Forking in Differential Fields of Positive Characteristic

(joint with Omar León Sánchez and Amador Martin-Pizarro)

Piotr Kowalski

Instytut Matematyczny
Uniwersytetu Wrocławskiego

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Three notions of independence

There are very natural notions of independence in the following three classical cases.

1. Pure set X: the independence relation is non-equality.
2. A vector space V over a fixed field F: the independence relation is the F-linear independence.
3. An algebraically closed field K: the independence relation is the algebraic independence over the prime field.

In these cases the independence relation yields a ternary relation

$$A \underset{C}{\downarrow} B$$

(A, B, C subsets of the given universe) defined as: for any tuple a in A, if a is independent over C, then a is independent over $B \cup C$.

Good properties

Each of the ternary relations $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}$ from the previous slide satisfies the following "seven good properties" ($C \subseteq D \subseteq B, C \subseteq A$).

1. Invariance (under the automorphism group).
2. Extension: there is A' s.t. $\mathrm{tp}(A'/C) = \mathrm{tp}(A/C)$ and $A' \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C B$.
3. Symmetry: $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C B \iff B \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C A$.
4. Full Transitivity:

$$A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C D \text{ and } A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_D B \iff A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C B.$$

5. Finite Character: $A \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C B \iff \forall a \underset{\text{finite}}{\subseteq} A \quad a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_C B$.
6. Local Character: $\forall a \underset{\text{finite}}{\subseteq} A \ \exists B_0 \underset{\text{countable}}{\subseteq} B$ s.t. $a \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_{B_0} B$.
7. Stationarity over Models.

Forking in stable theories

These seven good properties axiomatize forking in stable theories in the following sense.

Theorem (Harnik-Harrington, Kim-Pillay)

Suppose that there is a ternary relation $\mathop{\smile\!\!\!\!\!\mid}$ on small subsets of a monster model of a complete theory T satisfying the seven properties from the previous slide.
Then, T is a stable theory and $\mathop{\smile\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\mid}^T$, where $\mathop{\smile\!\!\!\!\!\mid}^T$ is the forking-independence relation in T defined as:

$$A \mathop{\smile\!\!\!\!\!\mid}_C^T B \iff \mathrm{tp}^T(A/B \cup C) \text{ does not fork over } C.$$

Moreover, $\mathop{\smile\!\!\!\!\!\mid}^T$ satisfies these seven properties.

Forking in DCF_0

- Let DCF_0 denote the model companion of the theory of differential fields of characteristic 0.
- The theory DCF_0 is ω-stable and for small subsets A, B, C (such that $C \subseteq A \cap B$) of a monster model of DCF_0, we have

$$A \underset{C}{\overset{\mathrm{DCF}_0}{\downarrow}} B \iff \mathrm{acl}^{\mathrm{DCF}_0}(A) \underset{\mathrm{acl}^{\mathrm{DCF}_0}(C)}{\overset{\mathrm{ACF}_0}{\downarrow}} \mathrm{acl}^{\mathrm{DCF}_0}(B),$$

  which follows from quantifier elimination for DCF_0 (Blum).
- So, the forking independence relation in DCF_0 is explained by
  1. the algebraic closure in DCF_0,
     and
  2. the forking independence in ACF_0.

p-independence

- Let us fix (for the entire talk) a prime number p.
- Let F be a field of characteristic p and $A, B, C \subseteq F$.
- A is p-independent over C in F if for all $a \in A$ we have
$$a \notin F^p\left(C \cup (A \backslash \{a\})\right).$$
- A maximal set S which is p-independent over C in F is called a p-basis of F over C.
- Assume that $C \subseteq A \cap B$. Then, A is p-independent from B over C in F, if for all $\bar{a} \subseteq A$ we have:
$$\bar{a} \text{ is } p\text{-indep. over } C \text{ in } F \quad \Longrightarrow \quad \bar{a} \text{ is } p\text{-indep. over } B \text{ in } F.$$
- Unlike the algebraic disjointness or the linear disjointness, the p-independence relation depends on the ambient field F!

Forking in $\mathrm{SCF}_{p,\infty}$

- Let $\mathrm{SCF}_{p,\infty}$ denote the theory of separably closed fields K of characteristic p such that $[K : K^p] = \infty$.
- The theory $\mathrm{SCF}_{p,\infty}$ is complete (Ershov) and stable (Macintyre, Shelah, Wood). Fix a monster $\mathbb{U} \models \mathrm{SCF}_{p,\infty}$.
- Let $F \subset \mathbb{U}, K \subset \mathbb{U}, M \subset \mathbb{U}$ be separable field extensions such that $F \subseteq K \cap M$. Srour showed the following:

$$K \underset{F}{\overset{\mathrm{SCF}_{p,\infty}}{\downarrow}} M \iff K \underset{F}{\overset{p}{\downarrow}} M \text{ and } K \underset{F}{\overset{\mathrm{ACF}_p}{\downarrow}} M,$$

where $K \downarrow_F^p M$ means K is p-indep. from M over F in $\mathbb{U}$.

The theory DCF_p (Shelah, Wood)

- The theory of differential fields of characteristic p has a model companion denoted DCF_p.
- Wood axioms for $(\mathbb{U}, \partial) \models \mathrm{DCF}_p$:
  1. $\mathbb{U}^p = \ker(\partial) (=: C_{\mathbb{U}})$ (the differential perfectness of $\mathbb{U}$), and
  2. the necessary conditions on systems of algebraic differential (non-)equations in one variable over $\mathbb{U}$ to have solutions.
- The theory DCF_p is stable, complete, and expands $\mathrm{SCF}_{p,\infty}$.
- The theory DCF_p has quantifier elimination after adding a function symbol for the inverse of Frobenius.

My wrong claim and counterexample

- I wrongly claimed in my JSL2005 paper that, analogously to the DCF_0 case, the forking independence relation in DCF_p is explained by the algebraic closure in DCF_p and the forking independence relation in $\mathrm{SCF}_{p,\infty}$.
- Amador and Omar provided the following counterexample last year. Let us take $a, b \in \mathbb{U} \models \mathrm{DCF}_p$ such that

$$\partial(a) = 1 = \partial(b) \quad \text{and} \quad a \underset{}{\overset{\mathrm{DCF}_p}{\downarrow}} b.$$

Then, we have

$$a, b \notin \mathbb{U}^p \quad \text{and} \quad a - b \in C_{\mathbb{U}} = \mathbb{U}^p.$$

Hence a is not p-independent from b (over $\mathbb{F}_p$ in $\mathbb{U}$), thus

$$a \overset{\mathrm{SCF}_{p,\infty}}{\not\downarrow} b!$$

p-independence in the differential context

- The following is folklore. If $F \subseteq K \subseteq \mathbb{U}$ and $F \subseteq M \subseteq \mathbb{U}$ are all $\mathrm{acl}^{\mathrm{DCF}_p}$-closed differential fields, then

$$K \underset{F}{\overset{p}{\downarrow}} M \quad \Longleftrightarrow \quad \text{the compositum } KM \text{ is differentially perfect.}$$

- However, we know that the p-independence is too strong to describe the forking independence in DCF_p.
- Hence, we cannot demand KM to be differentially perfect in attempts of such a description.
- Therefore, we are looking for an appropriate weakening of the notion of a differentially perfect field (the next slide).

Differential Transcendental Imperfectness

Definition

A differential subfield $K \subseteq \mathbb{U}$ has (differentially) transcendental (differential) imperfectness, abbreviated trim, if there is a p-basis S of $C_K^{1/p}$ over K such that the tuple $(\partial(s) \mid s \in S)$ is differentially algebraically independent over K, that is the tuple

$$\left(\partial^i(t) \mid t \in S, i > 0\right)$$

is algebraically independent over K.

The following is crucial.

Theorem (Kolchin, slightly restated)

If K has trim and S is a p-basis of $C_K^{1/p}$ over K, then $K\langle S\rangle$ (differential field generated by S over K) is differentially perfect.

Our independence notion

- The notion of trim depends on a chosen embedding into $\mathbb{U}$.
- In the definition of trim, "there is a p-basis" can be replaced with "for all p-bases".
- Obviously, each differentially perfect field has trim.

Definition ($A, B, C \subset \mathbb{U}, C \subseteq A \cap B$)

We say that the ternary relation $A \mathop{\downarrow}_C^* B$ holds if

$$\mathrm{acl}^{\mathrm{DCF}_p}(A) \mathop{\downarrow}_{\mathrm{acl}^{\mathrm{DCF}_p}(C)}^{\mathrm{ACF}_p} \mathrm{acl}^{\mathrm{DCF}_p}(B),$$

and the following differential compositum field has trim

$$\mathrm{acl}^{\mathrm{DCF}_p}(A)\,\mathrm{acl}^{\mathrm{DCF}_p}(B).$$

Main Theorem

We showed the following.

Theorem

The ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{*}$ satisfies the "seven good properties", where Stationarity also holds over $\mathrm{acl}^{\mathrm{DCF}_p}$-closed sets (in the home sort). Therefore, we have

$$\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{DCF}_p} = \mathop{\smile\!\!\!\!\!\!|}^{*}$$

and the DCF_p-types over algebraically closed sets (in the home sort) are stationary.

Over some special differential fields, the (wrong in general) description from the JSL2005 paper still holds (the next slide).

Result of Shelah

While proving the stability of DCF_p, Shelah proved the following.

Theorem (Shelah)

Let $M \subseteq K, M \subseteq F$ be differential subfields of $\mathbb{U}$ such that $M \prec \mathbb{U}$ (we observed that M being "PV-closed" is enough). If K and F are differentially perfect and K is linearly disjoint from F over M, then KF is differentially perfect.

It implies the over models, or even "PV-closed" differential fields, we have the following for $M \subseteq A \cap B$ (A, B are $\mathrm{acl}^{\mathrm{DCF}_p}$-closed):

$$A \underset{M}{\overset{\mathrm{SCF}_{p,\infty}}{\downarrow}} B \iff A \underset{M}{\overset{\mathrm{DCF}_p}{\downarrow}} B \iff A \underset{M}{\overset{\mathrm{ACF}_p}{\downarrow}} B.$$

Bernoulli differential equation

- General form (Bernoulli, 1695).

$$y' + P(x)y = Q(x)y^n,$$

where we can consider P and Q as polynomials.

- Leibniz used (also in 1695) the following substitution

$$u = y^{1-n}$$

to reduce any Bernoulli equation to a linear differential equation (for $n \neq 1$).

- Hence, in DCF_0, the solution set of any Bernoulli equation is almost internal to the field of constants.
- We just consider the case of $y' = y^n$.

Bernoulli differential equation in positive characteristic

- If we consider $y' = y^n$ in characteristic $p > 0$, then Leibniz method works for $n \leqslant p$ and gives a finite-to-one definable function from the set of solutions of $y' = y^n$ onto $\mathbb{U}$.
- This method stops working at the case of $y' = y^{p+1}$ and there are interesting model-theoretic reasons for that.

Theorem

Let us define

$$V := \{a \in \mathbb{U}\backslash\{0\} \mid \partial(a) = a^{p+1}\}.$$

Then, V is strictly disintegrated, that is V has the induced structure of a pure set. In particular, V is strongly minimal.

Note that $\mathrm{SCF}_{p,\infty}$-definable infinite sets are not superstable.

Bernoulli in positive characteristic and transcendence

- In particular, DCF_p-forking on V comes from equality.
- By the description of forking in DCF_p, we obtain the following.

Theorem

If $a_1, \ldots, a_n \in \mathbb{U}\setminus\{0\}$ are pairwise different and such that

$$\partial(a_1) = a_1^{p+1}, \ldots, \partial(a_n) = a_n^{p+1},$$

then we have

$$\mathrm{trdeg}_{\mathbb{F}_p}\mathbb{F}_p(a_1, \ldots, a_n) = n$$

(and more, since $a_1, \ldots, a_n$ are also p-independent in this case).

Can we obtain any interesting Ax-Schanuel statements in positive characteristic using such transcendence results?

$\mathrm{DCF}_p\mathrm{A}$

Theorem (Scanlon-Li)

Any strongly minimal and geometrically trivial set definable in DCF_0 *remains minimal in* $\mathrm{DCF}_0\mathrm{A}$*, which is the model companion of the theory of differentially closed fields of characteristic 0 with a differential automorphism.*

Theorem (Ino-León Sánchez)

The theory $\mathrm{DCF}_p\mathrm{A}$ *exists.*

One can ask whether a positive characteristic version of the result of Scanlon-Li holds, which may be tested on the solution set of the differential equation $\partial(x) = x^{p+1}$.

General criterion for stability

Let $L \subseteq L'$, T be an L-theory, T' be an L'-theory, and $T \subseteq T'$.

Theorem (inspired by Shelah, expanding on a result from JSL2005)

The conditions listed below imply that the theory T' is stable.

1. *The theory T is stable with forking-independence relation $\mathop{\smile\!\!\!\!\!\!\,|}$.*
2. *The theory T' has quantifier elimination.*
3. ($\mathop{\smile\!\!\!\!\!\!\,|}$*-coproduct condition) Suppose that*
   - $M, M', N \models T'$;
   - $M \cap N = M' \cap N =: N_0$;
   - $M \mathop{\smile\!\!\!\!\!\!\,|}_{N_0} N$ *and* $M' \mathop{\smile\!\!\!\!\!\!\,|}_{N_0} N$;
   - *there is an L'_{N_0}-isomorphism over* $f : M \xrightarrow{\cong} M'$.

   Then, f extends to an L'_N-isomorphism $\tilde{f} : MN \xrightarrow{\cong} M'N$.

It gives stability of $T' = \mathrm{DCF}_{p,m}$ (for $T = \mathrm{SCF}_{p,\infty}$) and maybe applies to "all stable theories of fields with operators".