# Geometric axioms for existentially closed Hasse fields

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**Abstract**

We give geometric axioms for existentially closed Hasse fields. We prove a quantifier elimination result for existentially closed $n$-truncated Hasse fields and characterize them as reducts of existentially closed Hasse fields.

## 0 Introduction

The purpose of this paper is to give geometric axioms for $\mathrm{SCH}_{p,e}$ (the model companion of the theory of fields of positive characteristic $p$ with $e$ commuting iterative Hasse derivations) in the spirit of the axiomatizations of ACFA [CH], $\mathrm{DCF}_0$ [PP] or $\mathrm{DCF}_{p,n}$ [K]. Axioms for $\mathrm{SCH}_{p,e}$ were given in [Zi1] and they are actually pretty simple, i.e. a field as above is existentially closed if and only if it is strict and separably closed of inseparability degree $e$, see [Zi1, 1.1].

One could wonder why do we need geometric, possibly (and actually) more complicated axioms, than Ziegler's ones. However, Ziegler's axioms do not say much about solvability of Hasse-differential equations. Specifically, we know that in models of $\mathrm{SCH}_{p,e}$ all solvable (in an extension) systems of

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Hasse-differential equations and inequalities can be solved. But we still do not know which systems are solvable. As was pointed out to me by Margit Messmer and Anand Pillay, even very simple Hasse-differential equations may have no generic solutions, i.e. the equation $X' = X$ has 0 as its unique solution, so the system $X' = X \wedge X \neq 0$ is not solvable.

Therefore the situation is different than in the differential case. From Blum's axioms [Bl] for differentially closed fields of characteristic 0 we know that each non-explicitly inconsistent differential equation has a generic solution in a differentially closed field of characteristic 0. In the case of differentially closed fields of positive characteristic, Wood's axioms [Wo1, 2.5] tell us that each differential equation satisfying a certain separant condition (needed to get rid of inseparability issues) has a generic solution. The same is true in the case of derivations of Frobenius [K].

Our axioms for $\mathrm{SCH}_{p,e}$ give certain geometric criteria to decide whether a given system of Hasse-differential equations and inequalities has a solution. We also work with $n$-truncated Hasse derivations (as in [PZ]) and prove that their theory has a model companion, $\mathrm{SCH}_{p,e,n}$, which is an intermediate theory between the theory of separably closed fields of inseparability degree $e$ and $\mathrm{SCH}_{p,e}$. Therefore $\mathrm{SCH}_{p,e,n}$ is stable and not superstable. We also show that $\mathrm{SCH}_{p,e,n}$ does not have quantifier elimination in the Hasse field language, but it eliminates quantifiers after adding the $p$-th root function as in the case of $\mathrm{DCF}_p$ .

# 1 Basic notions about Hasse derivations

We do not use any symbol for multiplication as well as composition. It is always clear from the context which operation is taken. For example $f(a)^n$ denotes the $n$-th power of $f(a)$ and $f^n(a)$ denotes the $n$-th composition of $f$ with itself applied to $a$. To avoid a possible confusion with the set of the $n$-th powers, the $n$-th Cartesian product of a set $A$ is denoted by $A^{\times n}$.

All the rings considered in this paper are commutative, with unity and of prime characteristic $p > 0$. Let $e$ be a fixed positive integer. In this section we recall some definitions, mainly from [Zi1]. We use the term *Hasse field* instead of $\mathcal{D}$-field.

A sequence $\mathbf{D} = (D_i : R \to R)_{i \geqslant 0}$ is a Hasse derivation on a ring $R$, if

$D_0 = \mathrm{id}$ and the corresponding map

$$R \ni a \mapsto \sum_{i=0}^{\infty} D_i(a)X^i \in R[[X]]$$

is a ring homomorphism. $\mathbf{D}$ is *iterative*, if for any $i, j$

$$D_i D_j = \binom{i+j}{i} D_{i+j}.$$

It is clear that it is enough to check the above equality on a set of generators of $R$. For the rest of the paper we will denote the binomial coefficient $\binom{i+j}{i}$ by $c_{i,j}$. Note that

$$c_{i,j} \equiv 0 (\mod p)$$

for $i, j < p^n$ such that $i + j \geqslant p^n$ ($n \in \mathbb{N}$). We will regard $c_{i,j}$ as an element of $\mathbb{F}_p$ (the field of $p$ elements).
For two sequences $\mathbf{i} = (i_1, \dots, i_e), \mathbf{j} = (j_1, \dots, j_e)$ of natural numbers, we define

$$c_{\mathbf{i},\mathbf{j}} := c_{i_1,j_1} \dots c_{i_e,j_e}.$$

For each $n$ we have also an $R$-linear map:

$$c_n : R^{\times p^n} \to R^{\times p^{2n}}, \quad c_n(a_0, \dots, a_{p^n-1}) = (c_{i,j}a_{i+j})_{0 \leqslant i,j < p^n}.$$

Note that $c_{i,j}a_{i+j}$ always makes sense, since for $i, j < p^n$ and $i + j \geqslant p^n$, $c_{i,j} = 0$, so we put $c_{i,j}a_{i+j} = 0$ in that case.
For example, if $n = 1$, $p = 3$, then we get a map of the form:

$$R^{\times 3} \ni (a_0, a_1, a_2) \mapsto \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & 2a_2 & 0 \\ a_2 & 0 & 0 \end{pmatrix} \in R^{\times 9}.$$

For any $m$, let us denote $(D_0, \dots, D_{m-1})$ by $\mathbf{D}_{<m}$. In the sequel, we also consider $\mathbf{D}_{<m}$ as a map from $R$ to $R^{\times m}$ and $\mathbf{D}$ as a map from $R$ to $R^{\times \omega}$.
Let $\mathbf{D}_1, \mathbf{D}_2$ be Hasse derivations on $R$. We say that $\mathbf{D}_1, \mathbf{D}_2$ commute if for any $i, j$,

$$D_{1,i} D_{2,j} = D_{2,j} D_{1,i}.$$

It is again clear that the above equality needs to be checked on a set of generators of $R$ only. A *Hasse ring* is a ring with a finite number of commuting iterative Hasse derivations. For a Hasse ring $(R, \mathbf{D}_1, \ldots, \mathbf{D}_e)$, the *ring of constants of* $R$ is the ring of common zeroes of $D_{1,1}, \ldots, D_{e,1}$ and the *ring of absolute constants of* $R$ is the ring of common zeroes of $\mathbf{D}_1, \ldots, \mathbf{D}_e$. It is always true that $R^p$ is a subring of constants and $R^{p^\infty}$ is a subring of absolute constants. $(R, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ is *strict* if the constants of $R$ coincide with $R^p$. It follows from equation (2.2) in the proof of [Zi1, 2.4] that if $(K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ is a strict Hasse field, then for each $n$, common zeroes of $\mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n}$ coincide with $K^{p^n}$.

For the notions of $p$-independence, $p$-basis and the inseparability degree of a field of characteristic $p$, see [De]. If $(K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ is a Hasse field, then a $p$-basis $\{b_1, \ldots, b_e\}$ of $K$ is *canonical* if

$$D_{i,m}(b_j) = \begin{cases} 1 & \text{if } m = 1 \text{ and } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Following [Zi2], we say that a $p$-basis is *canonical of depth* $k$, if the above holds for $m < p^k$. It is shown in [Zi2] that every strict Hasse field of degree of imperfection $e$ has a canonical $p$-basis of depth $k$ for any natural number $k$. Hence ([Zi2]) every $\omega$-saturated strict Hasse field of degree of imperfection $e$ has a canonical $p$-basis.

We will also need truncated Hasse derivations (as in [PZ]).
A finite sequence of maps $(D_i : R \to R)_{i<p^n}$ is called an *$n$-truncated Hasse derivation* (always $n \geqslant 1$) on a ring $R$, if $D_0 = \mathrm{id}$ and the corresponding map

$$R \ni a \mapsto \sum_{i=0}^{p^n-1} D_i(a) X^i \in R[X]/(X^{p^n})$$

is a ring homomorphism.
We get a natural notion of an $n$-truncated *iterative* Hasse derivation:

$$D_i D_j = \begin{cases} c_{i,j} D_{i+j} & \text{for } i + j < p^n \\ 0 & \text{for } i + j \geqslant p^n, \end{cases}$$

since again $c_{i,j} = 0$ if $i, j < p^n$ and $i + j \geqslant p^n$.
For example an iterative 1-truncated Hasse derivation is given by a sequence

$$(\mathrm{id}, D, D^2/2, \ldots, D^{p-1}/(p-1)!)$$

for a derivation $D$ such that $D^p = 0$.
We define an *$n$-truncated Hasse ring* in an obvious way. Obviously, the notion of a canonical $p$-base (of depth $k$) makes sense for an $n$-truncated Hasse field (if $k \leqslant n$). Also constants and absolute constants are defined in the same way as in the Hasse case (e.g. for 1-truncated Hasse ring constants coincide with absolute constants).

Let $\mathbf{i} \in \mathbb{N}^{\times e}$ (we consider $\mathbb{N}^{\times e}$ as a monoid, i.e. addition makes sense). For a Hasse ring $(R, \mathbf{D}_1, \dots, \mathbf{D}_e)$, we define

$$D_{\mathbf{i}} := D_{1,i_1} \dots D_{e,i_e}.$$

From now on we denote a cartesian power of any map $f$ by the same symbol $f$. In particular it applies to the maps $c_n, \mathbf{D}_{<p^n}$ in the next lemma.

**Lemma 1.1** *Let $(R, \mathbf{D}_1, \dots, \mathbf{D}_e)$ be a Hasse ring ($e = 1$ in (ii) and (iii)).*
*(i) For $\mathbf{i}, \mathbf{j} \in \mathbb{N}^{\times e}$ we have*

$$D_{\mathbf{i}} D_{\mathbf{j}} = c_{\mathbf{i},\mathbf{j}} D_{\mathbf{i}+\mathbf{j}}.$$

*(ii) If $b \in R^{\times m}$, then*

$$\mathbf{D}_{<p^n} \mathbf{D}_{<p^n}(b) = c_n \mathbf{D}_{<p^n}(b).$$

*(iii) Assume $R = K$ is a field. Let $\mathbf{D}'_{<p^n}$ be an $n$-truncated Hasse derivation on a field $L$ such that $(K, \mathbf{D}_{<p^n}) \subset (L, \mathbf{D}'_{<p^n})$. If there exists a tuple $a \subset L$ such that $L = K(\mathbf{D}'_{<p^n}(a))$ and $\mathbf{D}'_{<p^n}$ is iterative on $a$, then $\mathbf{D}'_{<p^n}$ is iterative.*

**Proof** (i)

$$D_{\mathbf{i}} D_{\mathbf{j}} = D_{1,i_1} D_{1,j_1} \dots D_{e,i_e} D_{e,j_e} = c_{i_1,j_1} \dots c_{i_e,j_e} D_{1,i_1+j_1} \dots D_{e,i_e+j_e} = c_{\mathbf{i},\mathbf{j}} D_{\mathbf{i}+\mathbf{j}}.$$

(ii) It is a straightforward generalization of the iterativity condition.
(iii) Since $c_n$ is defined by a matrix over $\mathbb{F}_p$, $\mathbf{D}'_{<p^n}$ and $c_n$ commute. Therefore (by (ii)):

$$\mathbf{D}'_{<p^n} \mathbf{D}'_{<p^n} \mathbf{D}'_{<p^n}(a) = \mathbf{D}'_{<p^n} c_n \mathbf{D}'_{<p^n}(a) = c_n \mathbf{D}'_{<p^n} \mathbf{D}'_{<p^n}(a).$$

Hence $\mathbf{D}'_{<p^n}$ is iterative on the set $K \cup \{\mathbf{D}'_{<p^n}(a)\}$, therefore it is iterative on $K[\mathbf{D}'_{<p^n}(a)]$. By formula (2.1) in the proof of [Ok, 2.3], $\mathbf{D}'_{<p^n}$ is iterative on $L$. $\square$

For a Hasse ring $(R, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ and a tuple of variables $X$, there exists a Hasse ring of Hasse polynomials $R\{X\}$ (see Section 3 of [Ok]). As a ring

$$R\{X\} := R[X^{\mathbf{i}} \mid \mathbf{i} \in \mathbb{N}^{\times e}] \quad (\text{let } X = X^{(0,\ldots,0)}),$$

where

$$D_{\mathbf{i}}(X^{\mathbf{j}}) = c_{\mathbf{i},\mathbf{j}} X^{\mathbf{i}+\mathbf{j}}.$$

If $e = 1$, then $R\{X\}$ (as a ring) is just $R[X, X', \ldots, X^{(n)}, \ldots]$ .
It is easy to see that for each $n$, the ring

$$R\{X\}_n := R[X^{\mathbf{i}} \mid \mathbf{i} \in \{0, 1, \ldots, p^n - 1\}^{\times e}]$$

is an $n$-truncated Hasse ring with $\mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n}$.

The notion of an ($n$-truncated) *Hasse ideal* is the obvious one and one can see that quotients of ($n$-truncated) Hasse rings by ($n$-truncated) Hasse ideals have a natural structure of ($n$-truncated) Hasse rings.

# 2 Truncated Hasse fields

In this section we deal with truncated Hasse fields. The main purpose is to provide a step in the proof of the main theorem (4.3). We also characterize existentially closed truncated Hasse fields as reducts of existentially closed Hasse fields.
We will need a fact about expansions of $n$-truncated iterative Hasse derivations to Hasse derivations (2.3). First we need a version of Lemma 4 from [Zi2].

**Lemma 2.1** *Let $(K, \mathbf{D}_{1,\leqslant p^n}, \ldots, \mathbf{D}_{e,\leqslant p^n}, \mathbf{D}_{e+1,<p^n}, \ldots, \mathbf{D}_{e',<p^n})$ satisfy its part of the axioms for strict truncated Hasse fields ($n \geqslant 1,\ e \leqslant e'$). Let $a \in K$ be such that for all $m < n$*

$$D_{i,p^n} D_{j,p^m}(a) = 0$$

*for $i \leqslant e, j \leqslant e'$. Then there is $a' \in K$ such that*

$$\begin{cases} D_{j,p^n}(a') = 0 & \text{for all } j \leqslant e \\ D_{j,<p^n}(a') = D_{j,<p^n}(a) & \text{for all } j \leqslant e'. \end{cases}$$

Actually Lemma 4 in [Zi2] is formulated for strict Hasse fields but the proof also works in the truncated case.
The next lemma slightly generalizes [Zi2, 2], however its proof follows from the proof of [Zi1, 2.1].

**Lemma 2.2** *Let $D_1, \ldots, D_e$ be commuting derivations on a field $K$ such that $D_1^p = \ldots D_e^p = 0$ and let $C$ denote the constants of $(K, D_1, \ldots, D_e)$. Then the following are equivalent:*
*(i) There exist $a_1, \ldots, a_e \in K$ such that $D_i(a_j) = \delta_j^i$.*
*(ii) $[K : C] = p^e$.*
*Moreover, each sequence as in (i) generates $K$ over $C$.*

**Proof** By [Zi2, 2], we need only to prove that (i) implies (ii).
For $i \leqslant e$ let $B_i$ denote the field of constants of $(K, D_1, \ldots, D_i)$ (so $B_e = C$). For $i < e$, we have $D_{i+1}(B_i) \subset B_i$ and $a_{i+1} \in B_i \setminus B_{i+1}$. Since $[K : C] \leqslant p^e$ [Zi1, 2.1], and $a_{i+1}^p \in B_i$, the result follows. $\square$

Abusing the language a little bit, we will say that for a field extension $K \subset L$ a subset $B \subset L$ is a *p-basis of $L$ over $K$*, if

$$\{b_1^{\alpha_1} \ldots b_n^{\alpha_n} \mid \alpha_i < p \ , i \leqslant n < \omega, b_1, \ldots, b_n \in B\}$$

is a basis of $L$ over $KL^p$ (as a linear space).

**Theorem 2.3** *Assume that $(K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ is a Hasse field which has a canonical p-basis. Suppose that $L$ is a finitely generated extension of $K$ and there are commuting iterative $n$-truncated Hasse derivations $\mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n}$ on $L$ such that*

$$(K, \mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n}) \subset (L, \mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n}).$$

*Then there is $(\bar{\mathbf{D}}_1, \ldots, \bar{\mathbf{D}}_e)$, a Hasse field structure on $L$ such that*

$$(K, \mathbf{D}_1, \ldots, \mathbf{D}_e), (L, \mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n}) \subset (L, \bar{\mathbf{D}}_1, \ldots, \bar{\mathbf{D}}_e),$$

*i.e. $(L, \bar{\mathbf{D}}_1, \ldots, \bar{\mathbf{D}}_e)$ is an extension of $(K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ and an expansion of $(L, \mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n})$.*

**Proof** By definition of canonical $p$-basis, $K$ has inseparability degree $e$. By 2.2, $(K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ is strict.

We will use [Zi1, 2.1] several times. It is stated for extensions of Hasse fields, however what is used in the proof is only that $D_{1,1},\dots,D_{e,1}$ is a sequence of commuting derivations such that each of them composed with itself $p$ times vanishes.

Let $C$ denote the constants of $L$ and $B$ be a canonical $p$-basis of $K$. By [Zi1, 2.1], $[L:C]\leqslant p^e$ and $C$ is linearly disjoint from $K$ over $K^p$, so $[L:C]=p^e$. Therefore $B$ is also a $p$-basis of $L$ over $C$. Let $B_0$ be a $p$-basis of $C$ over $L^p$. Then $B_1:=B\cup B_0$ is a $p$-basis of $L$. It is finite, since $L$ is finitely generated over $K$. By [Zi1, 4.1], there exists $(\mathbf{D''}_1,\dots,\mathbf{D''}_{e'})$, a Hasse structure on $L$ having $B_1$ as its canonical $p$-basis which implies that

$$(K,\mathbf{D}_1,\dots,\mathbf{D}_e),(L,\mathbf{D}'_{1,<p},\dots,\mathbf{D}'_{e,<p})\subset(L,\mathbf{D''}_1,\dots,\mathbf{D''}_e).$$

Now we want to find a new $p$-basis on $L$ which will be a canonical $p$-basis of depth 2 for

$$\mathbf{L}:=(L,\mathbf{D}'_{1,\leqslant p},\dots,\mathbf{D}'_{e,\leqslant p},\mathbf{D}''_{e+1,<p},\dots,\mathbf{D}''_{e',<p}).$$

Let $a\in B_0$. It satisfies the assumption of 2.1 for $\mathbf{L}$. Let $B_{00}=\{a':a\in B\}$, where each $a'$ is as in the conclusion of 2.1 . Then $B_2:=B\cup B_{00}$ is a canonical $p$-basis of depth 2 for $\mathbf{L}$. We use this basis to get $(\mathbf{D'''}_1,\dots,\mathbf{D'''}_{e'})$, a Hasse structure on $L$ such that

$$(K,\mathbf{D}_1,\dots,\mathbf{D}_e),(L,\mathbf{D}'_{1,<p^2},\dots,\mathbf{D}'_{e,<p^2})\subset(L,\mathbf{D'''}_1,\dots,\mathbf{D'''}_e).$$

We continue doing the last step of the proof until we get $\mathbf{D}_1^{(n+1)},\dots,\mathbf{D}_{e'}^{(n+1)}$. $(\bar{\mathbf{D}}_1,\dots,\bar{\mathbf{D}}_e):=(\mathbf{D}_1^{(n+1)},\dots,\mathbf{D}_e^{(n+1)})$ works. $\square$

Recall that every strict Hasse field ($e$ Hasse derivations) of degree of imperfection $e$ has a canonical $p$-basis of depth $k$ for any natural number $k$ [Zi2]. Actually, the proof from [Zi2] gives also a version in the truncated case:

**Theorem 2.4** *Let $(K,\mathbf{D}_{1,<p^n},\dots,\mathbf{D}_{e,<p^n})$ be a strict $n$-truncated Hasse field. If $e$ is the inseparability degree of $K$, then $(K,\mathbf{D}_{1,<p^n},\dots,\mathbf{D}_{e,<p^n})$ has a canonical $p$-basis of depth $n$.*

We are interested in extensions of $n$-truncated Hasse fields.

**Fact 2.5** *Any $n$-truncated Hasse field has a strict extension.*

**Proof** Let $(K, \mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n})$ be an $n$-truncated Hasse field and $C$ denote its field of constants. Obviously, $K^p \subset C$. By the usual chain procedure, it is enough to prove the following claim.

**Claim**

If there is an $a \in C \setminus K^p$, then there exists an $n$-truncated Hasse field extension $K \subset L$ such that $a \in L^p$ and $L^p \subset K$.

**Proof of Claim** In this proof we exercise the general tactic of this paper, i.e. first we prove the statement in the case of one (truncated) Hasse derivation, then we point out how the proof extends to the general case.

We work with the ring $K\{X\}_n$ of $n$-truncated Hasse polynomials (in one variable). In this case $K\{X\}_n = K[X, X', \ldots, X^{(p^n-1)}]$. Let $I$ be the ideal of $K\{X\}_n$ generated by the set

$$\{X^{(p^{n-1}+k)}, (X^{(i)} - D_{pi}(a)^{1/p})^{p_i} \mid 0 \leqslant k < (p-1)p^{n-1}, 0 \leqslant i < p^{n-1}\},$$

where

$$p_i = \begin{cases} 1 & \text{if } D_{pi}(a) \in K^p \\ p & \text{if } D_{pi}(a) \notin K^p \end{cases}.$$

It is enough to show that $I$ is a Hasse ideal, since it is clearly maximal and we can set $L := K\{X\}_n/I$ (note that $a^{1/p} \in L$ and $L$ is generated over $K$ by elements from $K^{1/p}$, so $L^p \subseteq K$).

We will use formula (2.2) from the proof of [Zi1, 2.4] several times. It says that for $i < p^{n-1}$ and $a \in K$, $D_{ip}(a^p) = D_i(a)^p$. Actually, it is formulated for Hasse fields but is true in the truncated case as well.

We need to apply the Hasse derivation to the generators only. Let

$$0 \leqslant i < p^{n-1},\ \ 0 \leqslant j < n-1,\ \ 0 < l < p^n-1,\ \ 0 \leqslant k < (p-1)p^{n-1},\ \ 0 < m < p^n.$$

We have:

$$D_l(X^{(p^{n-1}+k)}) = c_{l,p^{n-1}+k} X^{(l+p^{n-1}+k)}.$$

In the case $p_i = p$, we have

$$D_1((X^{(i)})^p - D_{pi}(a)) = -D_{pi} D_1(a) = 0,$$

$$D_{p^{j+1}}((X^{(i)})^p - D_{pi}(a)) = D_{p^j}(X^{(i)})^p - D_{p^{j+1}} D_{pi}(a) = c^p_{p^j,i} X^{(p^j+i)p} - c_{p^{j+1},pi} D_{p(p^j+i)}(a).$$

Now we want to check that if $p^j + i \geqslant p^{n-1}$, then $c_{p^j,i} = c_{p^{j+1},pi} = 0$, and that $l + p^{n-1} + k \geqslant p^n$ implies $c_{l,p^{n-1}+k} = 0$. But this clearly follows from the fact

that $\alpha, \beta < p^\gamma$, $\alpha + \beta \geqslant p^\gamma$ implies $c_{\alpha,\beta} = 0$.
To finish this case we need to notice that for all $k, l$:

$$k^p \equiv k \pmod{p}, \quad c_{k,l} \equiv c_{pk,pl} \pmod{p}.$$

In the case $p_i = 1$ we compute

$$D_m(X^{(i)} - D_{pi}(a)^{1/p})^p = D_m(X^{(i)})^p - D_m(D_{pi}(a)^{1/p})^p$$
$$= c_{i,m}^p(X^{(i+m)})^p - D_{pm}D_{pi}(a) = c_{i,m}^p(X^{(i+m)})^p - c_{pm,pi}D_{p(i+m)}(a)$$
$$= c_{i,m}[(X^{(i+m)})^p - D_{p(i+m)}(a)],$$

since

$$c_{i,m}^p = c_{i,m} = c_{pi,pm} = c_{pm,pi}.$$

Hence if $c_{i,m} = 0$ (e.g. if $i + m \geqslant p^n$), we get

$$D_m(X^{(i)} - D_{pi}(a)^{1/p})^p = 0 = D_m(X^{(i)} - D_{pi}(a)^{1/p}),$$

and if $c_{i,m} \neq 0$, then

$$D_{p(i+m)}(a) = (1/c_{i,m})D_m(D_{pi}(a)^{1/p})^p \in K^p.$$

Therefore $p_{i+m} = 1$ and (since the Frobenius map is injective)

$$D_m(X^{(i)} - D_{pi}(a)^{1/p}) = c_{i,m}(X^{(i+m)} - D_{p(i+m)}(a)^{1/p}) \in I. \quad \square$$

In the case of several $n$-truncated Hasse derivations the proof goes through using 1.1(i). $\square$

Note that the field extension we have constructed in the previous fact is purely inseparable and smallest, but an extension of the Hasse derivation may be defined in many ways (it becomes evident in the proof of the next lemma). In the case of a Hasse field $D_{p^{n-1}}(a^{1/p}) = D_{p^n}(a)^{1/p}$ for each $n$, which gives the smallest strict extension of a Hasse field as in [Zi1, 2.4].

**Lemma 2.6** *The class of $n$-truncated Hasse fields does not have the amalgamation property.*

**Proof** Note that in the ideal $I$ (in the proof of 2.5), $X^{(p^{n-1})}$ could have been replaced by $X^{(p^{n-1})} - b$, for any absolute constant $b$. Therefore the value of $D_{p^{n-1}}(a^{1/p})$ can be any absolute constant and we easily get two extensions which can not be amalgamated. $\square$

**Lemma 2.7** *Each strict $n$-truncated Hasse field $(K, \mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n})$ embeds into a strict $n$-truncated Hasse field of inseparability degree $e$.*

**Proof** Let $e'$ be the inseparability degree of $K$. By [Zi1, 2.1], $e' \leqslant e$. Let $e = e' + f$.
By [Zi1, 2.1] and [Zi2, 2], we can assume that there is $\{b_1, \ldots, b_{e'}\} \subset K$, a canonical $p$-basis of depth 1 of $(K, \mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e',<p^n})$.
If $f = 0$, there is nothing to do. Let $f > 0$ and consider the field of rational functions $L = K(X_1, \ldots, X_f)$. By formula (2.1) in the proof of [Ok, 2.3], there are $\mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n}$, Hasse derivations on $L$ extending $\mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n}$ respectively, such that

$$D'_{i,1}(X_j) = \begin{cases} 1 & \text{if } i = j + e' \\ 0 & \text{otherwise.} \end{cases}$$

We want to check that $(L, \mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n})$ is an $n$-truncated Hasse field. We have already noticed in Section 1 that we only need to check the iterativity and commutativity conditions on a set of ring generators of $L$. By the uniqueness clause in [Ok, 2.3] (or formula (2.1) there), we need to check only field generators.
Let $1 \leqslant i, i' \leqslant e$, $1 \leqslant j, j' < p^n$ and $1 \leqslant k \leqslant f$. We have

$$D_{i,j} D_{i',j'}(X_k) = 0 = D_{i',j'} D_{i,j}(X_k),$$

$$D_{i,j} D_{i,j'}(X_k) = 0 = \begin{cases} 0 & \text{if } j + j' \geqslant p^n \\ D_{i,j+j'}(X_k) & \text{if } j + j' < p^n \end{cases}.$$

Therefore $(L, \mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n})$ is an $n$-truncated Hasse field.

Note that $B := \{b_1, \ldots, b_{e'}, X_1, \ldots, X_f\}$ is a $p$-basis of $L$ (in particular $L$ has inseparability degree $e$).
Take $a \in L \setminus L^p$. We need to show that $a$ is not a constant.
**Case 1**: $a \in L^p(X_1, \ldots, X_f)$
By 2.2, constants of $(L^p(X_1, \ldots, X_f), \mathbf{D}'_{e'+1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n})$ coincide with $L^p$, so there is $i \leqslant f$ such that $D'_{e'+i,1}(a) \neq 0$.
**Case 2**: $a \notin L^p(X_1, \ldots, X_f)$
By 2.2, $L^p(X_1, \ldots, X_f)$ coincides with the constants of $(L, \mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e',<p^n})$. Therefore $a$ is not a constant. □

**Lemma 2.8** *Any $n$-truncated Hasse field structure extends uniquely to the separable closure.*

**Proof** We do it first for one $n$-truncated Hasse derivation (not necessarily iterative) $\mathbf{D}$ on a field $K$. For any $i < n$, $a \in K$ and $f \in K[X]$ we have:

$$D_{p^i}(f(a)) = f^{D_{p^i}}(a) + f'(a)D_{p^i}(a) + w,$$

where in $f^{D_{p^i}}$ each coefficient from $f$ is hit by $D_{p^i}$ and in $w$ only $\mathbf{D}_{<p^i}$ appears. This implies the uniqueness of the extension to the separable closure. The existence can be proved in the same way as in 2.5, after unravelling the formula for $w$.
Commutativity and iterativity assertions follow from the uniqueness as in the proof of [Zi1, 2.3]. $\square$

One can also see that the proof of the Hasse field version of 2.8 (see [Ok, 2.5]) works also in the truncated case.

**Fact 2.9** *Strict $n$-truncated Hasse fields have the amalgamation property.*

**Proof** As in [Zi1, 2.6], the amalgamation property follows from the uniqueness of extensions of $n$-truncated Hasse derivations to the separable closure (2.8) and the fact that an $n$-truncated Hasse field extension of a strict $n$-truncated Hasse field is separable [Zi1, 2.1]. $\square$

**Theorem 2.10** *Let $\mathbf{K}_n = (K, \mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n})$ be an $n$-truncated Hasse field. The following are equivalent.*
*(i) $\mathbf{K}_n$ is existentially closed*
*(ii) $\mathbf{K}_n$ is separably closed, strict and of inseparability degree $e$.*
*(iii) $\mathbf{K}_n$ is a reduct of an existentially closed Hasse field.*
*Therefore the theory of $n$-truncated Hasse fields has a model companion.*

**Proof**
(i)$\longrightarrow$(ii) Suppose that $\mathbf{K}_n$ is existentially closed. By 2.5, $\mathbf{K}_n$ is strict. By [Zi1, 2.1] any $n$-truncated Hasse field extension of $\mathbf{K}_n$ has the inseparability degree bounded by $e$. By 2.2, having inseparability degree $e$ is an existential condition, so $\mathbf{K}_n$ has the inseparability degree $e$ by 2.7. By 2.8, $\mathbf{K}_n$ is separably closed.
(ii)$\longrightarrow$(iii) Assume $\mathbf{K}_n$ is separably closed, strict and of inseparability degree $e$. By 2.4, $\mathbf{K}_n$ has a canonical $p$-basis of depth $n$. We use that $p$-basis to expand $\mathbf{K}_n$ to a Hasse field as in [Zi1, 4.1]. By [Zi1, 1.1], the Hasse field we have obtained is existentially closed.
(iii)$\longrightarrow$(i) Suppose $\mathbf{K}_n$ is a reduct of $\mathbf{K} = (K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$, an existentially

closed Hasse field. Take $\mathbf{L}_n$, an $n$-truncated Hasse field extension of $\mathbf{K}_n$. It is enough to embed $\mathbf{L}_n$ into $\mathbf{M}_n$, the $n$-truncated reduct of a Hasse field $\mathbf{M}$ extending $\mathbf{K}$. We can assume that $\mathbf{L}_n$ is finitely generated (as a field) over $K$.
Take $\mathbf{K}'$, an $\omega$-saturated elementary extension of $\mathbf{K}$. By 2.9, $\mathbf{K}'_n$ and $\mathbf{L}_n$ can be amalgamated over $\mathbf{K}_n$ into an $n$-truncated Hasse field $\mathbf{M}_n$. By [Zi1, 4.2], $\mathbf{K}'$ has a canonical $p$-basis. Since $\mathbf{L}_n$ is finitely generated over $K$, we can assume that $\mathbf{M}_n$ is finitely generated over $\mathbf{K}'$. By 2.3, $\mathbf{M}_n$ can be expanded to a Hasse field $\mathbf{M}$ extending $\mathbf{K}'$ (so extending $\mathbf{K}$, too) . $\square$

Let $\mathrm{SCH}_{p,e,n}$ denote the theory of existentially closed $n$-truncated Hasse fields, and $L_{n,e}$ denote the language of $n$-truncated Hasse fields. By 2.10, the axioms of $\mathrm{SCH}_{p,e,n}$ are exactly the axioms of $\mathrm{SCH}_{p,e}$ restricted to $L_{n,e}$. As in [Zi1], $\mathrm{SCH}_{p,e,n}$ is stable not superstable, since after choosing a canonical $p$-basis any model of $\mathrm{SCH}_{p,e,n}$ is biinterpretable with the underlying separably closed field.
Note that $\mathrm{SCH}_{p,1,1}$ coincides with 1-DCF, the Wood theory from [Wo2], which is the theory of existentially closed differential fields with nilpotent derivations of order $p$. Wood considers also theories $n$-DCF, where derivations have order $p^n$. Higher order (non-iterative) analogues of $n$-DCF were developed and studied in [MW].

**Fact 2.11** $\mathrm{SCH}_{p,e,n}$ *does not have quantifier elimination in* $L_{n,e}$ *and has quantifier elimination in* $L_{n,e} \cup \{p\text{-th root function}\}$.

**Proof** The first part follows from 2.6 .
For the second part, it is enough to see that strictness is a universal property with the $p$-th root function, so quantifier elimination follows from 2.9 . $\square$

Note that results of this section would be trivial if it was possible to expand any truncated Hasse field to a Hasse field. This is not the case though, see Example 2.2.3 in [Tr].

# 3 Geometric axioms for existentially closed Hasse fields (one derivation)

The results of this section form a proper subset of the results of the next section. However, we have the feeling that in the case of several Hasse derivations the presentation is somewhat blurred by a multiple usage of multi-indices. Therefore, we prefer to start with the case of a single Hasse derivation, and then just to point out how to generalize the results to the case of several Hasse derivations.

Our axioms involve prolongation spaces, and we start by recalling the necessary definitions. In [Sc], the general framework for prolongation spaces with respect to an arbitrary operator is given. Let $(K, \mathbf{D})$ be a fixed Hasse field with a single Hasse derivation. We assume that all the fields we consider are embedded into a big (pure) algebraically closed field $\Omega$.

We identify algebraic varieties with $\Omega$-points of algebraic varieties definable over $\Omega$. For example $\mathbb{A}^m = \mathbb{A}^m(\Omega) = \Omega^m$ is an $m$-affine space.

Let $V$ be an affine variety definable over $K$ and $I_K(V) \subset K[\bar{X}]$ be the zero-ideal of $V$ over $K$. We consider now an $n$-truncated Hasse derivation on $K\{\bar{X}\}_n$ as a sequence of maps $(D_i : K[\bar{X}] \to K\{\bar{X}\}_n)_{i<p^n}$.

**Definition 3.1** *The $n$-th prolongation of $V$, denoted by $\nabla_n(V)$, is an affine variety given by the ideal $(\mathbf{D}_{<p^n}(I_K(V))) \subset K\{\bar{X}\}_n$.*

The above definition differs from the general definition from [Sc] just by twisting the indices. Usually, $\nabla_n(V)$ denotes the zeros of $\mathbf{D}_{<n}(I_K(V))$. Since we are going to consider only the prolongation spaces corresponding to ideals of the form $\mathbf{D}_{<p^n}(I_K(V))$, we adopt the more convenient notation.

**Fact 3.2** *Let $(K, \mathbf{D}) \subset (L, \mathbf{D}')$ be a Hasse field extension, $V$ be an affine variety definable over $K$ and $a \in V(L)$. Then for each $n$, $\mathbf{D}'_{<p^n}(a) \in \nabla_n(V)$.*

**Proof** It is enough to see that for each $f \in K[X], d \in L$, we have

$$\mathbf{D}'_{<p^n}(f(d)) = \mathbf{D}_{<p^n}(f)(\mathbf{D}'_{<p^n}(d)). \quad \square$$

For a variety $V$ *defined* over $K$ (i.e. $I_K(V)$ generates $I_\Omega(V)$), $\nabla_n V(K)$ naturally corresponds to $V(K[X]/(X^{p^n}))$, where the $K$-algebra structure on $K[X]/(X^{p^n})$ is given by a homomorphism

$$K \ni a \mapsto \sum_{i<p^n} D_i(a)X^i \in K[X]/(X^{p^n}).$$

Therefore, on the level of coordinate rings, $\nabla_n$ is a left-adjoint functor to the functor taking a $K$-algebra $R$ to $R[X]/(X^{p^n})$, where the $K$-algebra structure on $R[X]/(X^{p^n})$ comes from the above homomorphism composed with the natural embedding $K[X]/(X^{p^n}) \subset R[X]/(X^{p^n})$.

The prolongation spaces are closely related to the arc spaces (see [DL]), which correspond to $V(K[X]/(X^{p^n}))$, where $K[X]/(X^{p^n})$ has the usual $K$-algebra structure (corresponding to the 0-Hasse derivation). In particular, $\nabla_1(V)$ projects onto a torsor of the tangent space to $V$, and $\nabla_1(V)$ projects onto the tangent space to $V$ if $V$ is defined over the constants (in the standard notation, for $V$ defined over the constants, $\nabla_1(V)$ is exactly the tangent space to $V$).

Let us fix $b = (b_0, ..., b_{p^n-1}) \in \Omega^{mp^n}$ and set $V = \operatorname{locus}_K(b_0), W = \operatorname{locus}_K(b)$. Recall that for a tuple $c \in \Omega^l$, $\operatorname{locus}_K(c)$ is an affine algebraic subvariety (not necessarily irreducible) of $\Omega^l$ defined by $I_K(c) := \{f \in K[\bar{X}] \mid f(c) = 0\}$.
We are interested in finding geometric conditions (i.e. in terms of $V$ and $W$) for extending $(K, \mathbf{D}_{<p^n})$ to an $n$-truncated Hasse field $(K(b), \mathbf{D}'_{<p^n})$ such that $\mathbf{D}'_{<p^n}(b_0) = b$.
For the proof of the next lemma we need the notion of an $n$-truncated Hasse derivation of a homomorphism of rings. It is quite natural, for a ring homomorphism $f : R \to S$, a sequence of maps $(D_i : R \to S)_{i<p^n}$ is an *$n$-truncated Hasse derivation of $f$*, if $D_0 = f$ and the following map

$$R \ni a \mapsto \sum_{i=0}^{p^n-1} D_i(a)X^i \in S[X]/(X^{p^n})$$

is a ring homomorphism. As an example we have $f : K[\bar{X}] \to K\{\bar{X}\}_n$ the natural embedding and $\mathbf{D}_{<p^n}$ being the restrictions of the $n$-truncated Hasse derivations on $K\{\bar{X}\}_n$. Note that if $R \neq S$, then it does not make much sense to talk about iterative $n$-truncated Hasse derivation of $f$.

**Lemma 3.3** *If $b \subset K(b_0)$ and $b \in \nabla_n(V)$, then there is $\mathbf{D}'_{<p^n}$, an $n$-truncated Hasse derivation (not necessarily iterative) on $K(b_0)$ such that $\mathbf{D}'_{<p^n}$ extends $\mathbf{D}_{<p^n}$ and $\mathbf{D}'_{<p^n}(b_0) = b$.*

**Proof** It is clear that

$$b \in \nabla_n(V) \quad \text{if and only if} \quad \mathbf{D}_{<p^n}(I_K(b_0))(b) = 0.$$

Therefore, the natural $n$-truncated Hasse derivation of the ring embedding $K[\bar{X}] \subset K\{\bar{X}\}_n$ factors through

$$K[b_0] = K[\bar{X}]/I_K(b_0) \to K\{\bar{X}\}_n/I_K(b) = K[b]$$

if and only if $b \in \nabla_n(V)$. Hence we get an $n$-truncated Hasse derivation of the ring embedding $K[b_0] \subset K[b]$.
By the formula (2.1) in the proof of [Ok, 2.3], any $n$-truncated Hasse derivation of an embedding of domains extends to an $n$-truncated Hasse derivation of the corresponding embedding of their fields of fractions. $\square$

The next fact is crucial for giving geometric axioms for $\mathrm{SCH}_{p,e}$. It is similar to [K, 1.7] which was motivated by [PP]. However there is an additional condition in [K, 1.7] (corresponding to the separant condition in [Wo1]) which is not present in [PP] because of the triviality of separability issues in the characteristic 0 case. A priori, our case here is even more complicated, since we also have to care about the iterativity condition. However, it turns out that the geometric iterativity condition already implies the additional assumption from [K, 1.7] and even the condition $W \subset \nabla_n(V)$.

**Fact 3.4** *The following are equivalent:*
*(i) There is an $n$-truncated Hasse field extension $(K, \mathbf{D}_{<p^n}) \subset (K(b), \mathbf{D}'_{<p^n})$ such that $\mathbf{D}'_{<p^n}(b_0) = b$.*
*(ii) There is an $n$-truncated Hasse field extension $(K, \mathbf{D}_{<p^n}) \subset (L, \mathbf{D}'_{<p^n})$ such that $b \subset L$ and $\mathbf{D}'_{<p^n}(b_0) = b$.*
*(iii) $c_n(W) \subset \nabla_n(W)$.*

**Proof**
(i)$\longrightarrow$(ii) Obvious.
(ii)$\longrightarrow$(iii) By 1.1(ii),

$$c_n(b) = c_n\mathbf{D}'_{<p^n}(b_0) = \mathbf{D}'_{<p^n}\mathbf{D}'_{<p^n}(b_0) = \mathbf{D}'_{<p^n}(b).$$

Therefore $c_n(W) \subset \nabla_n(W)$ (by 3.2).
iii) $\longrightarrow$ i) Since $c_n(b) \subset K(b)$, we get by 3.3 (with $b$ playing the role of $b_0$) an $n$-truncated Hasse derivation on $K(b)$ such that $\mathbf{D}'_{<p^n}(b) = c_n(b)$. In particular $\mathbf{D}'_{<p^n}(b_0) = b$.
$\mathbf{D}'_{<p^n}$ is iterative on $b_0$, therefore (by 1.1(iii)) $\mathbf{D}'_{<p^n}$ is iterative on $K(b)$. $\square$

We can give now the geometric axioms for existentially closed fields with one Hasse derivation.

**Geometric axioms for $\mathbf{SCH}_{p,1}$**

For any natural numbers $n, m$, suppose that $V \subset \mathbb{A}^m$, $W \subset \mathbb{A}^{mp^n}$ are $K$-irreducible $K$-varieties, and $T$ is a proper $K$-subvariety of $W$. If $W$ projects generically onto $V$, and $c_n(W) \subset \nabla_n W$, then there is an $a \in V(K)$ such that $\mathbf{D}_{<p^n}(a) \in W \setminus T$.

The proof of the next theorem is almost the same as the proof of [K, 2.1]. The only difference comes from the fact that we can not determine a Hasse field extension by finite data (which can be governed by a formula) as in the differential case. That could lead to serious troubles, but thanks to Theorem 2.10 the proof goes through.

**Theorem 3.5** $(K, \mathbf{D})$ *is an existentially closed Hasse field with one Hasse derivation if and only if* $(K, \mathbf{D})$ *is a model of the geometric axioms for* $\mathrm{SCH}_{p,1}$.

**Proof**

$\Longrightarrow$ Take $V$, $W$ and $T$ satisfying the axiom assumptions. Since $V$ and $W$ are $K$-irreducible $K$-varieties and $W$ projects generically on $V$, there are $b_0, b \subset \Omega$ such that $V = \mathrm{locus}_K(b_0)$, $W = \mathrm{locus}_K(b)$ and $b = (b_0, \ldots, b_{p^n-1})$.

By 3.4, there exists an $n$-truncated Hasse derivation $\mathbf{D}'_{<p^n}$ on $K(b)$ such that $\mathbf{D}'_{<p^n}(b_0) = b$. Obviously $b \in W \setminus T$. By 2.10, $(K, \mathbf{D}_{<p^n})$ is existentially closed as an $n$-truncated Hasse field, hence we can find $b'_0 \in V(K)$ such that $\mathbf{D}_{<p^n}(b'_0) \in W \setminus T$.

$\Longleftarrow$ We could check that models of our axioms are strict separably closed Hasse fields of inseparability degree 1, and quote [Zi1, 1.1]. Instead, we repeat an argument from [K, 2.1].

By axioms for Hasse fields, it is enough to show that for $x = (x_0, \ldots, x_{p^n-1})$ and $\zeta(x)$, any quantifier-free formula over $K$ in the language of fields, if $(M, \mathbf{D}') \models (\exists x_0)\zeta(\mathbf{D}'_{<p^n}(x_0))$ for some Hasse field $(M, \mathbf{D}')$ extending $(K, \mathbf{D})$, then $(K, \mathbf{D}) \models (\exists x_0)\zeta(\mathbf{D}_{<p^n}(x_0))$.

Take $b_0 \subset M$ such that $(M, \mathbf{D}') \models \zeta(\mathbf{D}'_{<p^n}(b_0))$ and let $V = \mathrm{locus}_K(b_0)$, $W = \mathrm{locus}_K(b)$ for $b = \mathbf{D}'_{<p^n}(b_0)$. We take $T$ such that the axiom assumptions concerning it are satisfied and $\zeta(x)$ is implied by "$x \in W \setminus T$".

By 3.4, the axioms assumptions hold. Hence we get $a_0 \in V(K)$ such that $\mathbf{D}_{<p^n}(a_0) \in W \setminus X$. Therefore $(K, \mathbf{D}) \models (\exists x_0)\zeta(\mathbf{D}_{<p^n}(x_0))$. $\square$

Coming back to our original motivation, the equation $D_1(X) = X$ corresponds to the following situation (assume for convenience that $p = 2$):

$$V = \mathbb{A}^1, \quad W = Z(X_1 - X_2) = \{(x, x) : x \in \mathbb{A}^1\} \subset \mathbb{A}^2 = \nabla_1(V),$$

$$c_1(W) = \{(x,x,x,0) : x \in \mathbb{A}^1\} \subset \mathbb{A}^4 = \nabla_1(\nabla_1(V)),$$
$$\nabla_1(W) = Z(X_1 - X_2, X_1' - X_2') = \{(x,x,x',x') : x,x' \in \mathbb{A}^1\}.$$

By 3.5, we can not find a generic point of $W$ in $W \cap \mathrm{graph}(D_1)$ (corresponding to a generic solution of $D_1 X = X$), since $c_1(W)$ is not a subset of $\nabla_1(W)$.

The axioms above can be used to find solutions of certain linear Hasse differential equations as in Section 6 of [PZ]. However such solutions can be also easily found just by using that the Hasse field in question is existentially closed and not caring about the particular axioms

It is easy to see that for a natural number $N$, the above geometric axioms for $n \leqslant N$ give us axioms for $\mathrm{SCH}_{p,1,N}$.

One could wonder whether the theory of non-iterative ($n$-truncated) Hasse fields is companionable. In my previous attempt, I was trying to find those axioms first, and then add the geometric iterativity condition. It did not work though. Of course, for $n = 1$ and $p = 2$, the non-iterative theory has a model companion which coincides with $\mathrm{DCF}_p$ [Wo1]. In the more general case of $e$ derivations (still $n = 1$ and $p = 2$), the non-iterative theory has a model companion which coincides with $\mathrm{DCF}_p^e$ (see [Pi, 2.1]).

**Question 1** *Is the non-iterative theory of (n-truncated) Hasse fields companionable?*

# 4 Geometric axioms for existentially closed Hasse fields (several derivations)

In this section we observe that after suitable generalizations, the results of the previous section hold for Hasse fields with several derivations. Almost everything is pretty straightforward, only forcing commutativity in the generalization of 3.4 causes some problems.

We fix $(K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$, a Hasse field. Let us denote $\{0, 1, \ldots, p^n - 1\}$ by $[p^n]$. For $\mathbf{i} = (i_1, \ldots, i_e) \in [p^n]^{\times e}$, we define

$$\mathbf{D}_{<p^n} := (D_{\mathbf{i}})_{\mathbf{i} \in [p^n]^{\times e}}$$

For an affine variety $V$ definable over $K$, and for $I \subset \{1, \ldots, e\}$ let

$$\nabla_{I,n}(V) := Z((D_{\mathbf{i}}(I_K(V)))_{\mathbf{i} \in [p^n]^I}).$$

For example if $I=\{1,2,e\}$, then in the definition of $\nabla_{I,n}$ only $\mathbf{D}_1,\mathbf{D}_2,\mathbf{D}_e$ matter. We also define

$$\nabla_n(V):=\nabla_{\{1,\dots e\},n}(V)\ ,\quad \nabla_{i,n}(V):=\nabla_{\{i\},n}(V).$$

For each $n$ there is a linear morphism

$$c_n:\mathbb{A}^{p^{ne}}\to\mathbb{A}^{p^{2ne}},\quad c_n((a_{\mathbf{i}})_{\mathbf{i}\in[p^n]^{\times e}})=(c_{\mathbf{i},\mathbf{j}}a_{\mathbf{i}+\mathbf{j}})_{\mathbf{i},\mathbf{j}\in[p^n]^{\times e}},$$

where $c_{\mathbf{i},\mathbf{j}}a_{\mathbf{i}+\mathbf{j}}$ makes sense in the same way as in Section 1.

Again we fix $b=(b_{\mathbf{i}})_{\mathbf{i}\in[p^n]^{\times e}}\in\Omega^{mp^{ne}}$ and set $V=\operatorname{locus}_K(b_0),W=\operatorname{locus}_K(b)$ ($b_0:=b_{(0,\dots,0)}$). We are interested in finding geometric conditions (i.e. in terms of $V$ and $W$) for extending $(K,\mathbf{D}_{1,<p^n},\dots,\mathbf{D}_{e,<p^n})$ to an $n$-truncated Hasse field $(K(b),\mathbf{D}'_{1,<p^n},\dots,\mathbf{D}'_{e,<p^n})$ such that $\mathbf{D}'_{<p^n}(b_0)=b$.

**Lemma 4.1** *If $b\subset K(b_0)$ and $b\in\nabla_n(V)$, then there are commuting $n$-truncated Hasse derivations (not necessarily iterative) $\mathbf{D}'_{1,<p^n},\dots,\mathbf{D}'_{e,<p^n}$ on $K(b_0)$ such that for each $i\leqslant e$, $\mathbf{D}'_{i,<p^n}$ extends $\mathbf{D}_{i,<p^n}$ and $\mathbf{D}'_{<p^n}(b_0)=b$.*

**Proof** For $I$, a subset of $\{1,\dots,e\}$ let

$$b_I:=(b_{\mathbf{i}})_{\mathbf{i}\in[p^n]^I}.$$

Let us fix for a moment $I\subset\{1,\dots,e\}$ and $i\in\{1,\dots,e\}\setminus I$.
Since $\nabla_{j,n}$ and $\nabla_{k,n}$ commute for any $j,k\leqslant e$, we get

$$\nabla_{i,n}(\nabla_{I,n}(V))=\nabla_{I\cup\{i\},n}(V).$$

Since $K[b_{I\cup\{i\}}]\subset K[b]\subset K(b_0)\subset K(b_I)$, we get (by 3.4) $\mathbf{D}^I_{i,<p^n}$, an $n$-truncated Hasse derivation on $K(b_0)$ extending $\mathbf{D}_{i,<p^n}$ and such that

$$\mathbf{D}^I_{i,<p^n}(b_I)=b_{I\cup\{i\}}.$$

In particular $\mathbf{D}^I_{i,<p^n}(b_0)=b_{\{i\}}$, hence all $n$-truncated Hasse derivations of the form $\mathbf{D}^I_{i,<p^n}$ (for fixed $i$ and any $I\subset\{1,\dots e\}\setminus\{i\}$) coincide. Let us denote this $n$-truncated Hasse derivation by $\mathbf{D}'_{i,<p^n}$. We get for each $i,j\leqslant e$,

$$\mathbf{D}'_{i,<p^n}\mathbf{D}'_{j,<p^n}(b_0)=\mathbf{D}'_{i,<p^n}(b_{\{j\}})=b_{\{i,j\}}=\mathbf{D}'_{j,<p^n}(b_{\{i\}})=\mathbf{D}'_{j,<p^n}\mathbf{D}'_{i,<p^n}(b_0).$$

Hence the $n$-truncated Hasse derivations $\mathbf{D}'_{i,<p^n},\mathbf{D}'_{j,<p^n}$ commute on the set $K\cup\{b_0\}$ which generates $K(b_0)$ as a field, therefore they commute everywhere. $\square$

Using 4.1, the next fact can be proved in the same way as 3.4.

**Fact 4.2** *The following are equivalent:*
*(i) There is an $n$-truncated Hasse field extension* $(K, \mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n}) \subset (K(b), \mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n})$ *such that* $\mathbf{D}'_{<p^n}(b_0) = b$.
*(ii) There is an $n$-truncated Hasse field extension* $(K, \mathbf{D}_{1,<p^n}, \ldots, \mathbf{D}_{e,<p^n}) \subset (L, \mathbf{D}'_{1,<p^n}, \ldots, \mathbf{D}'_{e,<p^n})$ *such that* $b \subset L$ *and* $\mathbf{D}'_{<p^n}(b_0) = b$.
*(iii)* $c_n(W) \subset \nabla_n(W)$.

The reader may become suspicious, because he or she may have a feeling that we did not add any geometric commutativity conditions. But such conditions are hidden in the definition of the map $c_n$. The matrix defining $c_n$ is symmetric, in particular we have:

$$c_{(i,0),(0,j)}a_{(i,0)+(0,j)} = c_{(j,0),(0,i)}a_{(j,0)+(0,i)}.$$

One could still doubt why do we get commutativity in Lemma 4.1, since no map $c_n$ was there. This is because the iterations between different $n$-truncated Hasse derivations are hidden in the definition of $\nabla_n$ and also different $\nabla_{i,n}$ and $\nabla_{j,n}$ commute.

We continue as in Section 3.

**Geometric axioms for $\mathbf{SCH}_{p,e}$**

> For any natural numbers $n, m$, suppose that $V \subset \mathbb{A}^m$, $W \subset \mathbb{A}^{mp^{ne}}$ are $K$-irreducible $K$-varieties, and $Z$ is a proper $K$-subvariety of $W$. If $W$ projects generically onto $V$, and $c_n(W) \subset \nabla_n W$, then there is an $a \in V(K)$ such that $\mathbf{D}_{<p^n}(a) \in W \setminus Z$.

The next theorem can be proved in the same way as Theorem 3.5.

**Theorem 4.3** $(K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ *is an existentially closed Hasse field if and only if* $(K, \mathbf{D}_1, \ldots, \mathbf{D}_e)$ *is a model of the geometric axioms for* $\mathrm{SCH}_{p,e}$.

Obviously, for $n \leqslant N$, the above axioms still axiomatize $\mathrm{SCH}_{p,e,N}$.

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