On the isotriviality of projective iterative δ-varieties

Piotr Kowalski* Wrocław University
Anand Pillay† University of Illinois

July 14, 2006

Abstract

We study algebraic varieties X over a universal iterative differential field (K, δ) (typically of positive characteristic), together with an extension of δ to an iterative derivation D of the structure sheaf of X. Our work is motivated by the conjecture that if X is projective then the pair (X, D) is isotrivial, namely isomorphic over K to a pair (Y, D_0) where Y is defined over the constants C of K and D_0 is the lifting to K of the trivial iterative derivation on Y_C. We prove that up to isomorphism there is at most one such D on X extending δ, thus answering the question when X is defined over C. Other special cases are also proved, including abelian varieties, and smooth curves.

1 Introduction

In this paper we attempt to generalize results of Buium (see [Bu]) on projective δ-varieties over differential fields of characteristic 0, to the positive characteristic case. In the characteristic 0-case, the ground field K is equipped with a derivation δ such that (K, δ) is differentially closed. A δ-structure on a variety X defined over K is an extension of δ to a derivation D of the structure sheaf of X. Giving X a δ-structure is equivalent to equipping X

*Supported by an NSF FRG postdoc and Polish KBN grant 2P03A 018 24
†Supported by NSF grants

with a regular section $s : X \to T_\delta(X)$ (defined over K) of a certain twisted version $T_\delta(X)$ of the tangent bundle of X. The pair (X, D) or (X, s) is called a δ-variety over K.

If X is defined over the field of constants C of K, then the structure sheaf of X over C can be equipped with the 0-derivation, which can be tensored with δ over K, to get a derivation D_0 of the structure sheaf of X. This corresponds to the 0-section of the tangent bundle of X. We call such a pair (X, D_0) a trivial δ-variety.

There is a natural notion of *morphism* of δ-varieties, and (X, D) is said to be isotrivial if it is isomorphic to a trivial δ-variety.

In [Bu], Buium proves that (in this characteristic 0 context), any δ-variety (X, D) over K such that X is projective, is isotrivial.

The work presented here is an attempt to generalize Buium's theorem to a suitable positive characteristic context. In characteristic 0 if we equip a function field $K = k(t)$ (say where k is algebraically closed) with the derivation d/dt then the field of constants is k. However in characteristic $p > 0$, the field of constants of K is K^p rather than k. The situation can be remedied by replacing the single derivation d/dt by a suitable sequence of maps (a Hasse derivation) whose common field of constants *will be* k.

So we will work with such generalized derivations.

Definition 1.1 *Let R be a ring. Then*
(i) A sequence $\delta = (\delta_n : n < \omega)$ of additive maps from R to itself is called a Hasse derivation if δ_0 is the identity, and for all $n > 0$ and $x, y \in R$,

$$\delta_n(xy) = \sum_{i+j=n} \delta_i(x)\delta_j(y).$$

(ii) We call the sequence δ an iterative Hasse derivation on R if in addition to (i) we have for all i, j

$$\delta_i \circ \delta_j = \binom{i+j}{i} \delta_{i+j}.$$

We will sometimes use the expression "iterative derivation" for "iterative Hasse derivation".

For an iterative derivation $\delta = (\delta_n : n < \omega)$ on a ring R, we denote by the same symbol the map

$$\delta : R \to R[[X]], \quad \delta(x) = \sum \delta_n(x) X^n.$$

Note that the first condition in the above definition says that $\delta : R \to R[[X]]$ is a ring homomorphism. Occasionally, we also denote by δ the composition of $\delta : R \to R[[X]]$ with the quotient map $R[[X]] \to R[X]/X^n$.

(K, δ) will usually denote a field K of characteristic $p > 0$ equipped with an iterative derivation δ. Ziegler [Zi] identified a complete first order theory $SCH_{p,1}$ (the theory of separably closed Hasse fields of characteristic p and Ershov invariant 1) whose models are appropriate to work over in our context. In fact it will usually be appropriate to take (K, δ) to be a "universal domain" namely a saturated model of $SCH_{p,1}$.

The field of (absolute) constants C of K consists of those $x \in K$ such that $\delta_i(x) = 0$ for all i, which coincides with the intersection of all the K^{p^n}.

In section 5 we introduce iterative δ-schemes over K. For now, an iterative δ-variety over K is a variety X over K together with an extension D of δ to an iterative derivation of the structure sheaf of X. If (X, D^1) and (Y, D^2) are such then we have the obvious notion of a morphism from (X, D_1) to (X, D_2): namely a morphism $f : X \to Y$ (defined over K) of varieties such that $f^*D_2 = D_1 f^*$ where f^* is the induced map from the structure sheaf of Y to the structure sheaf of X.

As in the characteristic zero case we have the notion of a trivial δ-variety over K. Isotrivial means again isomorphic over K to a trivial object. To have "enough" isomorphisms we really need here to assume that (K, δ) is a universal domain.

The notion of the set of $\sharp$-points of a δ-variety (X, D) will be important. If for example X is affine, and for the coordinate functions $x_1, .., x_n$, $(D(x_1), .., D(x_n))$ is given by a sequence $s_j(x_1, .., x_n)$ of polynomial functions, then $X^\sharp$ will be the set of points $a = (a_1, .., a_n)$ of $X(K)$ such that $\partial(a) = (s_j(a))_j$.

Our main result, proved in section 6, is:
$(*)$ if X is a projective iterative δ-variety over K, then X has at most one structure of a iterative δ-variety over K. Namely if D^1, D^2 are iterative δ-structures on X over K, then (X, D^1) and (X, D^2) are isomorphic.

It follows from $(*)$ that the analogue of Buium's theorem (isotriviality of projective δ-varieties) holds if we already know that X is defined over C.

A few words should be said about the proof of $(*)$. The idea is very simple but the technical difficulties around it may blur the picture. Our assumptions imply that the functor $\mathrm{Aut}(X)$ is representable, by a group scheme G say over K, whose connected component G^0 is an algebraic group. We will equip

G^0 with a canonical iterative δ-variety structure, and any $\sharp$-point of G^0 will give rise to an isomorphism of (X, D_1) with (X, D_2) (6.2). However defining this iterative δ-variety structure on G^0 will be somewhat indirect. In general $\mathrm{Aut}(X)$ is just a functor from the category of schemes to the category of sets, so we will be extending the definitions of and machinery around (iterative) δ-structures to the case of functors.

Probably our constructions also work in the context of an arbitrary operator as in [Sc] satisfying the condition 2.1(i) – existence of unique extensions to localizations (all the other properties from 2.1 are satisfied for an arbitrary operator). In particular Buium's p-derivations would be included.

We also prove (section 7) the full analogue of Buium's result in special cases, such as when X has ample canonical or anticanonical divisor. This proof does not use $(*)$.

We will also mention the work of Benoit [Be] which is very relevant to our main conjecture. Benoit proves that if the algebraic variety X (over a model (K, δ) of $SCH_{p,1}$) can be equipped with the structure of an iterative δ-variety over K, then K descends to K^{p^n} for all n. If X belongs to a family with a good "moduli space" one can conclude that X descends to C. By this means we can also, using $(*)$, conclude that the conjecture holds when for example X is an abelian variety.

2 More on iterative derivations

We have already defined the notion of an (iterative) Hasse derivation δ on a ring R. By a Hasse ideal of (R, δ) we mean an ideal of R which is closed under the δ_n's. Likewise a homomorphism between rings R, S equipped with Hasse derivations δ, δ' is a homomorphism which respects this differential structure.

Let us fix an iterative differential field (K, δ). By a δ-algebra over K, or a K-δ-algebra, we mean a K-algebra R together with an iterative derivation D on R such that for $x \in K$, $y \in R$ and $n > 0$,

$$D_n(xy) = \sum_{i+j=n} \delta_i(x) D_j(y).$$

Now suppose (R, δ) to be an iterative differential ring. By a δ-module over R, or an R-δ-module, we mean an R-module V together with a sequence

$D = (D_n : n < \omega)$ of endomorphisms of the abelian group V, such that D_0 is the identity,

$$D_n(rx) = \sum_{i+j=n} \delta_i(r) D_j(x)$$

(for $n > 0$, $r \in R$ and $x \in V$), as well as the iterativity property

$$D_i \circ D_j = \binom{i+j}{i} D_{i+j}.$$

We may sometimes speak for example of a K-δ-algebra R when we really mean (R, D).

We will need the following well-known facts about Hasse derivations, followed by some explanations and references.

Fact 2.1 *Assume R, R' are K-δ-algebras and V, V' are R-δ-modules. Then:*
(i) D uniquely extends to an iterative derivation D' on the quotient field Q of R (if the quotient field exists). If a subring of Q is a localization of R then it is closed under D'.
(ii) $R \otimes_K R'$ is a K-δ-algebra.
(iii) $V \oplus V'$, $V \otimes_R V'$, $\bigwedge^n V$, V^* are (naturally) R-δ-modules.
(iv) A direct limit of R-δ-modules is an R-δ-module.
(v) If I is a radical δ-ideal and R is noetherian, then the primary components of I are δ-ideals.

Proof (iii) Let us give the definition of the δ-structure on the dual space V^* of V. We leave the rest to the reader. Given a ring R and R-module V, V^* is the set of R-module maps from V to R, and V^* is naturally an R-module itself. To ease notation let us suppose (R, δ) to be an (iterative) differential ring, and (V, D) an (R, δ)-module. Define $D^* = (D^*_n : n < \omega)$ on V^* by the following formula:
for $\lambda \in V^*$ and $v \in V$,

$$\delta_n(\lambda(v)) = \sum_{i+j=n} (D^*_i \lambda)(D_j v)$$

It is then clear that (V^*, D^*) is an (R, δ)-module.
(v) This has the same proof as the characteristic 0 case: see Theorem 1.18 in [Ma]. □

Denote Spec(K) by A and Spec$(K[X]/X^n)$ by $A^{(n)}$. If a tensor product has no subscript it is taken over K and if a product of schemes has no subscript, it is taken over A. The same applies to isomorphisms.
For a K-algebra S, we will denote $S \otimes K[X]/X^n$ by $S^{(n)}$ and more generally for an A-scheme X, $X^{(n)}$ denotes $X \times A^{(n)}$. Let $S^\infty := \varprojlim S^{(n)}$ and $X^\infty := \varinjlim X^{(n)}$.
Note that $S^{(n)} \cong S[X]/X^n$ and $S^\infty \cong S[[X]]$, but usually $S^\infty \ncong S \otimes K[[X]]$ e.g. for $S = K[x]$ (x-transcendental), we have $\sum x^i X^i \in S^\infty \setminus SK[[X]]$.
For a K-algebra S, $S_\delta^{(n)}$ denotes $S[X]/X^n$ with the K-algebra structure given by $\delta : K \to K[X]/X^n$ (see remark after definition 1.1) composed with the natural inclusion $K[X]/X^n \subseteq S[X]/X^n$. As before, $S_\delta^\infty := \varprojlim S_\delta^{(n)}$.
Note that $S_\delta^{(n)}$ is isomorphic as a K-algebra to the tensor product $S \otimes K[X]/X^n$, where the K-algebra structure comes from the following composition

$$K \xrightarrow{\ \delta\ } K[X]/X^n \xrightarrow{\ \rho\ } S \otimes K[X]/X^n,$$

where ρ is the natural map $x \mapsto 1 \otimes x$.

Therefore, we can define $X_\delta^{(n)}$ for any scheme X, as the product $X \times A^{(n)}$, where the A-scheme structure comes from the following composition

$$X \times A^{(n)} \longrightarrow A^{(n)} \xrightarrow{\mathrm{Spec}(\delta)} A.$$

For $m < n$, we have quotient maps $K^{(n)} \to K^{(m)}$ which give natural transformations $X_\delta^{(m)} \to X_\delta^{(n)}$. We define X_δ^∞ as $\varinjlim X_\delta^{(n)}$.
Notice that $X_\delta^{(n)}$ is naturally an $A_\delta^{(n)}$-scheme and since for any scheme Y, $(X \times A^{(n)}) \times_{A^{(n)}} (Y \times A^{(n)})$ is isomorphic to $X \times Y \times A^{(n)}$ over $X \times Y$, we have an easy fact.

Fact 2.2 *There is an isomorphism over* $X \times Y$

$$X_\delta^{(n)} \times_{A_\delta^{(n)}} Y_\delta^{(n)} \cong (X \times Y)_\delta^{(n)}.$$

Note also that we cannot replace n by ∞ in the isomorphism above (e.g. it is not true that for K-algebras R, S, $R[[X]] \otimes_{K[[X]]} S[[X]]$ is isomorphic to $(R \otimes S)[[X]]$).

We need one natural transformation, which will encode the iterativity condition. For any ring L, consider the map

$$c : L[[X]] \to L[[X,Y]], \quad c(\sum a_i X^i) = \sum \binom{i+j}{i} a_{i+j} X^i Y^j.$$

Clearly c is a natural transformation and if L is a δ-algebra, then c is a natural transformation between functors $L \mapsto L_\delta^\infty$ and $L \mapsto (L_\delta^\infty)_\delta^\infty$. For each n there is a natural transformation as above

$$c_n : L[X]/X^n \to L[X,Y]/(X^n, Y^n)$$

such that $c = \varprojlim c_n$.
We also get natural transformations on the level of functors between categories of schemes, which we denote by the same letters:

$$c : (X_\delta^\infty)_\delta^\infty \to X_\delta^\infty, \quad c_n : (X_\delta^{(n)})_\delta^{(n)} \to X_\delta^{(n)}.$$

3 Extensions of iterative derivations

Given a ring L, we can consider a sequence of maps $D = (D_n : L \to L)$ as a map

$$D : L \to L[[X]], \quad D(x) = \sum D_i(x) X^i,$$

and we sometimes denote by the same symbol the map into $L[X]/X^n$

$$D : L \to L[X]/X^n, \quad D(x) = \sum_{i=0}^{n-1} D_i(x) X^i.$$

Note that $D = (D_i)$ is an iterative Hasse derivation on L if and only if $D : L \to L[[X]]$ is a ring homomorphism (see the remark after Definition 1.1) and the following diagram is commutative

$$\begin{tikzcd} L[[X,Y]] & L[[X]] \arrow[l, "D^\infty"'] \\ L[[X]] \arrow[u, "c"] & L \arrow[l, "D"] \arrow[u, "D"'] \end{tikzcd} ,$$

where D^∞ is the image of $D : L \to L[[X]]$ under the power series functor.
More generally, for a ring homomorphism $f : R \to S$, we denote by f_δ^∞ the corresponding map $f_\delta^\infty : R_\delta^\infty \to S_\delta^\infty$.
We will need an obvious characterization of δ-homomorphisms between δ-algebras.

Fact 3.1 *If (R, D_R) and (S, D_S) are δ-algebras, then an R-algebra homomorphism $f : R \to S$ is a δ-homomorphism if and only if the following diagram is commutative*

$$\begin{tikzcd} R^\infty_\delta \arrow[r, "f^\infty_\delta"] & S^\infty_\delta \\ R \arrow[u, "D_R"] \arrow[r, "f"'] & S \arrow[u, "D_S"'] \end{tikzcd} .$$

Assume R is a K-algebra. An iterative Hasse derivation D on R extends δ if and only if the K-algebra structure on $R[[X]]$ given by D coincides with R^∞_δ. Hence we obtain the following:

Fact 3.2 *Extensions of δ to R correspond to K-algebra homomorphisms*

$$D : R \to R^\infty_\delta, \quad D(x) = \sum D_i(x) X^i$$

such that the following diagram is commutative

$$\begin{tikzcd} (R^\infty_\delta)^\infty_\delta & R^\infty_\delta \arrow[l, "D^\infty_\delta"'] \\ R^\infty_\delta \arrow[u, "c"] & R \arrow[l, "D"] \arrow[u, "D"'] \end{tikzcd} .$$

So, we need to deal with algebras of the form $(R^\infty_\delta)^\infty_\delta$.

It is well-known (see e.g. [Tr, page 30]) that (not necessarily iterative) Hasse derivations considered as maps $D : L \to L[[X]]$ extend to ring automorphisms $e^D : L[[X]] \to L[[X]]$. One defines

$$e^D(\sum a_i X^i) = \sum_i (\sum_{n+m=i} D_n(a_m)) X^i.$$

Clearly e^D is the inverse limit of maps

$$e^D_m : L[[X]]/X^{m+1} \to L[[X]]/X^{m+1},$$

and it is also true that each e^D_m is an isomorphism, so e^D is an isomorphism as well.

If D is an iterative Hasse derivation on R extending δ, then e^D is a K-algebra isomorphism between $R[[X]]$ with and R_δ^∞.
Therefore $K_\delta^{(n)} \cong K[X]/X^n$ and $K_\delta^\infty \cong K[[X]]$. So, obviously $A_\delta^{(n)} \cong A^{(n)}$ and $A_\delta^\infty \cong A^\infty$.
From now on, for any ring L, we use the notation L^∞ for $L[[X]]$.
By a straightforward computation one can check that the iterative condition from 3.2 extends as in the following fact.

Fact 3.3 *Extensions of δ to R correspond to K-algebra homomorphisms*

$$D : R \to R_\delta^\infty, \quad D(x) = \sum D_i(x) X^i$$

such that the following diagram is commutative

$$\begin{tikzcd}
(R_\delta^\infty)_\delta^\infty & R_\delta^\infty \arrow[l, "D_\delta^\infty"'] \\
R_\delta^\infty \arrow[u, "c"] & R^\infty \arrow[l, "e^D"] \arrow[u, "e^D"'] .
\end{tikzcd}$$

Assume we are given an iterative Hasse derivation D on R extending δ. As noticed above we have an isomorphism

$$e^D : R^\infty \to R_\delta^\infty,$$

which induces an isomorphism

$$(e^D)_\delta^\infty : (R^\infty)_\delta^\infty \to (R_\delta^\infty)_\delta^\infty.$$

There is a Hasse derivation coming from the composition below

$$R^\infty \xrightarrow{e^D} R_\delta^\infty \xrightarrow{(D_0)_\delta^\infty} (R^\infty)_\delta^\infty,$$

where D_0 is the 0-derivation on R. We call this derivation (for lack of a better name) D_∞, since D^∞ already exists and is a map from R^∞ to $(R_\delta^\infty)^\infty$.
By an easy diagram chase we obtain.

Fact 3.4 *Extensions of δ to R correspond to K-algebra homomorphisms*

$$D : R \to R_\delta^\infty, \quad D(x) = \sum D_i(x) X^i$$

such that the following diagram is commutative

$$\begin{tikzcd}
(R_\delta^\infty)_\delta^\infty & (R^{(\infty)})_\delta^\infty \arrow[l, "(e^D)_\delta^\infty"'] \\
R_\delta^\infty \arrow[u, "c"] & R^\infty \arrow[l, "e^D"] \arrow[u, "D_\infty"'] .
\end{tikzcd}$$

Proof It is enough to look at the diagram

$$\begin{tikzcd}
 & (R^\infty)_\delta^\infty \arrow[dl, "(e^D)_\delta^\infty"'] \\
(R_\delta^\infty)_\delta^\infty & R_\delta^\infty \arrow[l, "D_\delta^\infty"] \arrow[u, "(D_0)_\delta^\infty"'] \\
R_\delta^\infty \arrow[u, "c"] & R^\infty \arrow[l, "e^D"'] \arrow[u, "e^D"'] .
\end{tikzcd}$$

Commutativity of the lower part is exactly 3.3. Commutativity of the upper part comes from applying the $B \mapsto B_\delta^\infty$ functor to the commutative diagram saying that e^D is an R-algebra isomorphism of R^∞ with R_δ^∞.
Since $D_\infty = (D_0)_\delta^\infty \circ e^D$, the fact follows. $\square$

Remark
We could have picked some other diagram (and there were more natural choices) instead of the one in 3.4. But we will need precisely the diagram above in the sequel.
It is also clear that extensions of δ as above are in 1-to-1 correspondence with isomorphisms $\phi : R_\delta^\infty \cong R^\infty$ such that a diagram similar the one above is commutative.

4 Buium's prolongations in the iterative case

We recall Buium's definition of prolongations. We start from a very general set-up.
In any category $\mathcal{C}$, for $X \in \mathcal{C}$ we denote by $\mathcal{C}_X$ the category of objects over X, i.e. arrows $Y \to X$. If we have 2 morphisms $p_1 : T' \to T_1$, $p_2 : T' \to T_2$ and an object $X \in \mathcal{C}_{T_2}$, then we can define a functor

$$\mathcal{G}_{X,p_1,p_2} : \mathcal{C}_{T_1} \to \mathbf{Set}, \quad \mathcal{G}_{X,p_1,p_2}(Y) = \mathrm{Hom}_{T_2}(Y \times_{T_1} T', X).$$

Since the association $X \mapsto \mathcal{G}_{X,p_1,p_2}$ is functorial, we obtain a functor

$$\{X \in \mathcal{C}_{T_2} \mid \mathcal{G}_{X,p_1,p_2} \text{ is representable}\} \to \mathcal{C}_{T_1}.$$

In our case we have $\mathcal{C} = \mathbf{Sch}$ (the category of schemes), $T_1 = T_2 = A$, $T' = A^{(n)}$, p_1 comes from the 0-derivation and p_2 comes from δ.
Note that a scheme Z represents the functor $\mathcal{G}_{X,p_1,p_2}$ if and only if there is a natural bijection

$$\mathrm{Hom}(Y_\delta^{(n)}, X) \leftrightarrow \mathrm{Hom}(Y, Z).$$

It is easy to see that the functor on K-algebras $R \mapsto R_\delta^{(n)}$ has a left-adjoint, which we call ∇_n which commutes with localizations. Therefore for any scheme X, the object representing the functor $\mathcal{G}_{X,p_1,p_2}$ exists and we call it $\nabla_n(X)$. Obviously, ∇_n is the right adjoint functor to $Y \mapsto Y_\delta^{(n)}$ and for a K-algebra R, we have

$$\mathrm{Spec}(\nabla_n(R)) \cong \nabla_n(\mathrm{Spec}(R)).$$

For $m < n$, we have quotient maps $K^{(n)} \to K^{(m)}$ which give an inverse system of functors $\nabla_n \to \nabla_m$. We define the *infinite* (or *total*) prolongation to be $\nabla := \varprojlim \nabla_n$.
Note that for schemes X, Y we get (since inverse limits commute with representable functors)

$$\nabla X(Y) = (\varprojlim \nabla_n X)(Y) \cong \varprojlim \nabla_n X(Y) = \varprojlim X(Y_n^\delta) \cong X(\varinjlim Y_n^\delta) \cong X(Y_\delta^\infty).$$

Hence ∇ represents the functor

$$\mathbf{Sch}_A \ni Y \mapsto X(Y_\delta^\infty) \in \mathbf{Sets}.$$

and is right-adjoint to the functor $Y \mapsto Y_\delta^\infty$.

5 Iterative δ-schemes

(K, δ) still denotes an iterative differential field.

An *iterative δ-scheme* (over K) is a scheme X over K together with an iterative derivation of its structure sheaf which extends δ. Namely, for each open $U \subseteq X$, there is an iterative Hasse derivation D_U on $\mathcal{O}_X(U)$ which extends δ and such that the restriction maps are iterative differential ring

homomorphisms.
The notion of a morphism f between iterative δ-schemes (X, D), (Y, D') is the obvious one – we require that f be a morphism of schemes such that $f^* : \mathcal{O}_Y \to \mathcal{O}_X$ respects the δ-structure. We call such a morphism a *δ-morphism*, and we obtain the category of iterative δ-schemes over K.
If we think about the category of schemes as an extension of the category of rings, where new objects arise by gluing rings along localizations, then the same analogy holds in the iterative Hasse case – we just need to replace rings with iterative Hasse ring extensions of (K, δ) and all the gluing maps have to be K-δ-isomorphisms.
A *δ-group* is a group object in the category of δ-schemes.

From now on we will assume that (K, δ) is a universal iterative differential field, in the sense discussed in section 1.

Fact 5.1 *The global section functor*

$$\Gamma : \textit{Affine iterative } \delta\textit{-schemes} \ \longrightarrow \ K\textit{-}\delta\textit{-algebras}$$

is an equivalence of categories.

Proof Take a K-δ-algebra R. Let $X = \mathrm{Spec}(R)$. We want to expand the Hasse derivation on $R = \mathcal{O}_X(X)$ to the entire structure sheaf $\mathcal{O}_X$. The affine scheme X has an open basis consisting of the localization schemes $X_f = \mathrm{Spec}(R_f)$ for $f \in R$. By 2.1(i), we can uniquely extend an iterative derivation on R to each R_f. Therefore we are in the situation from 7.3(iv) below and we can use the argument from the proof there. □

In other words, the fact above means that an iterative δ-scheme structure on an affine scheme $\mathrm{Spec}(R)$ is the same thing as an extension of δ to R. So, we can think of iterative δ-schemes structures on a scheme X as "extensions" of δ to X.
Let C be the field of absolute constants of (K, δ):

$$C = \{x \in K \mid \forall i \ \delta_i(x) = 0\}.$$

As in the characteristic 0 case, if X is a scheme defined over C, then there is a natural δ-structure on X. Namely, for an affine $X = \mathrm{Spec}(R)$, it is enough (by the fact above) to find a δ-K-algebra structure on R. But X is defined over C, so there is a C-algebra R_C such that $R = K \otimes_C R_C$. By 2.1(ii), we

can define an iterative derivation on R using 0 on R_C and δ on K. For an arbitrary scheme X over C it is enough to use an open affine cover of X by C-schemes.
We call iterative δ-schemes as above *trivial* and ones δ-isomorphic to them *isotrivial*. Clearly, $A = \operatorname{Spec}(K)$ with the iterative δ-scheme structure given by δ itself is trivial. We denote the morphism $\operatorname{Spec}(\delta) : A^\infty_\delta \to A$ just by δ.

We will need several equivalent conditions for the existence of an iterative δ-scheme structure on a given scheme. For the proof of one equivalence we need a purely category-theoretic (and probably well-known) lemma.

Lemma 5.2 *Let $\mathcal{C}$ be a category and $F, G : \mathcal{C} \to \mathcal{C}$ a pair of adjoint functors. Assume that $c : F^2 \to F$ is a natural transformation, and for a morphism $f : F(X) \to Y$, let $f^\dagger : X \to G(Y)$ denote the adjoint morphism. Then:*
(i) There is an "adjoint" natural transformation $G \to G^2$, which we still denote by c.
(ii) If there is a morphism $f : F(X) \to X$ such that the following diagram is commutative

$$\begin{array}{ccc} F^2(X) & \xrightarrow{F(f)} & F(X) \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle f} \\ F(X) & \xrightarrow[f]{} & X\ , \end{array}$$

then the following diagram is commutative

$$\begin{array}{ccc} G^2(X) & \xleftarrow{G(f^\dagger)} & G(X) \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle f^\dagger} \\ G(X) & \xleftarrow[f^\dagger]{} & X\ . \end{array}$$

Proof
(i) Note that there is no natural bijection between $\operatorname{Hom}(F^2(X), F(X))$ and $\operatorname{Hom}(G(X), G^2(X))$, so we really need to use the fact that c is a natural transformation. Let $X, Y \in \mathcal{C}$. We will define the "adjoint" of c using "Y-points" of $G(X)$, e.g. arrows $Y \to G(X)$.
We have a morphism $c : F^2(Y) \to F(Y)$, which gives a map

$$\operatorname{Hom}(F(Y), X) \to \operatorname{Hom}(F^2(Y), X).$$

By adjointness, we get a map

$$\mathrm{Hom}(Y, G(X)) \to \mathrm{Hom}(Y, G^2(X)).$$

By naturality of c and of the adjointness bijection, the above map is natural in X and Y. Hence, Yoneda's Lemma gives a natural transformation $c : G \to G^2$.

(ii) It is clearly enough to show that

$$(*) \qquad (f \circ c)^{\dagger\dagger} = c \circ f^{\dagger},$$

$$(**) \qquad (f \circ F(f))^{\dagger\dagger} = G(f^{\dagger}) \circ f^{\dagger}.$$

$(*)$ clearly follows from the way $c : G \to G^2$ was constructed.
For $(**)$ one needs to use two commutative diagrams

$$\begin{array}{ccccccc}
\mathrm{Hom}(FX, X) & \xrightarrow{\dagger} & \mathrm{Hom}(X, GX) & & \mathrm{Hom}(FX, X) & \xrightarrow{\dagger} & \mathrm{Hom}(X, GX) \\
{\scriptstyle F(f)^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} & & {\scriptstyle (f^{\dagger})_*}\big\downarrow & & \big\downarrow{\scriptstyle G(f^{\dagger})_*} \\
\mathrm{Hom}(F^2X, X) & \xrightarrow{\dagger} & \mathrm{Hom}(FX, GX), & & \mathrm{Hom}(FX, GX) & \xrightarrow{\dagger} & \mathrm{Hom}(X, G^2X)
\end{array}$$

plugging f in the upper-left corner of both of them. $\square$

We apply the above fact to the natural transformations c, c_n (between functors on schemes) defined at the end of Section 2. Hence we get natural transformations $c : \nabla \to \nabla \circ \nabla$ and $c_n : \nabla_n \to \nabla_n \circ \nabla_n$.
If X is an affine δ-scheme, then (by 3.2, 3.3 and 3.4) we have morphisms

$$D : X_\delta^\infty \to X, \quad e^D : X_\delta^\infty \cong X^\infty, \quad D_\infty : (X^\infty)_\delta^\infty \to X^\infty.$$

Note that if we have an open cover $X = \bigcup U_i$, then $X^\infty = \bigcup U_i^\infty$ and $X_\delta^\infty = \bigcup (U_i)_\delta^\infty$. Hence we can define the above morphisms for arbitrary δ-schemes using an open cover by affine sub-schemes. Similarly, we get a morphism $e_n^D : X_\delta^{(n)} \cong X^{(n)}$.

Fact 5.3 *Let X be a scheme over K. The following are equivalent*
(i) There is an iterative δ-scheme structure on X.

(ii) There is a morphism $D : X_\delta^\infty \to X$ such that the following diagram is commutative

$$\begin{array}{ccc}
(X_\delta^\infty)_\delta^\infty & \xrightarrow{D_\delta^\infty} & X_\delta^\infty \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle D} \\
X_\delta^\infty & \xrightarrow[D]{} & X\ .
\end{array}$$

(iii) There is a morphism $D : X_\delta^\infty \to X$ such that the following diagram is commutative

$$\begin{array}{ccc}
(X_\delta^\infty)_\delta^\infty & \xrightarrow{D_\delta^\infty} & (X^\infty)_\delta^\infty \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle D_\infty} \\
X_\delta^\infty & \xrightarrow[e^D]{} & X^\infty\ .
\end{array}$$

(iv) There is a map $s : X \to \nabla(X)$ such that the following diagram is commutative

$$\begin{array}{ccc}
\nabla(\nabla X) & \xleftarrow{\nabla(s)} & \nabla X \\
{\scriptstyle c}\uparrow & & \uparrow{\scriptstyle s} \\
\nabla X & \xleftarrow[s]{} & X\ .
\end{array}$$

Proof Let $X = \bigcup U_i$, where $U_i = \mathrm{Spec}(R_i)$.
(i)$\Rightarrow$(ii)
Then clearly each U_i has an iterative δ-scheme structure, so we have extensions of δ to R_i, call them D^i. By 2.1(i), all these derivations are compatible with respect to the gluing maps. Therefore, the corresponding K-algebra maps $D_i : R_i \to R_i^\infty$ are also compatible and satisfy the condition from (ii) Therefore, they give a global morphisms as in 2.
(ii)$\Leftrightarrow$(iii)
Follows from 3.4.
(ii)$\Leftrightarrow$(iv)
Follows from 5.2.
(ii)$\Rightarrow$(i)
By 3.2 and 2.1(i) we get compatible iterative δ-variety structures on each U_i, so we get an iterative δ-variety structure on X. $\square$

We define now the set of certain special points of an iterative δ-scheme.

Definition 5.4 *Let X be an iterative δ-scheme. Then*

$$X^\sharp := \{x \in X(K) \mid x : A \to X \text{ is a } \delta\text{-morphism } \} \ .$$

Note that $x \in X^\sharp$ if and only if $x^* : \mathcal{O}_{X,x} \to K$ is a δ-homomorphism.

We also need an analogue of the previous fact in the case of $\sharp$-points and morphisms. Note that for any scheme X (not necessarily a δ-scheme!), there is a map δ_X obtained as the following composition

$$X(K) \xrightarrow{\delta_*} X(K_\delta^\infty) \xrightarrow{\dagger} \nabla X(K),$$

where $\dagger$ is the adjoint map. This map does not come from a morphism between X and ∇X, since it can not be completed to a natural transformation. In the language of logic, δ_X is definable in the structure (K, δ) but not definable in the structure K.
Note however, that δ_X is natural in X meaning that for each morphism of schemes $f : X \to Y$ the following diagram is commutative

$$\begin{tikzcd}
\nabla X(K) \arrow[r, "\nabla(f)"] & \nabla Y(K) \\
X(K) \arrow[u, "\delta_X"] \arrow[r, "f"'] & Y(K) \arrow[u, "\delta_Y"']
\end{tikzcd}$$

(this is obvious, since δ induces a natural transformation between functors represented by A and A_δ^∞).
For an affine δ-variety X, the definition of δ_X is very simple – it is just $\delta = (\delta_n)$ applied coordinate-wise to $X(K)$.

Fact 5.5 *For an iterative δ-scheme X we have (where s as in 5.3(iv))*

$$X^\sharp = \{x \in X(K) \mid s(x) = \delta_X(x)\}.$$

Proof Take $x \in X(K)$. By adjointness we get a commutative diagram as in the proof of 5.2

$$\begin{tikzcd}
\mathrm{Hom}(X_\delta^\infty, X) \arrow[r, "\dagger"] \arrow[d, "(x_\delta^\infty)^*"'] & \mathrm{Hom}(X, \nabla X) \arrow[d, "x*"] \\
\mathrm{Hom}(A_\delta^\infty, X) \arrow[r, "\dagger"] & \mathrm{Hom}(A, \nabla X).
\end{tikzcd}$$

Plugging D in the upper-left corner we get

$$(D \circ x_\delta^\infty)^\dagger = [(x_\delta^\infty)^*(D)]^\dagger = x^*(D^\dagger) = x^*(s) = s(x).$$

By the equivalence of (i) and (ii) in 5.7, $x \in X^\sharp$ if and only if $x \circ \delta = D \circ x_\delta^\infty$. Since $(x \circ \delta)^\dagger = \delta_X(x)$ (see the definition of δ_X above), the result follows. □

The following is the essential use of the assumption that (K, δ) is a saturated model of $SCH_{p,1}$:

Fact 5.6 *If X is an iterative δ-variety (namely iterative δ-scheme over K whose underlying scheme is a K-variety), then $X^\sharp$ is Zariski dense in X.*

Proof We assume X to be irreducible and affine. Let $F = K(a)$ be the function field of X over K (where a is a generic point of X over K). The iterative derivation on the structure sheaf of X gives an iterative derivation D of F whose restriction to K is ∂. Let $D_n(a) = s_n(a)$ where s_n is a K-rational function. Then $s := (s_n)$ is as in 5.3(iv). By [Zi], (F, D) embeds over K in an elementary extension of (K, ∂). Hence for each n, there is $a_n \in X(K)$ such that $\delta_i(a_n) = s_i(a_n)$ for $i \leqslant n$. Note that (since X is affine) for $b \in X(K)$, $\delta_X(b) = (\delta_n(b))$. Let X and the s be defined over a countable subfield K_0 of K. As (K, ∂) is saturated, we may find $a_\infty \in X(K)$, generic over K_0 such that $\delta_X(a_\infty) = s(a_\infty)$. By 5.5, $a_\infty \in X^\sharp$, hence $X^\sharp$ is Zariski-dense in X. □

Fact 5.7 *Let X, Y be iterative δ-schemes with maps D_X and D_Y as in 5.3(ii) and s_X, s_Y as in 5.3(iv). Let $f : X \to Y$ be a morphism. Then, the following are equivalent*
(i) f is a δ-morphism.
(ii) The following diagram is commutative

$$\begin{tikzcd} X_\delta^\infty \arrow[r, "f_\delta^\infty"] \arrow[d, "D_X"'] & Y_\delta^\infty \arrow[d, "D_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} .$$

(iii) The following diagram is commutative

$$\begin{tikzcd} X_\delta^\infty \arrow[r, "f_\delta^\infty"] \arrow[d, "e^{D_X}"'] & Y_\delta^\infty \arrow[d, "e^{D_Y}"] \\ X^\infty \arrow[r, "f^\infty"] & Y^\infty \end{tikzcd}$$

(iv) The following diagram is commutative

$$
\begin{array}{ccc}
\nabla(X) & \xrightarrow{\nabla(f)} & \nabla(Y) \\
{\scriptstyle s_X}\uparrow & & \uparrow{\scriptstyle s_Y} \\
X & \xrightarrow[f]{} & Y\ .
\end{array}
$$

(v) $f(X^\sharp) \subseteq Y^\sharp$.

Proof
(i)⇔(ii) It is clear for affine X, Y by 3.1, therefore it follows for arbitrary X, Y.
(ii)⇔(iii)⇔(iv) Follows in a similar way as in the proof of 5.3.
(ii)⇒(v) Take $x \in X^\sharp$. It means that $x : A \to X$ is a δ-morphism, therefore we already know that x fits into a diagram as in (ii). Assuming (ii) for f we get a commutative diagram

$$
\begin{array}{ccccc}
A^\infty_\delta & \xrightarrow{x^\infty_\delta} & X^\infty_\delta & \xrightarrow{f^\infty_\delta} & Y^\infty_\delta \\
{\scriptstyle \delta}\downarrow & & {\scriptstyle D_X}\downarrow & & \downarrow{\scriptstyle D_Y} \\
A & \xrightarrow[x]{} & X & \xrightarrow[f]{} & Y\ .
\end{array}
$$

Since $(f \circ x)^\infty_\delta = f^\infty_\delta \circ x^\infty_\delta$, we get that $f \circ x$ is a δ-morphism, so $f(x) \in Y^\sharp$.
(v)⇒(iv) Take $x \in X^\sharp$. Since $f(x) \in Y^\sharp$ we get by 5.5 the following

$$s_Y(f(x)) = \delta_Y(f(x)) = \nabla f(\delta_X(x)) = \nabla f(s_X(x)).$$

The second equality is the fact that δ_X is a natural transformation with respect to X (see the remark before 5.5).
Therefore, the diagram in (iv) is commutative on $X^\sharp$. By 5.6 the diagram is commutative on the entire X. $\square$

6 The automorphism group functor and the first isotriviality theorem

We again begin in quite a general setting. Let $\mathcal{C}$ be a category and $f : X \to Y$ an arrow in $\mathcal{C}$. For $A \in \mathcal{C}$, let $\mathcal{C}_A$ denote the category of morphisms into A.

Then f induces a pull-back (or fibre-product) functor

$$f^* : \mathcal{C}_Y \to \mathcal{C}_X, \quad f^*(Z) = Z \times_Y X,$$

where for a morphism over Y, $\phi : Z \to Z'$,

$$f^*(\phi) : Z \times_Y X \to Z' \times_Y X, \quad f^*(\phi) := \phi \times_Y \mathrm{id}_X .$$

We can extend the group of automorphisms of X to the following contravariant functor

$$\mathrm{Aut}_X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Gps}, \quad \mathrm{Aut}_X(Y) = \mathrm{Aut}_Y(X \times Y)$$

where for a morphism $f : Y \to Z$ we define

$$\mathrm{Aut}_X(f) : \mathrm{Aut}_Z(X \times Z) \to \mathrm{Aut}_Y(X \times Y), \quad \mathrm{Aut}_X(f)(\phi) := f^*(\phi),$$

since $(X \times Z) \times_Z Y$ is canonically isomorphic to $X \times Y$.

In our case $\mathcal{C} = \mathbf{Sch}_A$ (we omit the subscript A in the sequel) and X is an iterative δ-scheme. Let D be as in 5.3(ii).
Let us denote the functor Aut_X by $\mathcal{G}$.
Note that for any contravariant functor (in particular $\mathcal{G}$)

$$\mathcal{F} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets},$$

we can define contravariant functors

$$\nabla_n(\mathcal{F}) : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}, \quad \nabla_n(\mathcal{F})(Y) := \mathcal{F}(Y_\delta^{(n)}).$$

$$\nabla(\mathcal{F}) : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Sets}, \quad \nabla(\mathcal{F})(Y) := \mathcal{F}(Y_\delta^{\infty}).$$

Our aim is to define a natural map

$$\mathcal{G} \to \nabla(\mathcal{G}),$$

which will give an iterative δ-structure on the group scheme G representing the functor $\mathcal{G}$ (if such a group scheme exists).
We will use the following sequence of isomorphisms. Note that these isomorphisms are both over X and $Y_\delta^{(n)}$ and are natural with respect to Y. They are also compatible with the maps induced from the direct system $(Y_\delta^{(n)})$.

$$(X \times Y)_\delta^{(n)} \cong X_\delta^{(n)} \times_{A_\delta^{(n)}} Y_\delta^{(n)} \quad \text{from 2.2}$$

$$e^D : X_\delta^{(n)} \times_{A_\delta^{(n)}} Y_\delta^{(n)} \cong X^{(n)} \times_{A_\delta^{(n)}} Y_\delta^{(n)} \qquad \text{from 5.3(iii)}$$
$$X^{(n)} \times_{A_\delta^{(n)}} Y_\delta^{(n)} \cong (X \times A_\delta^{(n)}) \times_{A_\delta^{(n)}} Y_\delta^{(n)} \qquad \text{since } A_\delta^{(n)} \cong A^{(n)}$$
$$(X \times A_\delta^{(n)}) \times_{A_\delta^{(n)}} Y_\delta^{(n)} \cong X \times Y_\delta^{(n)} \qquad \text{obvious}$$

(by e^D in the second isomorphism we mean obviously $e^D \times_{A_\delta^{(n)}} \mathrm{id}_{A_\delta^{(n)}}$). Note that the existence of an iterative δ-structure on X was used only to get the second isomorphism.
Therefore, we have obtained an isomorphism of schemes

$$D_Y^{(n)} : (X \times Y)_\delta^{(n)} \cong X \times Y_\delta^{(n)}.$$

Note that we can not pass to a direct limit, since product does not usually commute with direct limits.

Assume now that we have two iterative δ-scheme structures on X, and let us call them D and $\bar{D}$. We get an inverse system of maps

$$\Delta_Y^{(n)} : \mathcal{G}(Y) \to \mathcal{G}(Y_\delta^{(n)}) = \nabla_n(\mathcal{G})(Y), \quad \Delta_Y^{(n)}(\phi) := \bar{D}_Y^{(n)} \circ \phi_\delta^{(n)} \circ (D_Y^{(n)})^{-1}$$

and therefore a map

$$\Delta_Y = \varprojlim(\Delta_Y^{(n)}) : \mathcal{G}(Y) \to \varprojlim \nabla_n(\mathcal{G})(Y) \cong \nabla(\mathcal{G})(Y).$$

Since $D_Y^{(n)}$, $\bar{D}_Y^{(n)}$ are natural in Y, Δ_Y is also natural in Y, so we get a natural transformation

$$\Delta : \mathcal{G} \to \nabla(\mathcal{G}).$$

We want to check, if Δ satisfies the condition from 5.3(iv). Note that this is the only one of the equivalent conditions in 5.3 which makes sense on the level of functors. We could define in such a way, a notion of an iterative δ-structure on any contravariant functor from the category of schemes to the category of sets. However, we have no idea how we could use it, since the existence of $\sharp$-points (to be defined in this context later) is problematic, when a functor is not representable. Actually, we will see later that for some automorphism functors $\sharp$-points do not exist at all.

For our purposes, it will be enough to check 5.3(iv) on $\mathcal{G}(A)$ only, since later on $\mathcal{G}$ will be replaced by an irreducible K-variety G and the set $G(K)$ is Zariski dense in G.

Fact 6.1 *For each n, the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{G}(A) & \xrightarrow{\Delta_A^{(n)}} & \nabla_n\mathcal{G}(A) \\ {\scriptstyle \Delta_A^{(n)}}\big\downarrow & & \big\downarrow{\scriptstyle \nabla_n(\Delta_A^{(n)})} \\ \nabla_n\mathcal{G}(A) & \xrightarrow{c_n} & \nabla_n\nabla_n\mathcal{G}(A)\ . \end{array}$$

Therefore, we also get a diagram as in 5.3(iv).

Proof
We will omit the composition symbol on the level of "points" of $\mathcal{G}$. We start with the "clockwise" composition.

$$[\nabla_n(\Delta_A^{(n)}) \circ \Delta_A^{(n)}](\phi) = \Delta^{(n)}_{A_\delta^{(n)}}(\bar{D}_A^{(n)}\phi_\delta^{(n)}(D_A^{(n)})^{-1})$$

$$= \bar{D}^{(n)}_{A_\delta^{(n)}}(\bar{D}_A^{(n)}\phi_\delta^{(n)}(D_A^{(n)})^{-1})_\delta^{(n)}(D^{(n)}_{A_\delta^{(n)}})^{-1}$$

$$= [\bar{D}^{(n)}_{A_\delta^{(n)}}(\bar{D}_A^{(n)})_\delta^{(n)}][(\phi_\delta^{(n)})_\delta^{(n)}][D^{(n)}_{A_\delta^{(n)}}(D_A^{(n)})_\delta^{(n)}]^{-1}.$$

We will now identify domains and targets, which will be useful later.

$$\bar{D}^{(n)}_{A_\delta^{(n)}}(\bar{D}_A^{(n)})_\delta^{(n)} : (X_\delta^{(n)})_\delta^{(n)} \to X \times (A_\delta^{(n)})_\delta^{(n)} \cong (X^{(n)})^{(n)}$$

$$(\phi_\delta^{(n)})_\delta^{(n)} : (X_\delta^{(n)})_\delta^{(n)} \to (X_\delta^{(n)})_\delta^{(n)}$$

The third map just acts in the opposite direction as the first one.

Now, we deal with the "counterclockwise" composition. In the bottom arrow of the diagram from our statement, c_n is understood as a natural transformation $\nabla_n \to \nabla_n \circ \nabla_n$. It will be convenient for us to understand c_n rather as a map $A_\delta^{(n)} \to (A_\delta^{(n)})_\delta^{(n)}$. Then the former interpretation (i.e. the bottom arrow from the diagram) becomes $\mathcal{G}(c_n)$. The reader is advised to recall the connection between the pull-back functor c_n^* and the automorphism group functor.

$$[\mathcal{G}(c_n) \circ \Delta_A^{(n)}](\phi) = c_n^*(\bar{D}_A^{(n)}\phi_\delta^{(n)}(D_A^{(n)})^{-1})$$
$$= [c_n^*(\bar{D}_A^{(n)})][c_n^*(\phi_\delta^{(n)})][c_n^*(D_A^{(n)})]^{-1}.$$

Of course we want to show that

$$(*) \qquad c_n^*(\phi_\delta^{(n)}) = (\phi_\delta^{(n)})_\delta^{(n)},$$

$$(**)\qquad c_n^*(D_A^{(n)}) = D^{(n)}_{A_\delta^{(n)}}(D_A^{(n)})_\delta^{(n)}.$$

There is a slight problem since domains and targets do not quite match:

$$c_n^*(D_A^{(n)}) : X_\delta^{(n)} \times_{A_\delta^{(n)}} (A_\delta^{(n)})_\delta^{(n)} \to (X \times A_\delta^{(n)}) \times_{A_\delta^{(n)}} (A_\delta^{(n)})_\delta^{(n)}.$$

But we have the following isomorphism

$$(X \times A_\delta^{(n)}) \times_{A_\delta^{(n)}} (A_\delta^{(n)})_\delta^{(n)} \cong X \times (A_\delta^{(n)})_\delta^{(n)} \cong (X^{(n)})^{(n)},$$

and a commutative diagram (naturality of c_n)

$$\begin{tikzcd}
(X_\delta^{(n)})_\delta^{(n)} \arrow[r, "c_n"] \arrow[d, "(\iota_\delta^{(n)})_\delta^{(n)}"'] & X_\delta^{(n)} \arrow[d, "\iota_\delta^{(n)}"] \\
(A_\delta^{(n)})_\delta^{(n)} \arrow[r, "c_n"] & A_\delta^{(n)}
\end{tikzcd}$$

($\iota : X \to A$ is the structure map) giving another isomorphism

$$c_n \times_{A_\delta^{(n)}} (\iota_\delta^{(n)})_\delta^{(n)} : (X_\delta^{(n)})_\delta^{(n)} \to X_\delta^{(n)} \times_{A_\delta^{(n)}} (A_\delta^{(n)})_\delta^{(n)}.$$

So, to show $(*)$, it is enough to see that the following diagram is commutative

$$\begin{tikzcd}
(X_\delta^{(n)})_\delta^{(n)} \arrow[r, "(\phi_\delta^{(n)})_\delta^{(n)}"] \arrow[d, "c_n \times_{A_\delta^{(n)}} (\iota_\delta^{(n)})_\delta^{(n)}"'] & (X_\delta^{(n)})_\delta^{(n)} \arrow[d, "c_n \times_{A_\delta^{(n)}} (\iota_\delta^{(n)})_\delta^{(n)}"] \\
X_\delta^{(n)} \times_{A_\delta^{(n)}} (A_\delta^{(n)})_\delta^{(n)} \arrow[r, "c_n^*(\phi_\delta^{(n)})"] & X_\delta^{(n)} \times_{A_\delta^{(n)}} (A_\delta^{(n)})_\delta^{(n)}
\end{tikzcd}$$

and this is just abstract nonsense – since

$$c_n^*(\phi_\delta^{(n)}) = \phi_\delta^{(n)} \times_{A_\delta^{(n)}} \mathrm{id}_{(A_\delta^{(n)})_\delta^{(n)}},$$

after projecting to $X_\delta^{(n)}$, commutativity of the diagram follows from naturality of c_n and after projecting to $(A_\delta^{(n)})_\delta^{(n)}$ commutativity just means that $(\phi_\delta^{(n)})_\delta^{(n)}$ is a morphism over $(A_\delta^{(n)})_\delta^{(n)}$.

To show $(**)$ we need (at last) to use the fact that D is iterative.
Again, it is enough to show that the following diagram is commutative

$$\begin{tikzcd}
(X_\delta^{(n)})_\delta^{(n)} \arrow[rr, "D^{(n)}_{A_\delta^{(n)}}(D_A^{(n)})_\delta^{(n)}"] \arrow[d, "c_n \times_{A_\delta^{(n)}} (\iota_\delta^{(n)})_\delta^{(n)}"'] & & X \times (A_\delta^{(n)})_\delta^{(n)} \arrow[d, "\cong"] \\
X_\delta^{(n)} \times_{A_\delta^{(n)}} (A_\delta^{(n)})_\delta^{(n)} \arrow[rr, "c_n^*(D_A^{(n)})"] & & (X \times A_\delta^{(n)}) \times_{A_\delta^{(n)}} (A_\delta^{(n)})_\delta^{(n)}
\end{tikzcd}$$

So, again we can consider projections to $(A_\delta^{(n)})_\delta^{(n)}$ and $X \times A_\delta^{(n)}$.
Since all the maps above are over $(A_\delta^{(n)})_\delta^{(n)}$, the commutativity in the first case is trivial.
For the commutativity in the second case, we first make the following identification

$$X \times A_\delta^{(n)} \cong X^{(n)}.$$

Then $D_A^{(n)}$ becomes e_n^D and $D^{(n)}_{A_\delta^{(n)}}$ becomes $D_{(n)}$, where $D_{(n)}$ is D_∞ restricted to $(X^{(n)})_\delta^{(n)}$. Therefore the commutativity in the second case is equivalent to the commutativity of the following diagram

$$\begin{tikzcd}
(X_\delta^{(n)})_\delta^{(n)} \arrow[r, "(e_n^D)_\delta^{(n)}"] \arrow[d, "c_n"'] & (X^{(n)})_\delta^{(n)} \arrow[d, "D_{(n)}"] \\
X_\delta^{(n)} \arrow[r, "e_n^D"] & X^{(n)}
\end{tikzcd}$$

which is exactly 5.3(iii) with ∞ replaced by n and it is clear that commutativity of the diagram in 5.3(iii) is equivalent to commutativity of the diagram above for each n. $\square$

We now try to define the set of "$\sharp$-points" of $\mathcal{G}$ as

$$\mathcal{G}^\sharp := \{\phi \in \mathcal{G}(A) \mid \mathcal{G}(\delta)(\phi) = \Delta_A(\phi)\},$$

where δ is understood as a morphism $\delta : A_\delta^\infty \to A$.

Fact 6.2 *Let ϕ be an automorphism of X. Then $\phi \in \mathcal{G}^\sharp$ if and only if ϕ is a δ-isomorphism between the iterative δ-schemes (X, D) and $(X, \bar{D})$.*

Proof For each n, let $\delta^{(n)}$ denote the morphism $\delta^{(n)} : A_\delta^{(n)} \to A$. We again identify $X \times A_\delta^{(n)}$ with $X^{(n)}$. Then

$$\mathcal{G}(\delta^{(n)})(\phi) = \phi^{(n)}, \qquad \Delta_A^{(n)}(\phi) = e_n^{\bar{D}} \phi_\delta^{(n)} (e_n^D)^{-1}.$$

Therefore

$$\mathcal{G}(\delta)(\phi) = \Delta_A(\phi)$$

if and only if for each n the following diagram is commutative

$$\begin{tikzcd} X_\delta^{(n)} \arrow[r, "\phi_\delta^{(n)}"] \arrow[d, "e_n^D"'] & X_\delta^{(n)} \arrow[d, "e_n^{\bar{D}}"] \\ X^{(n)} \arrow[r, "\phi^{(n)}"] & X^{(n)}, \end{tikzcd}$$

and we finish as in the proof of the previous fact using 5.7(iii). $\square$

By the above fact, we can give examples of functors with no $\sharp$-points – it is enough to give two non-isomorphic iterative δ-structures on a given scheme X, which is easy to do for instance if X is an affine line.

We now have the main result of this paper.

Theorem 6.3 *Let X be a projective variety over K, and suppose D and $\bar{D}$, are iterative δ-structures on X. Then (X, D) is isomorphic to $(X, \bar{D})$. In particular, if X is defined over C, then any iterative δ-structure on X is isotrivial.*

Proof
It is well-known that for any projective variety X, the functor $\mathcal{G} = \mathrm{Aut}_X$ is representable, by a group scheme G whose connected component G^0 is an algebraic group. So let G be such for our given X. Let D and $\bar{D}$ be two iterative δ-structures on X (as in 5.3(ii)). By 6.1, Yoneda's Lemma and 5.3, G has a δ-scheme structure (but not necessarily a δ-group scheme one!) coming from D and $\bar{D}$ and this δ-scheme structure is iterative on $G(K)$. It is rather easy to see (using e.g. 2.1(v) which did not use iterativity) that each irreducible component of G is a δ-subscheme. In particular this equips G^0 with an iterative δ-variety structure, since $G^0(K)$ is dense in G^0. By Fact 5.3, $(G^0)^\sharp$ is nonempty. By 6.2, any $\phi \in (G^0)^\sharp$ is a δ-isomorphism between (X, D) and $(X, \bar{D})$.
For the final clause, we take for $\bar{D}$ the trivial iterative δ-structure. $\square$

7 δ-sheaves and the Second Isotriviality theorem

In this section (K, δ) is still a universal iterative differential field.

A *δ-vector group* is a vector group together with a δ-group structure. A *δ-vector space* is a δ-vector group such that the δ-structure preserves also scalar multiplication. If V is a δ-module, then $V^\sharp = \{x \in V \mid \delta(x) = 0\}$.

Fact 7.1
(i) The notion of a finite-dimensional δ-module is equivalent to the notion of a δ-vector space.
(ii) Any finite-dimensional δ-module has a basis consisting of $\sharp$-points of the correspondent δ-vector space. Such a basis gives isomorphism between the given δ-module and a trivial δ-module.
(iii) If X is an iterative δ-scheme, then $\mathcal{O}_X(X)^\sharp$ corresponds to δ-morphisms from X to the trivial δ-variety $\mathbb{A}^1$.

Proof (i) and (ii) are well-known (e.g. (ii) is part of the theory of linear iterative differential equations).
(iii) follows from 5.7(v). $\square$

We need to extend the notion of a δ-module to the context of sheaves.

Definition 7.2 *(i) $\mathcal{F}$ is a δ-(pre)sheaf, if it is a (pre)sheaf of δ-$\mathcal{O}$-modules, i.e. for each $U \subseteq X$ open, $\mathcal{F}(U)$ is a $\mathcal{O}_X(U)$-module and there is δ_U, a δ-$\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$ s.t. restriction maps are δ-morphisms.*
(ii) A δ-sheaf is δ-trivial if it is δ-isomorphic to $\mathcal{O}_X^{\oplus n}$ (some n).
(iii) A δ-sheaf $\mathcal{F}$ is locally δ-trivial if for any $x \in X$, there is $U \ni x$ open such that $\mathcal{F}|_U$ is δ-trivial.

Fact 7.3 *(i) If $\mathcal{F}$ is a δ-presheaf, $x \in X$, then $\mathcal{F}_x$ (a stalk of $\mathcal{F}$ at x) has a natural δ-structure.*
(ii) If $\mathcal{F}$ is a δ-presheaf, then $\mathcal{F}^+$ (the sheafication of $\mathcal{F}$) is a δ-sheaf.
(iii) If $\mathcal{F}, \mathcal{G}$ are δ-sheaves, then $\mathcal{F} \otimes \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F}^*$, $\bigwedge^n \mathcal{F}$ are δ-sheaves.
(iv) Let $(U_i)_{i\in I}$ be an open basis of X, $\mathcal{F}$ a sheaf of $\mathcal{O}$-modules and assume that for each U_i there is an iterative derivation δ_i on $\mathcal{F}(U_i)$ such that the restriction maps are δ-maps. Then the δ_i's extend to make $\mathcal{F}$ a δ-sheaf.

Proof (i) By 2.1(iv), the direct limit of δ-modules is a δ-module. Therefore any $\mathcal{F}_x$ is a δ-module.

(ii) By (i), for each open U, the module of functions from U into $\bigcup_{x\in U}\mathcal{F}_x$ (being sections of the natural projection) has a δ-module structure.
It is easy to see that $\mathcal{F}^+$ is a δ-submodule of the above δ-module of functions (see [Ha, Section 2.1]).
(iii) By (ii), it follows from 2.1(iii).
(iv) Take U, open in X, $s\in\mathcal{F}(U)$ and let $U=\bigcup_{i\in I'}U_i$ (some $I'\subseteq I$). Let $\mathbf{s}_i:=\delta_i(s|_{U_i})$ (a sequence of sections over U_i). Then for each $i,j\in I'$, and each $x\in U_i\cap U_j$ there is $k\in I$ such that $x\in U_k\subseteq U_i\cap U_j$ and

$$s_i|_{U_k}=\delta_i(s|_{U_i})|_{U_k}=\delta_k(s|_{U_k})=\delta_j(s|_{U_j})|_{U_k}=s_j|_{U_k}.$$

Since $\mathcal{F}$ is a sheaf a collection $(\mathbf{s}_i)$ comes from a sequence of sections of $\mathcal{F}$ over U which is our $\delta(s)$.
Since $\mathcal{F}$ is a sheaf, we can check all the respective axioms of iterative derivations locally, so we get a sheaf of δ-modules (resp. δ-algebras) indeed. $\square$

Fact 7.4 *If $\mathcal{F}$ is a very ample invertible δ-sheaf on a projective iterative δ-variety X, then*
(i) $\mathcal{F}$ is locally δ-trivial.
(ii) $\mathcal{F}(X)$ has a basis consisting of elements of $\mathcal{F}(X)^\sharp$ and the map into δ-trivial projective space defined by this basis is a δ-map.

Proof
(i) Let $X=\bigcup_{i\in I}U_i$ be an open cover of X such that for each i there is an isomorphism of sheaves of $\mathcal{O}$-modules

$$f_i:\mathcal{F}|_{U_i}\cong\mathcal{O}_{U_i}.$$

Since X is projective, $\mathcal{F}(X)$ is a finite-dimensional δ-module. By 7.1(ii), there is $\{s_0,\ldots,s_n\}$, a basis of $\mathcal{F}(X)$ contained in $\mathcal{F}(X)^\sharp$. Let

$$U_{ij}:=U_i\setminus Z(f_i(s_j|_{U_i})),\quad f_{ij}:=f_i|_{U_{ij}},\quad s_{ij}:=s_i|_{U_{ij}}.$$

Since $\mathcal{F}$ is very ample, (U_{ij}) is a cover of X. For each $i,j\in I$, $\{s_{ij}\}$ is a basis of $\mathcal{F}(U_{ij})$, since $\{f_{ij}(s_{ij})\}$ is a basis of $\mathcal{F}(U_{ij})$ (being an invertible element). But $s_{ij}\in\mathcal{F}(U_{ij})^\sharp$. Therefore the map

$$f_{ij}(s_{ij})^{-1}f_{ij}:\mathcal{F}|_{U_{ij}}\to\mathcal{O}_{U_{ij}}$$

is a δ-sheaf isomorphism.

(ii) Let $X = \bigcup U_i$ be an open cover of X such that for each i there is an isomorphism of δ-sheaves

$$f_i : \mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}$$

(such a cover exists by (i)).
If $s \in \mathcal{F}(X)^\sharp$, then $f_i(s|_{U_i}) \in \mathcal{O}_{U_i}(U_i)^\sharp$. Therefore (by 7.1(iii)),

$$f_i(s|_{U_i}) : U_i \to \mathbb{A}^1$$

is a δ-map (where $\mathbb{A}^1$ has the trivial δ-variety structure).
Let $B = \{s_0, \ldots, s_n\} \subset \mathcal{F}(X)^\sharp$ be a basis of $\mathcal{F}(X)$ (it exists by 7.1(ii)). Then clearly the map

$$f_B : X \to \mathbb{P}^n$$

is a δ-map, where $\mathbb{P}^n$ has the trivial δ-variety structure. □

Fact 7.5 *If $A \to B$ is a δ-map of δ-rings and $f \in B$, then*
(i) $\Omega_{B/A}$ is naturally a δ-module.
(ii) The map $\Omega_{B/A} \to \Omega_{B_f/A}$ is a δ-map.

Proof (i) By [Ha, II.8.1A.], $\Omega_{B/A}$ is isomorphic to I/I^2, where

$$I = \ker(B \otimes_A B \to B).$$

I has clearly the δ-module structure, so has I^2, hence I/I^2 gets the quotient δ-module structure.
(ii) Since the localization map $B \to B_f$ is a δ-map its tensor square is a δ-map as well and it clearly preserves the kernel of multiplication, so the result follows. □

Proposition 7.6 *If X is an iterative δ-scheme, then Ω_X is a δ-sheaf.*

Proof We will use 7.3(iv). Take $(U_i)_i$ the open base of X consisting of open affine subvarieties. By the 7.5(i), each $\Omega_X(U_i)$ has a natural δ-module structure. We need to check that the restriction map preserves the δ-module structure. Since any affine variety has an open basis consisting of subsets corresponding to localizations, it is enough to use 7.5(ii). □

Corollary 7.7 *The canonical and anticanonical sheaves are locally δ-trivial invertible δ-sheaves.*

Proof By the previous proposition, 7.3(iii) and 7.4 $\square$

Theorem 7.8 *If V is a projective iterative δ-variety and the canonical or the anticanonical divisor of V is ample, then V is δ-isotrivial.*

Proof By 7.7, and 7.4. (after taking a suitable tensor power), V is δ-isomorphic to a δ-subvariety of a trivial δ-variety (namely $\mathbb{P}^n$), so V is isotrivial. $\square$

Corollary 7.9 *If V is a smooth projective δ-curve, then V is δ-isotrivial.*

Proof By [Ha, IV.3.3] a divisor X on V is ample if and only if $\deg(X) > 0$. Hence, by 7.8, we are done in the cases when the degree of the canonical divisor is non-zero, i.e. when V is not an elliptic curve. But the case of elliptic curve is solved in [Be]. $\square$

8 Further remarks

In [Be] Benoit proves the following:

Proposition 8.1 *Let X be an algebraic variety over K (where (K,δ) is a universal iterative differential field). Then X has an iterative ∂-structure if and only if for each n there is an isomorphism α_n between X and a variety X_n defined over K^{p^n} such that for each n the isomorphism $\alpha_n^{-1} \circ \alpha_n$ between X_{n+1} and X_n is defined over K^{p^n}.*

So we ask here whether any projective variety X over K satisfying the conditions of the proposition descends to $C = \bigcap_n K^{p^n}$.

On the other hand, if X belongs to a family with a fine moduli space, then simply the fact that X is (isomorphic to something) defined over each K^{p^n} implies that X descends to the intersection. See [Ha-Mo] for a discussion of fine moduli spaces and the related issues.

In particular the class of "principally polarized abelian varieties of dimension g with level n-structure" has a fine moduli space (see [Mu-Fo-Ki]). Together with Theorem 6.3 we obtain:

Corollary 8.2 *Let (X,D) be an iterative δ-variety over K, where X is an abelian variety. Then (X,D) is δ-isotrivial.*

It is also possible to combine the results of Section 6 together with [Be] to obtain Theorem 7.8. It is just because a projective variety polarized by its canonical or anticanonical divisor belongs to a family with a fine moduli space. But note that our proof of Theorem 7.8 does not use results of Section 6 at all.

References

[Be] F. Benoit, *Ph.D. thesis*, 2005.

[Bu] A. Buium, *Differential Function Fields and Moduli of Algebraic Varieties*, *Lecture Notes in Math.*, 1226, Springer-Verlag, 1986.

[Ha] R. Hartshorne, **Algebraic Geometry**, Springer-Verlag (1977).

[Ha-Mo] J. Harris and I. Morrison, *Moduli of curves*, Springer-Verlag, 1998.

[Ma] D. Marker, Model Theory of Differential Fields, *Model Theory of Fields, 2nd edition*, (D. Marker, M. Messmer, A. Pillay), Lecture Notes in Logic 5, ASL, 2006.

[Mu-Fo-Ki] D. Mumford, Fogarty, and F. Kirwan, *Geometric Invariant Theory, 3rd edition*, Springer-Verlag, 1994.

[Sc] Model Theory of Valued D-Fields, *Ph.D. thesis*, Harvard University, May 1997.
`http://math.berkeley.edu/~scanlon/papers/thesis.pdf`

[Tr] W. N. Traves, Differential Operators and Nakai's Conjecture, *Ph.D. thesis*, University of Toronto (1998).
`http://mathweb.mathsci.usna.edu/faculty/traveswn/publications/thesis.ps`

[Zi] M. Ziegler, Separably closed fields with Hasse derivations, *Journal of Symbolic Logic* **68** (2003), 311-318.