Subvarieties of commutative meromorphic groups

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Abstract

We prove that if X is a meromorphic subvariety of a commutative meromorphic group G and X has finite stabilizer, then X is contained in a translate of an algebraic subgroup of G. We give an application with a "diophantine" flavour.

1 Introduction and preliminaries

In this paper we study definable subsets or equivalently meromorphic subvarieties of commutative meromorphic groups. Meromorphic groups appear in [2], and were later studied in [9]. They are essentially complex (usually connected) Lie groups G which have a "good" compactification G^* (see below). Both complex algebraic groups and complex tori are meromorphic groups. A "meromorphic subvariety" or Zariski closed subset X of G is then an analytic subvariety of the form $Y \cap G$ where Y is an analytic subvariety of G^*. The work here was motivated by the following question of the first author:
(1) Suppose G is a commutative meromorphic group, X is an irreducible

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meromorphic subvariety of G, and Γ is a cyclic subgroup of G, such that $X \cap \Gamma$ is "Zariski-dense" in X. Must X be a translate of a meromorphic subgroup of G?

(1) is known in the case where G is a commutative complex algebraic group. The proof goes via the "Skolem-Chabauty" trick of embedding the situation in the p-adics, and does not generalize to arbitrary complex Lie groups. We will prove the general case of (1) by *reducing* to the algebraic case. This is done by proving the following result which is maybe of interest in its own right:

(2) Suppose X is a meromorphic subvariety of the commutative meromorphic group G and suppose X has finite stabilizer. Then, up to translation, X is contained in an algebraic subgroup A of G, (meaning that A is a meromorphic subgroup of G meromorphically isomorphic to an algebraic group).

(2) is known in the case that G is compact (namely when G is a complex torus). Our proof of (2) is influenced by Hrushovski's "socle argument" [4]. The latter takes place in the category of "commutative groups G of finite Morley rank". A certain definable subgroup, $s(G)$, the socle of G, is identified. (In the case that G is meromorphic $s(G)$ will be an almost direct sum of an algebraic subgroup and a "totally nonalgebraic" complex torus.) Hrushovski proves, under a certain "rigidity" assumption on $s(G)$ that any definable subset X of G with finite stabilizer is (essentially) contained in a translate of $s(G)$. This rigidity assumption will in general *not* hold for arbitrary meromorphic G (essentially because the algebraic part of $s(G)$ may have a nontrivial unipotent subgroup). We get around this problem by a slightly involved inductive argument, using, among other things, the fact (coming from GAGA) that if $f : X \longrightarrow Y$ is a fibration of a compact complex space X over a projective algebraic variety Y with general fibre $\mathbf{P}^n$, then X is (essentially) algebraic.

In the rest of this section we recall the category of meromorphic groups from [9], discuss the socle theory for groups of finite Morley rank, and identify the socles in the meromorphic case.

For us, a group G will typically be a *structure*, namely G equipped with the group operation, as well as various other distinguished subsets of G, $G \times G$,.... Meromorphic groups are certain complex Lie groups G equipped with certain analytic subsets of G, $G \times G$,.. Recall that $\mathcal{A}$ is the many sorted structure consisting of reduced compact complex spaces $(X_i)_i$, with relations for analytic subsets of the various $X_{i_1} \times ... \times X_{i_n}$. $\mathcal{A}$ has quantifier-elimination, finite Morley rank (sort-by-sort) and elimination of imaginaries. The Zariski closed subsets of a compact complex space X are by definition its analytic subsets. If U is a Zariski open subset of X, and f a holomorphic map from U to some Y, f is said to be meromorphic if there is an analytic subset Z of $X \times Y$ such that $Z \cap (U \times Y)$ is the graph of f. A Zariski closed subset of U is precisely the intersection of an analytic subset of X with U. There are various equivalent descriptions of meromorphic groups:

(1) A meromorphic group is a group G interpretable in $\mathcal{A}$.
As such G has definably the structure of a complex Lie group which has a finite covering by Zariski open subsets U_i of various irreducible compact complex spaces X_i, such that the transition maps as well as the group operation are meromorphic when read in the charts. G, $G \times G$, etc. are then equipped with their own Zariski topologies, and this makes G into a structure (which has quantifier elimination and finite Morley rank). From the model-theoretic point of view, we usually just work with G as a definable group in $\mathcal{A}$.

(2) A meromorphic group is a complex Lie group G with an analytic embedding into a compact complex space G^* such that G is Zariski-dense and Zariski open in G^*, and such that the group operation is a meromorphic from $G^* \times G^*$ to G^*. As a structure we again equip G, $G \times G$, etc. with their Zariski closed sets (induced from G^*,...).

(3) A meromorphic group is a complex Lie group G whose connected component G^0 has finite index and such that G^0 is (as a complex Lie group) an extension of a complex torus T by a connected linear algebraic group L.

Something is missing in (3) as we are not on the face of it specifying the additional structure on G. Neverthless the three descriptions are equivalent in the following sense. It is evident that if G (as a structure) is meromorphic in the sense of (2) then it is meromorphic in the sense of (1). On the other hand, it is proved in [9], that if G is meromorphic in the sense of (1) and connected then there is a meromorphic homomorphism from G onto a complex torus T whose kernel is meromorphically isomorphic to a connected linear algebraic group L, yielding (3). Finally, suppose G is a connected complex Lie group of type (3). G is then a principal bundle over T with typical fibre L. As in Remark 2.3 of [2], L has a compactification L^* giving L the structure of a meromorphic group in the sense of (2). Hence the action of L on L^* is meromorphic, and the corresponding bundle G^* is acted on by

G meromorphically, too. Then G is meromorphic.
Let us note:
(4) if G is connected, commutative and meromorphic, then a compactification G^* of G as in (2) can be chosen such that if U is the maximum connected linear unipotent subgroup of G, then the closure of U in G^* as well as the closure of any translate of U in G^* is biholomorphic to $\mathbf{P}^n$.

Let us now pass on to the model-theoretic theory of socles. This was introduced by Hrushovski in [4], but we will take the opportunity to expand a little on the theory. Let M be a structure whose theory has finite Morley rank. Assume M to be saturated if you want, but this is not essential. A definable set X in M can be considered as a structure in its own right (with the structure induced by M with additional parameters if one wants). A definable set X is said to be *almost strongly minimal*, if there is a strongly minimal set Y and a definable finite-to-one map f from X into Y^n for some n. X is then definably a finite cover of a definable set of the form Z/E for Z a definable subset of Y^n and E a definable equivalence relation on Z. If G is an almost strongly minimal connected definable group, then one can choose the strongly minimal set Y to be a subset of G and f to be $1-1$. An *almost pluriminimal* definable set X is defined just as above, except that Y is now just a set of Morley rank 1 (so with Morley degree possibly > 1). Again if G is an almost pluriminimal group we can choose Y and f with f an injection.

The socle theory can be generalized to arbitrary groups of finite Morley rank but we restrict ourselves here to the commutative case. Until we say otherwise, G denotes a connected commutative group definable in the structure M of finite Morley rank.

Definition 1.1 *The socle of G, $s(G)$ is the sum of all connected almost strongly minimal definable subgroups of G.*

By finite Morley rank considerations (or Zilber's indecomposability theorem) $s(G)$ is definable and connected (and moreover definable over the same parameters as G).

Lemma 1.2 *The following are equivalent:*
(i) G is a sum of connected almost strongly minimal subgroups,
(ii) G is almost pluriminimal.

(iii) After possibly adding parameters, $Th(G)$ has the properties
(a) weak elimination of imaginaries, and
(b) any algebraically closed subset of a model of $Th(G)$ is an elementary submodel.

Proof . (i) implies (ii) is immediate.
(ii) implies (iii): As G is almost pluriminimal, there is a definable subset Y of G of Morley rank 1, and a definable bijection f of G with some Z/E where $Z \subset Y^n$. It is well-known that working over a large enough set of parameters Y has weak elimination of imaginaries and every algebraically closed subset of Y is an elementary substructure. The same is then true of Z/E and so of G.
(iii) implies (i): Work over a set of parameters such that $Th(G)$ satisfies (iii)(a) and (b). We have to show that $s(G) = G$. Suppose otherwise. Let $T = G/s(G)$, and $f : G \longrightarrow T$ the canonical surjective homomorphism. Let a be a generic point of T. Our assumptions imply that there is $c \in f^{-1}(a)$ with $c \in acl(a)$. So there is a generic (so infinite) definable subset X of T and a definable subset Y of G, such that $f(Y) = X$ and $f|Y$ is finite-to-one. It follows that there is a strongly minimal subset Y_0 of Y which meets each translate of $s(G)$ in at most finitely many points. Without loss of generality, Y_0 is indecomposable and contains the identity. Then by Zilber indecomposability, Y_0 generates an almost strongly minimal subgroup of H which is not contained in $s(G)$, a contradiction. □

We see from the above lemma that $s(G)$ is precisely the maximal almost pluriminimal subgroup of G. (This is actually the definition of the socle in [4], and the equivalence of (i) and (ii) above appears already there.)

Lemma 1.3 *Let H be a definable connected subgroup of G. The following are equivalent:*
(i) H contains $s(G)$.
(ii) the surjective homomorphism $q : G \longrightarrow G/H$ has no definable multisection over any infinite subset of G/H.
(iii) Any strongly minimal subset X of G is contained in a single translate of H, up to finite.

Proof. First we explain (ii). Let $T = G/H$ and $q : G \longrightarrow T$ the canonical surjective homomorphism. By a "definable multisection of q over an infinite

subset of T", we mean some definable relation $R \subset T \times G$ such that for any $(t, g) \in R$, $q(g) = t$, for any $t \in T$ there are at most finitely many g such that $(t, g) \in R$ and the projection of R on T is infinite.
The lemma is proved easily using lines of argument as in the proof of 1.2. We give a sketch.
(i) implies (ii): if (ii) fails, we find as in the proof of Lemma 1.2, an indecomposable strongly minimal subset X of G which contains 0 and meets each translate of H in at most finitely many points. The group generated by X is almost strongly minimal, so contained in $s(G)$, but not contained in H.
(ii) implies (iii): If X witnesses the failure of (iii) then X clearly gives rise to a multisection of q over an infinite subset of T.
(iii) implies (i): Suppose (i) fails. So $H \cap s(G)$ has infinite index in $s(G)$. By the proof of Lemma 1.2, the surjective homomorphism $s(G) \longrightarrow H \cap s(G)$ has a definable multisection R over an infinite subset of $s(G)/(H \cap s(G))$. As in the proof of Lemma 1.2, we obtain a strongly minimal subset X of $s(G)$ which meets each translate of $H \cap s(G)$, hence every coset of H, in a finite set, contradicting (iii). $\square$

Corollary 1.4 *(i) $s(G)$ is generated by strongly minimal subsets.*
(ii) If $s(G)$ is modular then it is a sum of strongly minimal subgroups.

Proof . The meaning in (i) is as usual that there are strongly minimal subsets $X_1, .., X_n$ of $s(G)$ such that $s(G) = X_1 \cdot X_2 .. \cdot X_n$. This is either obvious, or follows from Lemma 1.3: Let H be the subgroup of $s(G)$ generated by strongly minimal indecomposable subsets which contain 0. If $H \neq s(G)$ then by (iii) in the previous lemma, there is a strongly minimal subset X of $s(G)$ meeting any translate of H in at most finitely many points. Without loss, X is indecomposable and contains 0, contradiction.
(ii) If $s(G)$ is modular then a strongly minimal subset is a coset of a strongly minimal group, up to finite. $\square$

We now return to meromorphic groups, which we view as definable groups in the structure $\mathcal{A}$. $\mathcal{A}'$ will denote a saturated elementary extension of $\mathcal{A}$. "Generic points" will be found in $\mathcal{A}'$ but we will be working with groups, sets etc. defined over the base model $\mathcal{A}$. Also we will use "definable" interchangeably with "meromorphic". We will call a complex torus T *modular* if every irreducible analytic subvariety of $T \times .. \times T$ is a translate of a subtorus. (This agrees with the model-theoretic notion.) We now identify the socle.

Lemma 1.5 *Let G be a commutative connected meromorphic group. Then $s(G) = A+T$, where A is the maximal connected definable algebraic subgroup of G and T is the sum of all strongly minimal modular complex tori in G (and T is also a modular complex torus). Also $G/s(G)$ is a complex torus.*

Proof An almost strongly minimal subgroup B of G is by [9], either algebraic, or modular. If B is modular then by Corollary 1.4 above, B is a sum of strongly minimal modular groups, which by 4.2 of [9], are complex tori. There is clearly a greatest connected algebraic subgroup A of G, and thus $s(G) = A + T$ as required. Also as, by [9] G is an extension of a complex torus T' by a linear algebraic group L, $L < s(G)$ and so $G/s(G)$ is a quotient of T', so a complex torus. □

Remark 1.6 *There are modular tori T which are not almost pluriminimal (so $s(T) \neq T$).*

Proof . Let T_1, T_2 be say strongly minimal modular tori. By [1] there is some (in fact uncountably many) extension T of T_1 by T_2 which is not split, that is where T_2 does not have have a direct summand. Then T is modular but is not a sum of strongly minimal subtori.

Finally in this introduction we give a couple of remarks which will be useful for the proofs in the next section. Recall that a complex space is called Moishezon if it is bimeromorphic to an algebraic variety.

Lemma 1.7 *Let X and Y be irreducible compact complex spaces, and $f : X \longrightarrow Y$ be a holomorphic surjection, such that Y is Moishezon and for some dense Zariski open subset U of X and some n, for all $x \in U$, $f^{-1}(x)$ is isomorphic to $\mathbf{P}^n$. Then X is Moishezon.*

Proof This is it seems part of the folklore of the subject. In any case, by Lemma 11 of [3], f is Moishezon. By GAGA, X is Moishezon, that is, bimeromorphic to a projective algebraic variety. □

Corollary 1.8 *Let G be a connected commutative meromorphic group which is (definably) an extension of a connected algebraic group A by a connected linear algebraic group B. Then G is algebraic.*

Proof A good compactification G^* of G is clearly a fibration over a compactification of A with general fibre $\mathbf{P}^n$ for some n. By Lemma 1.6, G^* is bimeromorphic to a projective algebraic variety. It follows that G is definably isomorphic to an algebraic group. $\Box$

Note that the Corollary can not be generalized to arbitrary extensions of algebraic groups by algebraic groups. In [11] an example is given of a complex torus which is an extension of an elliptic curve by an elliptic curve, but which is not an abelian variety. In fact "almost all" such extensions are not algebraic (see [1]).

2 Main results

We work in the structure $\mathcal{A}$. All the definitions needed are in [8]. Since we are dealing in this section with meromorphic groups, we can (see [6]) work over a countable sublanguage of the language of compact complex spaces, and assume that all the meromorphic groups under consideration are saturated. G will denote a connected commutative meromorphic group, unless stated otherwise. As mentioned in the introduction, G has a unique maximal definable connected linear algebraic group L and G/L is a complex torus T. (In fact T is precisely the "Albanese" of G.) L is uniquely a direct product of a unipotent group (definably isomorphic to some power of G_a) and an algebraic torus (definably isomorphic to a power of G_m).

Lemma 2.1 *Suppose G is definably an extension of a complex torus T by a unipotent group U. Let $\pi : G \longrightarrow T$ be the canonical surjection. Then there is a unique smallest connected definable subgroup $\tilde{G}$ such that $\pi(\tilde{G}) = T$. Moreover $G = \tilde{G} \times \tilde{U}$ for some definable subgroup $\tilde{U}$ of U.*

Proof Suppose that $\pi(G_i) = T$ for $i = 1, 2$. Then G/G_i is isomorphic to $U_i = U/U \cap G_i$. So $G/G_1 \cap G_2$ embeds in the unipotent group $U_1 \times U_2$, so has no compact factor. Thus $\pi(G_1 \cap G_2) = T$. This argument gives us a unique smallest $\tilde{G}$ as required. Let $\tilde{U}$ be a definable complement in U to $U \cap \tilde{G}$. $\Box$

Remark 2.2 *There is a nice theory of extensions of abelian varieties by unipotent groups ([10]) using the existence of rational sections. In particular in this case, $\tilde{G}$ above has dimension at most twice that of T. We do not*

know of any similar theory for the general case of extensions of complex tori by unipotent groups. But note:
(i) If G is an extension of a strongly minimal modular complex torus T by a unipotent group U, then G definably splits as $T \times U$ iff $\pi : G \longrightarrow T$ has a definable multisection.
(ii) If every extension of a strongly minimal modular torus by a unipotent group definably splits, then for any commutative meromorphic G and unipotent subgroup U of G, $s(G/U) = s(G)/U$.

Proof (i) If $\pi : G \longrightarrow T$ has a definable multisection, then by Lemma 1.3, $G = s(G)$, so clearly splits.
(ii) $s(G/U)$ is by Lemma 1.5 of the form $A/U + B/U$ where A, B are definable subgroups of G containing U, A/U is algebraic and B/U is an almost direct sum of strongly minimal modular tori. By 1.8, A is algebraic. By our assumptions $B = U + B_0$ for B_0 an almost direct sum of modular strongly minimal tori. So $s(G) = A + B_0$. $\square$

In any case we obtain from Lemma 2.1:

Corollary 2.3 *Given G (commutative, connected, meromorphic and 0-definable), for any infinite definable subgroup H of G, either H is unipotent, or H contains an infinite connected $acl(0)$ definable group.*

Proof Let

$$0 \longrightarrow L \longrightarrow G \longrightarrow T \longrightarrow 0$$

where L is linear algebraic and T is a complex torus. Let $L = U + B$ where U is unipotent, B an algebraic torus (and both are 0-definable). If H is an infinite definable subgroup of G and $H \cap B$ is infinite, then its connected component is $acl(0)$-definable. So we may assume that $L = U$.
Let T_i, for $i \in I$ be the subtori of T. Each T_i is $acl(0)$-definable. Let $S_i < G$ be the preimage of T_i under $\pi : G \longrightarrow T$. S_i is connected and $acl(0)$-definable. Let $\tilde{S}_i$ be as given by Lemma 2.1, (so also $acl(0)$-definable, by uniqueness).
Take any definable infinite subgroup H of G. We may assume H to be connnected. So $\pi(H) < T$ is connected. If $\pi(H) = 0$ then $H < L$ so H is unipotent. Otherwise $\pi(H) = T_i$, whereby $H < S_i$. But then H contains $\tilde{S}_i$. $\square$

We now state and prove the main theorem of this paper. The statement will be in the language of types. We give the geometric interpretation in a subsequent corollary..

Theorem 2.4 *Denote by S the socle of G. Let $p(x)$ be a stationary type (over some given set of parameters) of an element of G. Let $Stab(p)$ denote the model-theoretic stabilizer of p. Then either $Stab(p) \cap S$ is infinite, or all realizations of p are in a single translate of S.*

Proof Note first that if $RM(p) = n$ and that if X is a definable set in p of Morley rank n and Morley degree 1, then the conclusion says: either $Stab(p) \cap S$ is infinite, or X is contained in a single translate of S up to a set of Morley rank $< n$.

Note secondly that by 1.3 any definable subgroup H of G is infinite if and only if $H \cap S$ is infinite. So the first disjunct of the conclusion is equivalent to $Stab(p)$ is infinite.

We will prove the theorem by induction on $RM(G)$ (or even $dim(G)$). Note that the truth of the theorem for G implies truth of the analogous statement for any principal homogeneous space for G.

Now fix a compactification G^* of G as in (4) in section 1. It follows that (*) the closure in G^* of any translate of a unipotent subgroup of G is biholomorphic with some $\mathbf{P}^k$.
Let $T = G/S$. There is a (0-definable) surjective homomorphism from T to T_1 where T_1 is a simple complex torus. By 4.5 from [8], T_1 is either a simple abelian variety or a strongly minimal modular torus. Denote the kernel of $\pi : G \longrightarrow T_1$ by S_1. Note that $s(S_1) = s(G) = S$. We will work over some algebraically closed set of parameters C such that p is over C and S satisfies (a) and (b) from 1.2(iii) over C (so adjoin elements of C as constants.)

Let $p(x) = tp(a)$ where $a \in G$. We will assume that $Stab(p)$ is finite, and we aim to prove that all realizations of p are contained in a single translate of S.

Let $b = \pi(a)$. (So $b \in T_1$.) If $b \in acl(\emptyset)$, then all realizations of p are in a single translate of S_1, and we can apply induction to finish. So we will assume that $b \notin acl(\emptyset)$ and work for a contradiction. Note that $tp(b)$ is the generic type of an algebraic variety (in the case that T_1 is an abelian variety), or $tp(b)$ is modular, so orthogonal to $\mathbf{P}^1$ (in the case that T_1 is modular).

Let G_b be the fibre over b of $\pi : G \longrightarrow T_1$. $q_b = tp(a/acl(b))$. So q_b is a stationary complete type of an element in G_b, a principal homogeneous space for S_1. The induction hypothesis applies, yielding either:
(I). $Stab_{S_1}(q_b)$ $(= Stab_G(q_b))$ is infinite, or
(II). All realizations of q_b are contained in a single S-orbit (i.e. translate of S).
Case (I). Suppose (I) holds. Let $A \subseteq S_1$ be $Stab_{S_1}(q_b)$. So A is infinite, and by Corollary 2.3, either A contains an infinite $acl(\emptyset)$-definable subgroup D, or A is unipotent. In the first case D does not depend on b, so it easily follows that D is contained in $Stab(tp(a))$ so the latter is infinite, contradiction. So let us suppose A to be unipotent, hence contained in $S = s(S_1)$. Note that A is $acl(b)$-definable. Let $\hat{q}_b = tp((a/A)/acl(b))$ in G/A. So $Stab(\hat{q}_b)$ in G/A is trivial, hence by induction hypothesis all realizations of $\hat{q}_b$ are in a single translate of $s(G/A)$. Write $s(G/A)$ as the sum of N and T' where N is an algebraic group, and T' a modular torus. As in [4], modularity of T' implies that all realization of $\hat{q}_b$ are contained in a single translate of N. Thus back in G all realizations of q_b are contained in a single translate of an extension B of the algebraic group N by the unipotent group A. B is, by Corollary 1.8, also an algebraic group, hence contained in S, by 1.5. So (II) holds.

Case II. Assume (II) holds: all realizations of q_b are in a single S-orbit. This orbit, which we will call W_b is clearly $acl(b)$-definable. Let X_b be some definable set in q_b of least Morley rank ($= RM(q_b)$), of Morley degree 1, and contained in W_b. So $Stab(X_b) = Stab(q_b)$. Let Z_b be a nonempty $acl(b)$-definable subset of X_b of least (Morley rank, Morley degree). Let $U := \{s \in S : s + Z_b = Z_b\}$, an $acl(b)$-definable subgroup of S. We claim that Z_b is a U-orbit. As S acts transitively on W_b, it is enough to show that for each $g \in S$, $g + Z_b = Z_b$ or $(g + Z_b) \cap Z_b = \emptyset$. So suppose towards a contradiction that for some $s \in U$, $(s + Z_b) \cap Z_b$ is a proper nonempty subset of Z_b. We now make use of our choice of a base set of parameters over which 1.2(iii) (a) and (b) hold for S. That is, $acl(b) \cap S$ is an elementary substructure of S. Thus as everything going on is $acl(b)$-definable, there is $s \in acl(b) \cap S$, such that $(s + Z_b) \cap Z_b$ is a proper nonempty ($acl(b)$-definable) subset of Z_b, contradicting least choice of (Morley rank, Morley degree). Thus Z_b is a U-orbit as claimed. If U is finite, so is Z_b, whereby $G_b \cap acl(b)$ is nonempty. This contradicts Lemma 1.3 (we get a multisection of π over an infinite subset of T_1).

Thus U is infinite. An argument like the one above making use of the fact that $S \cap acl(B)$ is an elementary substructure of S, shows that X_b is a union of translates of Z_b by elements of S, each such translate again being a U-orbit. Thus U stabilizes X_b set-theoretically. It follows that U is contained in $Stab(q_b)$. We again apply Corollary 2.3. If U contains an infinite $acl(\emptyset)$-definable subgroup D say, then as remarked in Case (I) above, D is contained in $Stab(p)$, a contradiction.
Thus U is unipotent (and infinite).
We separate into two cases (a) and (b) depending on whether T_1 is a modular torus, or an algebraic group. In case (a), note that U is both definable over parameters from $\mathbf{P}^1$ as well as $acl(b)$. Modularity of T_1 implies that $tp(b)$ is orthogonal to any type of a tuple from $\mathbf{P}^1$ and thus U is $acl(\emptyset)$-definable. As before U is contained in $Stab(p)$, a contradiction. So we are in Case (b): T_1 is an algebraic group. Fix a compactification G^* of G satisfying (*) at the beginning of the proof. So the closure of Z_b in G^* is biholomorphic to some $\mathbf{P}^n$. Let c be a canonical parameter for Z_b. Note that $c \in acl(b)$ and $b \in dcl(c)$. Let e be a generic point of Z_b over c. Note that $c \in dcl(e)$. ($b = \pi(e)$ and c is the canonical parameter of the unique irreducible component of the locus of e over b which contains e.) Let Y be the compact complex space of which c is the generic point. As c is interalgebraic with b and b is a point in the algebraic variety T_1, Y is Moishezon. Let Z be the compact complex subspace of G^* of which e is the generic point. c being in $dcl(e)$ yields a meromorphic dominant map f from Z to Y, with generic fibre biholomorphic to $\mathbf{P}^n$. The "graph" of f is a compact complex space Z_1 mapping onto Y with general fibre biholomorphic to $\mathbf{P}^n$. By Lemma 1.7, Z_1 is Moishezon. As Z is bimeromorphic to Z_1, Z is Moishezon. Translating, we may assume that Z contains the identity. The definable subgroup of G generated by Z is then Moishezon, so algebraic, and is not contained in S. This contradicts Lemma 1.5. This contradiction completes the proof of Theorem 2.4. □

Here is the geometric restatement as well as the main result stated in the abstract.

Corollary 2.5 *Suppose X is an irreducible Zariski closed subset of G. Let $Stab(X)$ denote the set-theoretic stabilizer of X ($= \{g \in G : g + X = X\}$). Suppose $Stab(X)$ is finite. Then X is contained in a translate of the socle $s(G)$ of G. Moreover X is contained in a translate of an algebraic subgroup of G.*

Proof Let p be the generic type of X (over some given algebraically closed set of parameters over which everything is defined). Note that the model-theoretic stabilizer $Stab(p)$ of p is equal to $Stab(X)$ as defined in the corollary. Also note that any definable set in p contains a nonempty Zariski-open subset of p. By Theorem 2.4, all realizations of p are contained in a single translate of $s(G)$. Thus the intersection of X with that translate is a definable set in p. Hence some nonempty Zariski open subset of X is contained in a translate of $s(G)$. As this translate is Zariski closed, all of X must be contained in that translate. The moreover clause follows as in [4]: After translation we may assume that $X \subseteq s(G)$. By 1.5 $s(G) = A + B$ where A is a connected algebraic group and B is a modular torus. The full orthogonality of A and B implies that $X = X_1 + X_2$, where $X_1 \subseteq A$ and $X_2 \subseteq B$ are irreducible Zariski-closed. By modularity of B X_2 is a translate of a subtorus B_0 of B. Clearly B_0 stabilizes X, so must be the identity. It follows that X_2 is a singleton, and so up to translation X is contained in A. This concludes the proof. $\square$

Remark 2.6 *The socle statement (i.e. Theorem 2.4) is not true for arbitrary commutative groups of finite Morley rank.*

Proof Here is the counterexample. It will actually be a reduct of an algebraic group. Let K be an algebraically closed field. Let G be an additive group $(K \times K \times K, +)$ equipped with a predicate S for $K \times K \times 0$ and with a predicate X for $\{(x, y \cdot x, y) : x \in K, y \in K^*\}$. Then $S = s(G)$ and $Stab(X) \cap S$ is trivial. $\square$

Question 2.7 *Is 2.4 true for differential algebraic groups of finite Morley rank?*

As an application of the above theorem we prove the analytic Mordell-Lang conjecture for cyclic subgroups of a meromorphic group. For a survey of the Mordell-Lang the reader is referred to [8]. By [5] the Mordell-Lang for cyclic subgroups of algebraic groups is equivalent to the truth of the 1-step conjecture introduced in [7] and [5]. By the same argument as in [5] the Mordell-Lang for cyclic subgroups of meromorphic groups is equivalent to the truth of the 1-step conjecture in T. The 1-step conjecture for a stable theory T says that for any group G interpretable in a saturated model of T, which is generated by realizations of a stationary type p, any infinite convergent sequence of powers of p converges to a generic type of G (an (independent) power of p is the type of the product of n independent realizations of p).

Theorem 2.8 *The 1-step conjecture is true in $Th(\mathcal{A})$, at least for groups definable over $\mathcal{A}$.*

Proof Suppose Γ is a cyclic subgroup of the connected meromorphic group G, X is an irreducible Zariski closed subset of G, and $X \cup \Gamma$ is Zariski dense in X. We want to prove that X is a coset of a meromorphic subgroup of G. We can assume that G is connected and commutative. By quotienting G by $Stab(X)$ we may assume $Stab(X)$ is trivial. By Corollary 2.5, up to translation, X is contained in a connected algebraic subgroup of G. So we can apply the truth of the Theorem for algebraic groups (see Theorem 3.2 in [5]). □

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