho \rangle, \quad \rho = (\rho_1, \ldots, \rho_m). \]

By a \textit{G-field}, we mean a field together with a \textit{G-action} by field automorphisms. Similarly, we have \textit{G-field extensions}, \textit{G-rings}, etc.

We consider a \textit{G-field} as a first-order structure in the following way:

\[ \mathcal{K} = (K, +, -, \cdot, \rho_1, \ldots, \rho_m). \]

Note that any \( \rho_i \) above denotes \textit{three} things at the same time:

- an element of \( \mathcal{G} \),
- a function from \( K \) to \( K \),
- a formal function symbol.
Existentially closed $G$-fields: definition

Let us fix a $G$-field $(K, \rho)$.

**Systems of $G$-polynomial equations**

Let $x = (x_1, \ldots, x_n)$ be a tuple of variables and $\varphi(x)$ be a system of $G$-polynomial equations over $K$

$$\varphi(x) : \quad F_1(g_1(x_1), \ldots, g_n(x_n)) = 0, \ldots, F_n(g_1(x_1), \ldots, g_n(x_n)) = 0$$

for some $g_1, \ldots, g_n \in G$ and $F_1, \ldots, F_n \in K[X_1, \ldots, X_n]$.

**Existentially closed $G$-fields**

The $G$-field $(K, \rho)$ is existentially closed (e.c.) if any system $\varphi(x)$ of $G$-polynomial equations over $K$ which is solvable in a $G$-extension of $(K, \rho)$ is already solvable in $(K, \rho)$. 
Existentially closed $G$-fields: first properties

- Any $G$-field has an e.c. $G$-field extension (a general property of inductive theories).

- For $G = \{1\}$, the class of e.c. $G$-fields coincides with the class of algebraically closed fields.

- For $G = \mathbb{Z}$, the class of e.c. $G$-fields coincides with the class of transformally (or difference) closed fields.

- An e.c. $G$-field is usually not algebraically closed.

- The complex field $\mathbb{C}$ with the complex conjugation automorphism is not an e.c. $C_2$-field. (By $C_n$, we denote the cyclic group of order $n$ written multiplicatively.)
Let $K$ be an e.c. $G$-field, and $F := K^G$ be the fixed field.

- Both $K$ and $F$ are perfect.
- Both $K$ and $F$ are pseudo algebraically closed (PAC), hence their absolute Galois groups are projective profinite groups.
- The profinite group $\text{Gal}(F^{\text{alg}} \cap K/F)$ coincides with the profinite completion $\hat{G}$ of $G$.
- The profinite group $\text{Gal}(F)$ (the absolute Galois group of $F$) coincides with the universal Frattini cover $\tilde{\hat{G}}$ of $\hat{G}$.
- The field $K$ is algebraically closed iff $\hat{G}$ is projective (iff $\tilde{\hat{G}} = \hat{G}$), more precisely:

$$\text{Gal}(K) \cong \ker \left( \tilde{\hat{G}} \rightarrow \hat{G} \right).$$
The theory $G$-TCF

**Definition**

If the class of existentially closed $G$-fields is *elementary*, then we call the resulting theory $G$-TCF and say that $G$-TCF exists.

**Example**

- For $G = \{1\}$, we get $G$-TCF = ACF.
- For $G = F_m$ (free group), we get $G$-TCF = ACFA$_m$.
- If $G$ is finite, then $G$-TCF exists (Sjögren, independently Hoffmann-K.)
- $(\mathbb{Z} \times \mathbb{Z})$-TCF does *not* exist (Hrushovski).
We fix now a difference field \((K, \sigma)\), i.e. \((G, \rho) = (\mathbb{Z}, 1)\) (or, for technical reasons, \((G, \rho) = (\mathbb{Z}, 0, 1)\)).

- By a **variety**, we always mean an affine \(K\)-variety which is \(K\)-irreducible and \(K\)-reduced (i.e. a prime ideal of \(K[\bar{X}]\)).

- For any variety \(V\), we also have the variety \(\sigma V\) and the bijection (not a morphism!)

\[
\sigma_V : V(K) \rightarrow \sigma V(K).
\]

- A pair of varieties \((V, W)\) is called a **\(\mathbb{Z}\)-pair**, if \(W \subseteq V \times \sigma V\) and the projections \(W \rightarrow V, W \rightarrow \sigma V\) are dominant.

**Axioms for ACFA (Chatzidakis-Hrushovski)**

The difference field \((K, \sigma)\) is e.c. if and only if for any **\(\mathbb{Z}\)-pair** \((V, W)\), there is \(a \in V(K)\) such that \((a, \sigma_V(a)) \in W(K)\).
Let $G = \{\rho_1, \ldots, \rho_e\} = \rho$ be a finite group and $(K, \rho)$ be a $G$-field.

**Definition of $G$-pair**

A pair of varieties $(V, W)$ is a $G$-pair, if:

- $W \subseteq \rho_1 V \times \ldots \times \rho_e V$;
- all projections $W \to \rho_i V$ are dominant;
- **Iterativity Condition**: for any $i$, we have $\rho_i W = \pi_i(W)$, where

  $$\pi_i : \rho_1 V \times \ldots \times \rho_e V \to \rho_i \rho_1 V \times \ldots \times \rho_i \rho_e V$$

  is the appropriate coordinate permutation.

**Axioms for $G$-TCF, $G$ finite (Hoffmann-K.)**

The $G$-field $(K, \rho)$ is e.c. if and only if for any $G$-pair $(V, W)$, there is $a \in V(K)$ such that

$$((\rho_1)_V(a), \ldots, (\rho_e)_V(a)) \in W(K).$$
Our strategy 1

- Find a generalization of the known results (mentioned above) about free groups and finite groups.
- Natural class of groups for such a generalization: *virtually free* groups.
- For a fixed \((G, \rho)\), the general scheme of axioms should be as follows: for any \(G\)-pair \((V, W)\), there is \(a \in V(K)\) such that

\[
\rho_V(a) := ((\rho_1)_V(a), \ldots, (\rho_m)_V(a)) \in W(K).
\]

Hence one needs to find the right notion of a \(G\)-pair.

### G-pairs in general (looking for this “right notion”)

A pair of varieties \((V, W)\) will be called a **\(G\)-pair**, if:

- \(W \subseteq \rho V := \rho_1 V \times \ldots \times \rho_m V\);
- all projections \(W \rightarrow \rho_i V\) are dominant;
- **Iterativity Condition** (to be found) is satisfied.
Recall that, we need to find a good Iterativity Condition for a virtually free, finitely generated group \((G, \rho)\).

- \(G\) free: trivial Iterativity Condition.
- \(G\) finite: Iterativity Condition as before.

We need a convenient procedure to obtain virtually free groups from finite groups. Luckily, such a procedure exists and gives the right Iterativity Condition.

**Theorem (Karrass, Pietrowski and Solitar)**

Let \(H\) be a finitely generated group. Then TFAE:

- \(H\) is virtually free;
- \(H\) is isomorphic to the **fundamental group** of a finite graph of **finite groups**.
### Graph of groups (slightly simplified)

A graph of groups $G(\cdot)$ is a connected graph $(V, E)$ together with:

- a group $G_i$ for each vertex $i \in V$;
- a group $A_{ij}$ for each edge $(i, j) \in E$ together with monomorphisms $A_{ij} \to G_i, A_{ij} \to G_j$.

### Fundamental group

For a fixed maximal subtree $T$ of $(V, E)$, the fundamental group of $(G(\cdot), T)$ (denoted by $\pi_1(G(\cdot), T)$) can be obtained by successively performing:

- one free product with amalgamation for each edge in $T$;
- and then one HNN extension for each edge not in $T$.

$\pi_1(G(\cdot), T)$ does not depend on the choice of $T$ (up to $\cong$).
Iterativity Condition for amalgamated products

- Let $G = G_1 * G_2$, where $G_i$ are finite. We take $\rho = \rho_1 \cup \rho_2$, where $\rho_i = G_i$ and the neutral elements of $G_i$ are identified in $\rho$. We also define the projection morphisms $p_i : \rho V \to \rho_i V$. Let $W \subseteq \rho V$ satisfy the dominance conditions.

**Iterativity Condition for $G_1 * G_2$**

$(V, p_i(W))$ is a $G_i$-pair for $i = 1, 2$ (up to Zariski closure).

- Let $G = \pi_1(G(\mathcal{G}))$, where $G(\mathcal{G})$ is a tree of groups. We take $\rho = \bigcup_{i \in \mathcal{V}} G_i$, where for $(i, j) \in \mathcal{E}$, $G_i$ is identified with $G_j$ along $A_{ij}$.

**Iterativity Condition for the fundamental group of tree of groups**

$(V, p_i(W))$ is a $G_i$-pair for all $i \in \mathcal{V}$ (up to Zariski closure).
HNN extensions

Let us fix:

- a presentation $H = \langle X \mid R \rangle$ of a group $H$;
- two subgroups $H_1, H_2 \leq H$;
- an isomorphism $\alpha : H_1 \to H_2$.

The HNN-extension of $H$ relative to $\alpha$, denoted by $H \ast_\alpha$, is:

$$H \ast_\alpha = \langle X, t \mid R, h_1 t = t^{\alpha}(h_1); \ \forall h_1 \in H_1 \rangle.$$ 

$H$ is a subgroup of $H \ast_\alpha$ (a theorem of Graham Higman, B. H. Neumann and Hanna Neumann), and $\alpha$ is given by an inner automorphism of $H \ast_\alpha$ in the “most free” way.

Example

- $H \ast_{\text{id}\{1\}} = H \ast \mathbb{Z}$, in particular $\mathbb{Z} = \{1\} \ast_{\text{id}}$.
- For $\alpha \in \text{Aut}(G)$, we have $H \ast_\alpha = H \rtimes_\alpha \mathbb{Z}$, e.g. $H \ast_{\text{id}} = H \rtimes \mathbb{Z}$.
Let $C_2 \times C_2 = \{1, \sigma, \tau, \gamma\}$ and consider the following:

$$\alpha : \{1, \sigma\} \cong \{1, \tau\}, \quad G := (C_2 \times C_2) * \alpha.$$ 

Then the crucial relation defining $G$ is $\sigma t = t\tau$. We take:

- $\rho := (1, \sigma, \tau, \gamma, t, t\sigma, t\tau, t\gamma)$;
- $\rho_0 := (1, \sigma, \tau, \gamma)$;
- $t\rho_0 := (t, t\sigma, t\tau, t\gamma)$.

Let $W \subseteq \rho V$ satisfy the dominance conditions.

**Iterativity Condition for $(C_2 \times C_2) * \alpha$**

- $t (p_{\rho_0}(W)) = p_{t\rho_0}(W)$.
- $(V, p_{\rho_0}(W))$ is a $(C_2 \times C_2)$-pair.
Main Theorem

We find a complicated Iterative Condition for virtually free groups using the two previous conditions as the building blocks.

Theorem (Beyarslan-K.)

If $G$ is finitely generated and virtually free, then $G$-TCF exists.

Properties of $G$-TCF

- If $G$ is finite, then $G$-TCF is supersimple of finite rank ($= |G|$).
- If $G$ is infinite and free, then $G$-TCF is simple (not supersimple, for non-cyclic $G$).
- As we already know (Sjögren), for any $G$, if $(K, \rho)$ is an e.c. $G$-field then $K$ is PAC and $K^G$ is PAC.
- Chatzidakis: for a PAC field $K$, the theory $\text{Th}(K)$ is simple iff $K$ is bounded (i.e. the profinite group $\text{Gal}(K)$ is small).
New theories are not simple

**Theorem (Beyarslan-K.)**

Assume that $G$ is finitely generated, virtually free, infinite and not free. Then the following profinite group

$$\ker \left( \hat{G} \to \hat{G} \right)$$

is not small.

**Corollary**

Putting everything together, we get the following.

- If $G$ is finitely generated and virtually free, then the theory $G\text{-TCF}$ is simple if and only if $G$ is finite or $G$ is free.
- If $G$ is finitely generated, virtually free, infinite and not free, then the theory $G\text{-TCF}$ is not even NTP$_2$. 

Kowalski (joint with Beyarslan) 

Model theory of Galois actions
Neo-stability hierarchy

It looks possible that if the group $G$ is finitely generated and virtually free, then the theory $G$-TCF is $\text{NSOP}_1$ (Nick Ramsey communicated a sketch of an argument to us).

Non-finitely generated groups

- The theory $\mathbb{Q}$-TCF exists (Medvedev’s $\mathbb{Q}$ACFA).
- After a discussion with Alice Medvedev, we seem to have an argument showing that the theory $C_p\infty$-TCF exists, where $C_p\infty$ is the Prüfer $p$-group.
Conjecture \((G \text{ finitely generated})\)

The theory \(G\text{-TCF}\) exists if and only if \(G\) is virtually free.

- There is a long list of equivalent conditions characterising the class of finitely generated, virtually free groups e.g.:
  - fundamental groups of finite graphs of finite groups;
  - groups that are recognized by pushdown automata;
  - groups whose Cayley graphs have finite tree width.

- It would be interesting to have one more equivalent condition (as in the conjecture above) coming from model theory!

- Main challenge for a proof of this conjecture: infinite Burnside groups (finitely generated and of bounded exponent).