

PAC FIELDS AND DIFFERENCE FIELDS WITH GENERIC AUTOMORPHISMS

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ABSTRACT. We study the existence of model companions after adjoining an automorphism to field theories in two complementary settings. For bounded PAC fields of characteristic zero containing all roots of unity and admitting a cyclic Galois quotient of prime order, a suitable choice of countably many constants gives a model-complete field theory which has no model companion after adding an automorphism fixing the named constants. This applies, in particular, when the absolute Galois group is free profinite or free pro- p of finite positive rank. Secondly, for every completion of ACFA in every characteristic, adjoining a second commuting automorphism has no model companion. The two explicit obstructions use a common ordered product of inertia automorphisms along an infinite orbit, combined respectively with an equivariant PAC completion and an algebraically closed difference-field embedding.

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1. INTRODUCTION

The existence of ACFA, the model companion of the theory of fields with an automorphism, naturally raises the question of adjoining further automorphisms. Hrushovski showed in the 1990s that the theory of fields with two commuting automorphisms has no model companion; accounts of his argument appear in [12, Section 3] and [5, Section 1.11]. Chatzidakis and Pillay asserted in the last two bullet points of [7, Section 3.11] that analogous nonexistence results hold for ACFA and for the theory of pseudofinite fields expanded by an automorphism.

Chatzidakis subsequently gave a variant of Hrushovski’s proof in [5, Section 1.11], but emphasized in [5, Remark 1.12] that this argument does not by itself settle the questions for pseudofinite fields or completions of ACFA. The difficulty is to realize the required difference equations in an extension whose reduct still satisfies the prescribed theory. Our proofs follow Chatzidakis’s strategy, comparing an existential condition forbidden at certain

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finite periods with its realization along an infinite orbit. In the ACFA case, the construction requires extending two commuting automorphisms to an algebraically closed field while preserving commutativity. In the pseudofinite case, the corresponding construction must also respect the procyclic Galois action.

We now state the two main results. Recall that a field is *PAC* if every absolutely irreducible variety over it has a rational point. A field F is *bounded* if it has only finitely many finite separable extensions of each degree inside a fixed separable closure F^{sep} . We write $\text{Gal}(F)$ for the absolute Galois group of F . For a prime ℓ , let C_ℓ denote the cyclic group of order ℓ .

Theorem 1.1 (No bounded PACA). *Let F be a bounded PAC field of characteristic zero containing all roots of unity. Suppose that, for some prime ℓ , there is a continuous surjection*

$$\text{Gal}(F) \twoheadrightarrow C_\ell.$$

Then there is a countable tuple $\mathbf{d}$ from F such that $\text{Th}(F, \mathbf{d})$ is model-complete and the theory

$$\text{Th}(F, \mathbf{d}) \cup \{\sigma \text{ is a field automorphism and } \sigma(\mathbf{d}) = \mathbf{d}\}$$

has no model companion.

Under the stated field hypotheses, the theorem includes absolute Galois groups that are free profinite or free pro- p of finite positive rank; see Corollary 4.9.

The pseudofinite case, corresponding to $\text{Gal}(F) \cong \widehat{\mathbb{Z}}$ and included here under the same hypotheses, is quite special and will be treated separately in Feyza Nur Okudan's PhD thesis.

Our second result concerns an additional automorphism of a model of a fixed completion of ACFA.

Theorem 1.2 (No ACFAA). *Let T be a completion of ACFA (arbitrary characteristic) and let σ be its distinguished automorphism. Then the theory*

$$T \cup \{\tau \text{ is a field automorphism and } \tau\sigma = \sigma\tau\}$$

has no model companion in the language expanded by a new function symbol τ .

For a theory T , we write T_τ for the theory of models of T equipped with an automorphism τ . Our results support the conjecture that T_τ has no model companion whenever T is unstable, complete, and model-complete [12, Introduction]. Kikyo proved this assuming amalgamation for T_τ [11, Theorem 3.3]. Failure of amalgamation for ACFA_τ is reported in [11, Section 4] and [13, Introduction], but these references do not establish it for every completion in every characteristic. Chatzidakis and Pillay asked whether the theory of pseudofinite fields has the amalgamation property for automorphisms (PAPA) [7, end of Section 3]. Kikyo and Shelah subsequently stated that it was unknown whether the theory of pseudofinite fields with an automorphism has the amalgamation property [13, Introduction]. A related difficulty concerning the existence of model companions for complete theories of pseudofinite fields with an automorphism is discussed in [5, Remark 1.12].

The paper is organized as follows. In Section 2, we develop the common algebraic tools. In Section 3, we recall Chatzidakis's variant of Hrushovski's argument and explain the proof strategy, including the requirement that realizing extensions preserve the prescribed reduct theory. In Sections 4 and 5, we prove Theorems 1.1 and 1.2, respectively. In Section 6, we discuss the transfer of model companions to completions and the relation between model-completeness and the existence of weakly saturated existentially closed models.

Use of AI. The authors used ChatGPT (GPT-6 Astra in Pro mode) to assist in identifying examples, developing proofs, and drafting portions of the text. The authors subsequently refined and streamlined the arguments and assume full responsibility for the final content.

2. COMMON CONVENTIONS AND ALGEBRAIC TOOLS

This section collects the algebraic tools used in the subsequent arguments. We first recall the required facts about Kummer extensions, Puiseux and Hahn series, and ramification. We then construct commuting lifts of automorphisms, describe the behavior of inertia under restriction and conjugation, and prove a convergence result for ordered products in profinite groups. Together, these tools allow us to prescribe the action on a chosen root in a Kummer extension while preserving commutation with another automorphism, providing the algebraic input for the realization arguments in Sections 3, 4, and 5.

2.1. Preliminaries from field theory and valuation theory. We collect here necessary classical algebraic facts focusing on Kummer theory and ramification theory.

Let F be a field. For every positive integer n , we denote

$$\mu_n(F) := \{a \in F^\times \mid a^n = 1\}, \quad \mu_\infty(F) := \bigcup_{n \geq 1} \mu_n(F).$$

Thus $\mu_\infty(F)$ is the multiplicative group of all roots of unity in F . In characteristic zero, we use the abbreviations

$$\mu_n := \mu_n(\mathbb{Q}^{\text{alg}}), \quad \mu_\infty := \mu_\infty(\mathbb{Q}^{\text{alg}}),$$

where $\mathbb{Q}^{\text{alg}}$ is a fixed algebraic closure of $\mathbb{Q}$.

A field is *bounded* if it has only finitely many separable extensions of each finite degree inside a fixed separable closure. Equivalently, its absolute Galois group is small, that is, it has only finitely many open subgroups of each finite index.

We recall two facts from Kummer theory. Let K be a field, let $\ell \neq \text{char}(K)$ be prime, and suppose that K contains a primitive ℓ -th root of unity ω .

- (i) If $c \in K^\times \setminus (K^\times)^\ell$ and $\beta^\ell = c$, then $K(\beta)/K$ is cyclic of degree ℓ , with automorphisms

$$\beta \mapsto \omega^j \beta, \quad j \in \mathbb{Z}/\ell\mathbb{Z}.$$

Conversely, every cyclic extension of K of degree ℓ is obtained in this way and

$$\text{Gal}(K) \twoheadrightarrow C_\ell \iff K^\times / (K^\times)^\ell \neq \{1\},$$

where $\text{Gal}(K)$ denotes the absolute Galois group and the surjection is required to be continuous [17, Section 9.24, Tag 09I6].

- (ii) Suppose that the classes of $f_1, \dots, f_r \in K^\times$ are linearly independent in the $\mathbb{F}_\ell$ -vector space $K^\times / (K^\times)^\ell$. Choose α_i with $\alpha_i^\ell = f_i$ and put $E = K(\alpha_1, \dots, \alpha_r)$. Then

$$[E : K] = \ell^r, \quad \text{Gal}(E/K) \cong (C_\ell)^r,$$

and the monomials

$$\alpha_1^{e_1} \cdots \alpha_r^{e_r}, \quad 0 \leq e_i < \ell,$$

form a K -basis of E . Every assignment $\alpha_i \mapsto \omega^{j_i} \alpha_i$ therefore defines a K -automorphism of E . For an infinite independent family $(f_i)_{i \in I}$, the same assertions apply to every finite subfamily, and arbitrary choices of the multipliers define an automorphism of the field generated by all the radicals. More generally, if $\gamma \in \text{Aut}(K)$ and $\gamma(f_i) = f_{\pi(i)}$ for a permutation π of I , then any assignments

$$\alpha_i \mapsto \xi_i \alpha_{\pi(i)}, \quad \xi_i \in \mu_\ell(K),$$

extend γ to an automorphism of that field. These are consequences of the Kummer correspondence [14, Theorem 5.30 and Remark 5.32].

Let k be a field and x an indeterminate. The *Puiseux series field* is

$$P(k, x) := \bigcup_{m \geq 1} k((x^{1/m})),$$

where the fractional powers are chosen compatibly. The *Hahn series field*

$$H(k, x) := k((x^{\mathbb{Q}}))$$

consists of formal sums $\sum_{r \in \mathbb{Q}} a_r x^r$ with $a_r \in k$ and well-ordered support $\{r \in \mathbb{Q} \mid a_r \neq 0\}$, equipped with the usual series operations. There is a natural embedding $P(k, x) \hookrightarrow H(k, x)$ whose image consists precisely of the series with support contained in $\frac{1}{m}\mathbb{Z}$ for some $m \geq 1$. If k is algebraically closed of characteristic zero, the Newton–Puiseux theorem says that $P(k, x)$ is an algebraic closure of the field of Laurent series $k((x))$. In characteristic $p > 0$, however, $P(k, x)$ is not algebraically closed: the equation $Y^p - Y = x^{-1}$ has the Hahn-series solution

$$\sum_{n \geq 1} x^{-1/p^n} \in H(k, x) \setminus P(k, x);$$

see [16, Section 1]. For algebraically closed k in any characteristic, $H(k, x)$ is algebraically closed, since its value group $\mathbb{Q}$ is divisible [16, Corollary 4].

Let k be algebraically closed, put $K = k(x)$, and fix $\Omega = K^{\text{alg}}$. For $a \in k$, the *point valuation* $v_a = \text{ord}_{x=a}$ is normalized by $v_a(x - a) = 1$. At infinity use $v_\infty = \text{ord}_{u=0}$, where $u = x^{-1}$. Their valuation rings are $k[x]_{(x-a)}$ and $k[u]_{(u)}$, respectively, and their residue fields are k .

For a valuation w on a field E in general, we write

$$\mathcal{O}_w = \{z \in E : w(z) \geq 0\}, \quad \mathfrak{m}_w = \{z \in E : w(z) > 0\}, \quad \kappa(w) = \mathcal{O}_w / \mathfrak{m}_w,$$

with $w(0) = \infty$. If $K \subseteq E$ is a field extension, then a *prolongation* of a valuation v of K to E is a valuation w with $\mathcal{O}_w \cap K = \mathcal{O}_v$. For a finite extension E/K , let w prolong a point valuation v of K , and normalize w so that $w(E^\times) = \mathbb{Z}$. The *ramification index* is the positive integer

$$e(w/v) := [w(E^\times) : w(K^\times)].$$

Since $v(K^\times) = \mathbb{Z}$, this means that

$$w(f) = e(w/v) v(f) \quad \text{for all } f \in K^\times.$$

In particular, at a finite point a we have $e(w/v_a) = w(x - a)$.

We often just say “point” for “point valuation”. Since the residue fields above a point are finite extensions of the algebraically closed field k , they are again k . In this setting the point is *unramified* in E/K if every prolongation has $e(w/v) = 1$, and *ramified* otherwise. It is *totally ramified* if there is a unique prolongation and its ramification index is $[E : K]$.

Every valuation on a field admits a prolongation to any algebraic field extension [17, Tag 00IA]. In an algebraic extension, the induced residue-field extension is algebraic; this follows from the finite case by passing to finite subextensions. Consequently, every prolongation of a point valuation of K to Ω has residue field k , since k is algebraically closed.

Choose a prolongation of a point valuation v on K to $\tilde{v}$ on Ω . For $G = \text{Gal}(\Omega/K)$, its *decomposition group* and *inertia group* are

$$D_{\tilde{v}} = \{g \in G : g(\mathcal{O}_{\tilde{v}}) = \mathcal{O}_{\tilde{v}}\}, \quad I_{\tilde{v}} = \ker(D_{\tilde{v}} \rightarrow \text{Aut}(\kappa(\tilde{v})/k)).$$

The same definitions apply to a finite Galois extension E/K , using a prolongation w on E and $G = \text{Gal}(E/K)$. An *inertia automorphism above a* is an element of $I_{\tilde{v}}$ for a chosen prolongation of v_a .

For a finite Galois extension E/K , the group $\text{Gal}(E/K)$ acts transitively on the prolongations of v_a to E , with g sending w to $w \circ g^{-1}$. These prolongations correspond to the points above a on the normalization of $\mathbb{P}_k^1$ in E , that is the nonsingular projective curve C with function field $k(C) = E$, together with the finite morphism $\pi : C \rightarrow \mathbb{P}_k^1$ corresponding to the inclusion $k(x) \subseteq E$.

The degree formula for the finite normalization of the curve gives

$$[E : K] = \sum_{w|v_a} e(w/v_a) [\kappa(w) : k].$$

All residue degrees are 1, and the ramification indices are equal. Consequently,

$$|I_w(E/K)| = |D_w(E/K)| = e(w/v_a).$$

In particular, inertia at an unramified point is trivial.

Let E/K be a finite Galois extension inside Ω , let $\tilde{v}$ be a prolongation to Ω of a point valuation v of K , and let w be its restriction to E . Restriction of automorphisms induces surjective homomorphisms

$$D_{\tilde{v}} \twoheadrightarrow D_w(E/K), \quad I_{\tilde{v}} \twoheadrightarrow I_w(E/K)$$

[17, Tag 0BSX]. In particular, every element of the finite inertia group $I_w(E/K)$ lifts to an element of $I_{\tilde{v}}$.

For a monic polynomial $P(T) \in K[T]$ of degree n , with roots $\alpha_1, \dots, \alpha_n$ in an algebraic closure of K , counted with multiplicity, its *discriminant* is

$$\text{disc}(P) := \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2.$$

We shall use the following standard discriminant criterion [17, Tags 0BJF, 0EXR, and 0BTF]. Let (K, v) be a discretely valued field with valuation ring $\mathcal{O}_v$, let $P \in \mathcal{O}_v[T]$ be monic and separable, and let E/K be its splitting field. If $v(\text{disc}(P)) = 0$, then $I_w(E/K) = \{1\}$ for every prolongation w of v to E , and hence E/K is unramified at v .

2.2. Some algebraic results. We need the following result about commuting lifts which will be used in the proof of Claim 1 from the proof of Lemma 4.8 (corresponding to (A) from Section 3). The following construction is a variant of the commuting actions on generalized power series used in [3, Theorem 7]; see also [1, Section 2.1].

Lemma 2.1. *Let k be algebraically closed of characteristic zero, and let $\rho, (\gamma_i)_{i \in I} \in \text{Aut}(k)$ satisfy $\rho\gamma_i = \gamma_i\rho$ for every i . Let x be an indeterminate, put $A := P(k, x)$, and let Ω be the relative algebraic closure of $k(x)$ in A . For any character $\epsilon : \mathbb{Q} \rightarrow k^\times$ whose image is fixed pointwise by every γ_i , the formulas*

$$s\left(\sum_r a_r x^r\right) = \sum_r \rho(a_r) \epsilon(r) x^r, \quad d_i\left(\sum_r a_r x^r\right) = \sum_r \gamma_i(a_r) x^r$$

define automorphisms of A extending ρ, γ_i and satisfying $sd_i = d_i s$. They preserve Ω , which is an algebraic closure of $k(x)$; their restrictions are denoted by the same letters.

If every γ_i fixes $\mu_\infty(k)$ pointwise, then ϵ may be chosen with values in $\mu_\infty(k)$ as follows:

- (i) *For any prescribed $\zeta \in \mu_\infty(k)$, one can arrange $\epsilon(1) = \zeta$, giving $s(x) = \zeta x$.*
- (ii) *If $\rho = \text{id}_k$, then for any prime ℓ one can arrange $\epsilon(1) = 1$ and $\epsilon(1/\ell^m)$ of exact order ℓ^m for every $m \geq 1$. Then s fixes $k(x)$ pointwise, and $b_m := x^{1/\ell^m} \in \Omega$ is fixed by every d_i and has exact s -period ℓ^m .*

Proof. Newton–Puiseux makes A algebraically closed, hence Ω is an algebraic closure of $k(x)$. The character identity makes s multiplicative; its inverse replaces each coefficient a_r by $\rho^{-1}(a_r \epsilon(r)^{-1})$, while d_i^{-1} acts coefficientwise by γ_i^{-1} . The hypotheses give $sd_i = d_i s$. Since $s(x) = \epsilon(1)x$ and $d_i(x) = x$, these maps preserve $k(x)$ and hence Ω .

For the additional assertions, $\mu_\infty(k)$ is divisible and hence injective as an abelian group, so every character from a subgroup of $\mathbb{Q}$ to $\mu_\infty(k)$ extends to $\mathbb{Q}$. Extending $n \mapsto \zeta^n$ from $\mathbb{Z}$ proves (i). For (ii), choose primitive ℓ^m -th roots ξ_m with $\xi_0 = 1$ and $\xi_{m+1}^\ell = \xi_m$, and extend the character $a/\ell^m \mapsto \xi_m^a$ from $\mathbb{Z}[1/\ell]$. Then $\epsilon(1) = 1$, and, since $\rho = \text{id}_k$,

$$d_i(b_m) = b_m, \quad s^n(b_m) = \xi_m^n b_m.$$

Thus each b_m has exact period ℓ^m . □

Remark 2.2. Lemma 2.1 holds in arbitrary characteristic after replacing A by the Hahn field $H := k((x^{\mathbb{Q}}))$ and requiring $\ell \neq \text{char}(k)$ in part (ii). Indeed, H is algebraically closed [16, Corollary 4], and the same formulas preserve supports and define automorphisms preserving the relative algebraic closure of $k(x)$ in H . The character constructions remain valid because $\mu_{\infty}(k)$ is divisible and, for $\ell \neq \text{char}(k)$, coherent primitive ℓ^m -th roots exist for all m .

We need the following results about inertia and finite ramification which will be used in the proof of Claim 2 from Lemma 4.8.

Lemma 2.3. *Let k be algebraically closed, x be transcendental over k , $K = k(x)$, and $\Omega = K^{\text{alg}}$.*

- (i) *Every finite Galois extension E/K inside Ω ramifies at only finitely many points of $\mathbb{P}^1(k)$.*
- (ii) *If a finite Galois extension E/K inside Ω is unramified at $a \in k$, then every inertia automorphism in $\text{Gal}(\Omega/K)$ above a fixes E pointwise.*
- (iii) *If $a \neq d$ and $\ell \neq \text{char}(k)$ is prime, then*

$$K(y)/K, \quad y^{\ell} = \frac{x-a}{x-d},$$

is cyclic of degree ℓ , totally ramified at a, d , and unramified elsewhere.

Moreover, for a prescribed primitive ℓ -th root $\zeta \in k$, an inertia automorphism above a can be chosen to send y to ζy .

- (iv) *Let $s \in \text{Aut}(\Omega)$ satisfy*

$$s(k) = k, \quad s(x) = \zeta x, \quad s(\zeta) = \zeta$$

for some $\zeta \in k^{\times}$. If g is an inertia automorphism above $a \in k$, then, for every $j \in \mathbb{Z}$, the conjugate $s^j g s^{-j}$ is an inertia automorphism above $\zeta^{-j} s^j(a)$.

Proof. For (i), choose an integral primitive element with monic minimal polynomial $P \in k[x][Y]$. Its discriminant is a nonzero polynomial in x . All finite ramified points lie among its zeros; there is only one additional possible point, infinity.

For (ii), restriction sends inertia into inertia. Since E/K is unramified at a , its inertia groups above a are trivial. Hence every inertia automorphism in $\text{Gal}(\Omega/K)$ above a fixes E pointwise.

For (iii), put

$$f := \frac{x-a}{x-d}, \quad E := K(y), \quad y^{\ell} = f.$$

Since $v_a(f) = 1$, the element f is not an ℓ -th power in K . The Kummer facts recalled in Section 2.1 therefore show that E/K is cyclic of degree ℓ and is the splitting field of $P(T) := T^{\ell} - f$.

Let v be either v_a or v_d , and let w be a prolongation to E , normalized by $w(E^{\times}) = \mathbb{Z}$. The valuation identity from Section 2.1 gives

$$\ell w(y) = w(f) = e(w/v)v(f).$$

Since $v_a(f) = 1$ and $v_d(f) = -1$, we obtain $\ell \mid e(w/v)$. As $e(w/v) \leq [E : K] = \ell$, it follows that $e(w/v) = \ell$. The degree formula then shows that there is a unique prolongation at each of a and d , so both points are totally ramified.

At every other point, including infinity, we have $v(f) = 0$. Thus $P \in \mathcal{O}_v[T]$ and

$$\text{disc}(P) = \pm \ell^{\ell} f^{\ell-1}$$

is a unit in $\mathcal{O}_v$, since $\ell \neq \text{char}(k)$. The discriminant criterion from Section 2.1 shows that E/K is unramified at every such point.

Finally, the inertia group above a is the whole group $\text{Gal}(E/K)$. In particular, it contains the automorphism sending y to ζy . By the surjectivity of restriction on inertia groups recalled in Section 2.1, this automorphism lifts to an inertia automorphism in $\text{Gal}(\Omega/K)$ above a .

For (iv), choose a prolongation v of v_a to Ω such that $g \in I_v$. Fix $j \in \mathbb{Z}$, and put

$$a_j := \zeta^{-j} s^j(a), \quad w := v \circ s^{-j}.$$

Since $s(K) = K$ and $s^j(x - a) = \zeta^j(x - a_j)$, we have

$$\mathcal{O}_w = s^j(\mathcal{O}_v), \quad \mathcal{O}_w \cap K = s^j(k[x]_{(x-a)}) = k[x]_{(x-a_j)}.$$

Thus w prolongs v_{a_j} . The conjugate $s^j g s^{-j}$ fixes K and preserves $\mathcal{O}_w$. Its residue action is trivial because $\kappa(w)$ is algebraic over the algebraically closed field k , hence equals k . Therefore $s^j g s^{-j} \in I_w$, as required. $\square$

The following lemma constructs an element fixed by a continuous automorphism of a profinite group as an ordered product along an orbit. Recall that a sequence in a profinite group converges to x if and only if its image in every finite continuous quotient is eventually equal to the image of x .

Lemma 2.4. *Let G be a profinite group, and let $s : G \rightarrow G$ be a continuous automorphism. Let $g_0 \in G$ and put $g_i = s^i(g_0)$ for $i \in \mathbb{Z}$. Suppose that, in every finite continuous quotient of G , all but finitely many g_i have trivial image. Then*

$$\eta = \lim_{m \rightarrow \infty} P_m, \quad P_m = g_{-m} g_{-m+1} \cdots g_m,$$

exists in G and satisfies $s(\eta) = \eta$. In any finite continuous quotient in which all g_i for $i \neq 0$ are trivial, the image of η is the image of g_0 .

Proof. In a fixed finite quotient only finitely many factors can be nontrivial, so the images of P_m eventually stabilize. These limits are compatible with the quotient maps and hence define an element η of the inverse limit G . The hypothesis also gives

$$\lim_{m \rightarrow \infty} g_m = 1 = \lim_{m \rightarrow \infty} g_{-m}.$$

Since $s(g_i) = g_{i+1}$, we have

$$\begin{aligned} s(P_m) &= s(g_{-m}) s(g_{-m+1}) \cdots s(g_m) \\ &= g_{-m+1} \cdots g_m g_{m+1} \\ &= g_{-m}^{-1} P_m g_{m+1}. \end{aligned}$$

Continuity and passage to the limit give $s(\eta) = \eta$. The final assertion follows by computing the product in the specified finite quotient. $\square$

The proof of Lemma 2.4 has a formal analogy with Kikyo's use of average types in [11, Theorems 3.1 and 3.3]. There, eventual truth along an indiscernible orbit yields a complete type invariant under the automorphism; this permits extending the automorphism so that it fixes a realization of that type. Here, eventual stabilization in every finite quotient yields an element $\eta \in G$ fixed by s : shifting the ordered products changes only their boundary factors, which tend to the identity. The related proof of [13, Theorem 1] instead stabilizes families of formulas and uses compatible types to extend an automorphism; it does not explicitly use average types. These model-theoretic constructions feed into contradictions involving existential closedness and compactness, while Lemma 2.4 supplies the commuting automorphism needed in our realization arguments.

3. CHATZIDAKIS–HRUSHOVSKI ARGUMENT FOR TWO COMMUTING AUTOMORPHISMS

In [5, Section 1.11], Chatzidakis gives a variant of Hrushovski's proof that the theory of fields with two commuting automorphisms has no model companion. We recall the argument in a form that isolates three ingredients used in our proofs:

- (A) realization at an infinite orbit;
- (B) an obstruction at certain finite orbits;
- (C) the construction, in a single model, of elements with arbitrarily large finite orbits to which the obstruction applies.

These are supplied by Lemmas 3.1, 3.2, and 3.3, respectively.

For the first two lemmas, we fix a field with two commuting automorphisms (K, σ, τ) of characteristic zero containing a primitive cube root of unity ω such that

$$\sigma(\omega) = \omega^2, \quad \tau(\omega) = \omega.$$

Using ω as a parameter, we define the following formula with free variable z :

$$\Phi_{\text{comm}}(z) := \exists x \exists y \left[y \neq 0 \wedge y^3 = x + z \wedge \tau(y) = \omega y \wedge \sigma(x) = x \wedge \tau(x) = x \right].$$

The condition $y \neq 0$ excludes the degenerate witness $x = -z$ and $y = 0$. We use the same letters for automorphisms and their extensions, and we also fix $b \in K$.

Lemma 3.1 (A: realization at an infinite orbit). *Suppose that $\tau(b) = b$ and that b has infinite σ -orbit. Then there is an extension*

$$(K, \sigma, \tau) \subseteq (L, \sigma, \tau)$$

such that $(L, \sigma, \tau) \models \Phi_{\text{comm}}(b)$.

Proof. Adjoin an indeterminate x fixed by both automorphisms. For each $i \in \mathbb{Z}$, choose α_i with $\alpha_i^3 = x + \sigma^i(b)$, and put

$$L := K(x)(\alpha_i : i \in \mathbb{Z}).$$

The classes of the elements $x + \sigma^i(b)$ are linearly independent in the $\mathbb{F}_3$ -vector space $K(x)^\times / (K(x)^\times)^3$. Indeed, the valuation at $x = -\sigma^i(b)$ takes value 1 on $x + \sigma^i(b)$ and value 0 on every other member of this family. Hence, Kummer theory allows us to define extensions of σ and τ to L by

$$\sigma(\alpha_i) = \alpha_{i+1}, \quad \tau(\alpha_i) = \sigma^i(\omega)\alpha_i.$$

These extensions commute on $K(x)$ and on every α_i , because

$$\sigma\tau(\alpha_i) = \sigma^{i+1}(\omega)\alpha_{i+1} = \tau\sigma(\alpha_i).$$

Thus (x, α_0) witnesses that $(L, \sigma, \tau) \models \Phi_{\text{comm}}(b)$. □

Lemma 3.2 (B: the finite-orbit obstruction). *Suppose that $\sigma^n(b) = b$ for some integer $n \geq 1$. If there is an extension*

$$(K, \sigma, \tau) \subseteq (L, \sigma, \tau)$$

such that $(L, \sigma, \tau) \models \Phi_{\text{comm}}(b)$, then n is even. In particular, if b has odd σ -period, the formula $\Phi_{\text{comm}}(b)$ cannot hold in any such extension.

Proof. Let $x, y \in L$ satisfy the conditions in $\Phi_{\text{comm}}(b)$. Since $\sigma^n(x+b) = x+b$ and $y^3 = x+b$, there is $j \in \{0, 1, 2\}$ such that

$$\sigma^n(y) = \omega^j y.$$

Commutativity gives

$$\omega^{j+1}y = \tau\sigma^n(y) = \sigma^n\tau(y) = \omega^{2^n+j}y.$$

As $y \neq 0$ and ω has order 3, we obtain $2^n \equiv 1 \pmod{3}$, so n is even. □

Put

$$J := \{n \in \mathbb{N} \mid n \geq 3 \text{ and } n \text{ is odd}\}.$$

Lemma 3.3 (C: existence of finite orbits). *There is a characteristic-zero difference field (K_0, σ, τ) containing a primitive cube root of unity ω such that*

$$\sigma(\omega) = \omega^2, \quad \tau = \text{id}_{K_0},$$

and, for every $n \in J$, there is $b_n \in K_0$ having exact σ -period n .

Proof. Choose a primitive cube root of unity ω and a family of algebraically independent elements

$$\{b_{n,j} \mid n \in J, 0 \leq j < n\}$$

over $\mathbb{Q}$. Let us set

$$K_0 := \mathbb{Q}(\omega)(b_{n,j} \mid n \in J, 0 \leq j < n)$$

and let $\tau = \text{id}_{K_0}$. Extend the automorphism $\omega \mapsto \omega^2$ of $\mathbb{Q}(\omega)$ to K_0 by setting

$$\sigma(b_{n,j}) = \begin{cases} b_{n,j+1}, & 0 \leq j < n-1, \\ b_{n,0}, & j = n-1. \end{cases}$$

These assignments define an automorphism because they permute an algebraically independent generating family. Since τ is the identity, σ and τ commute. For each $n \in J$, put $b_n := b_{n,0}$. The elements $b_{n,0}, \dots, b_{n,n-1}$ are distinct, so b_n has exact σ -period n . Moreover, $\tau(b_n) = b_n$ for every $n \in J$, as required. $\square$

Proof of non-existence of a model companion. Let S be the theory of fields with two commuting automorphisms, and suppose that it has a model companion S^* . The theory S is inductive, so the models of S^* are precisely existentially closed models of S .

Take (K_0, σ, τ) given by Lemma 3.3 and embed it into $(M, \sigma, \tau) \models S^*$. Lemma 3.2 gives

$$M \models \neg \Phi_{\text{comm}}(b_n) \quad (n \in J).$$

Let $\mathcal{U}$ be a nonprincipal ultrafilter on J , and let us define the corresponding ultraproduct and its element

$$(N, \sigma, \tau) := (M, \sigma, \tau)^J / \mathcal{U}, \quad b := [b_n]_{\mathcal{U}} \in N.$$

By Łoś's theorem,

$$N \models S^*, \quad \tau(b) = b, \quad N \models \neg \Phi_{\text{comm}}(b).$$

For each $m > 0$ and each $n \in J$ with $n > m$, we have $\sigma^m(b_n) \neq b_n$. Hence, by Łoś's theorem again, b has infinite σ -orbit. Lemma 3.1 supplies an extension of (N, σ, τ) satisfying $\Phi_{\text{comm}}(b)$, which contradicts the existential closedness of N . $\square$

Remark 3.4. Our two main arguments follow this scheme (as suggested in [5, Remark 1.12]), but they additionally require the following:

(D) *The realizing extension should preserve the prescribed theory of the reduct.*

In both settings, constructing the realizing extension within the prescribed class is the additional issue highlighted in [5, Remark 1.12]. Using the notation of Sections 4 and 5, the ingredients are as follows.

- For bounded PAC fields, Lemma 4.8 supplies (A), subject to its hypotheses on the fixed elementary base. Lemma 4.4 supplies (B), excluding $\Phi_{\text{PAC}}(b)$ whenever $\sigma^n(b) = b$ and $\gcd(n, q) = 1$. Lemma 4.7 supplies (C): it constructs a single model containing the fixed elementary base and elements of exact σ -period ℓ^m for every $m \geq 1$. Requirement (D) is ensured by Lemma 4.6, which produces an elementary PAC extension while preserving the prescribed automorphism action.
- For a fixed completion T of ACFA, Proposition 5.2 supplies (A) under its orbit and fixed-point hypotheses, and Lemma 5.3 supplies (B) for periods $n \equiv 1 \pmod{r}$. Lemma 5.4 supplies (C) by constructing τ -fixed elements of these periods whose σ -cycles are algebraically independent over A_0 . This independence ensures the required orbit conditions in the ultrapower. Finally, Lemma 5.1 supplies (D), preserving the completion T of the first difference-field reduct.

4. AUTOMORPHISMS OF BOUNDED PAC FIELDS

In this section we prove Theorem 1.1, following the scheme of Section 3. After naming suitable constants to obtain a model-complete field theory, we use a single Kummer coset to establish an obstruction at certain finite periods. The main construction realizes the corresponding existential condition at an infinite orbit: commuting lifts and a bilateral product of inertia elements produce the required automorphism action, and a PAC completion preserves the prescribed elementary field theory, supplying property (D). We also construct elements of arbitrarily large finite periods to which the obstruction applies. An ultrapower then turns these finite orbits into an infinite orbit, yielding the contradiction to the existence of a model companion.

Let us fix a field F and a prime ℓ such that

$$(1) \quad F \text{ is bounded and PAC,} \quad \text{char}(F) = 0, \quad \mu_\infty \subseteq F, \quad \text{Gal}(F) \twoheadrightarrow C_\ell.$$

Put $T = \text{Th}_{L_{\text{ring}}}(F)$, where $L_{\text{ring}} = \{0, 1, +, -, \cdot\}$. Since $\mu_\ell \subseteq F$, Kummer theory makes the last hypothesis equivalent to the existence of an element

$$(2) \quad c \in F^\times \setminus (F^\times)^\ell.$$

For a tuple $\mathbf{d}$ from F , let $T_{\mathbf{d}} = \text{Th}(F, \mathbf{d})$ in the language with names for its entries, and write

$$S_{\mathbf{d}} = (T_{\mathbf{d}})_\sigma = T_{\mathbf{d}} \cup \{\sigma \text{ is a field automorphism and } \sigma(\mathbf{d}) = \mathbf{d}\}.$$

We use the following three standard facts about PAC fields:

- absolute Galois groups of PAC fields are projective;
- algebraic extensions of PAC fields are PAC;
- a regular inclusion $K \subseteq L$ of perfect PAC fields is elementary if the restriction map $\text{Gal}(L) \rightarrow \text{Gal}(K)$ is an isomorphism.

References are [8, Corollary 11.2.5, Theorem 11.6.2, Corollary 20.3.4]; see also [7, Section 4.3] and the proof of [10, Theorem 3.25].

The base-change assertion in the following lemma is [1, Section 1.4, Fact 3]. We include its proof and record the resulting commuting lift of an automorphism.

Lemma 4.1. *If $M \preceq K$ are models of the theory of a bounded characteristic-zero field, then, with compatible algebraic closures,*

$$K^{\text{alg}} = KM^{\text{alg}}, \quad \text{Gal}(K) \xrightarrow{\sim} \text{Gal}(M).$$

In particular, if $\sigma \in \text{Aut}(K/M)$, there is $\rho \in \text{Aut}(K^{\text{alg}})$ extending σ , fixing M^{alg} pointwise, and commuting with every element of $\text{Gal}(K)$.

Proof. Elementarity makes K/M regular. Fix a positive integer n . There are finitely many Galois extensions of M of degree at most n ; choose primitive-element polynomials for all of them. The assertion that these polynomials exhaust the Galois extensions of degree at most n , up to isomorphism, is first-order in their coefficients. One can express it using the field algebras associated to irreducible polynomials and the existence of a root of one such polynomial in another associated algebra.

By elementarity the same list exhausts the corresponding extensions of K . Regularity preserves the degrees and the Galois groups of the listed extensions. Every finite Galois extension of K therefore lies in KM^{alg} , giving $K^{\text{alg}} = KM^{\text{alg}}$; regularity gives that the restriction map $\text{Gal}(K) \rightarrow \text{Gal}(M)$ is an isomorphism.

The fields K and M^{alg} are linearly disjoint over M . Hence σ on K and the identity on M^{alg} define ρ on their compositum, which commutes with every element of $\text{Gal}(K)$. $\square$

Proposition 4.2. *There is a countable tuple $\mathbf{d}$ from F for which $T_{\mathbf{d}}$ is model-complete. Both $T_{\mathbf{d}}$ and $S_{\mathbf{d}}$ are then inductive.*

Proof. By the model-completeness result for bounded PAC fields [7, Section 4.6], there is a countable tuple $\mathbf{d}$ from F such that $T_{\mathbf{d}}$ is model-complete; see also the paragraph immediately preceding [10, Theorem 4.10]. It is clear that both $T_{\mathbf{d}}$ and $S_{\mathbf{d}}$ are inductive. $\square$

From now on choose $\mathbf{d}$ as in Proposition 4.2, enlarged to contain c from (2) and every element of the countable group μ_{∞} . Naming these additional constants preserves model completeness, so $T_{\mathbf{d}}$ remains model-complete and both $T_{\mathbf{d}}$ and $S_{\mathbf{d}}$ are inductive. Fix $(M, \mathbf{d}) \models T_{\mathbf{d}}$. The constructions will start with $M \preccurlyeq K$ and $\sigma|_M = \text{id}$. Pointwise fixation of this whole base is a choice of models used in the proof, not an additional axiom of $S_{\mathbf{d}}$.

We choose a prime $q \neq \ell$, a primitive q -th root of unity $\zeta \in M$ and consider the following existential formula

$$\Phi_{\text{PAC}}(z) := \exists x \exists y [\sigma(x) = \zeta x \wedge (x - z)(\zeta x - z) \neq 0 \wedge x - z = c(\zeta x - z)y^{\ell}].$$

Remark 4.3. Let us note here the meaning of Φ_{PAC} . For $(K, \mathbf{d}, \sigma) \models S_{\mathbf{d}}$ and $b \in K$, we have that

$$(K, \mathbf{d}, \sigma) \models \Phi_{\text{PAC}}(b)$$

if and only if there is $x \in K$ such that the following hold:

- $(x - b) \neq 0 \neq (\zeta x - b)$;
- $\sigma(x) = \zeta x$;
- $(x - b)/(\zeta x - b) \in c(K^{\times})^{\ell}$: the fixed non-trivial coset of $(K^{\times})^{\ell}$.

For a related use of multiplicative power cosets to constrain automorphisms of pseudofinite fields, see [3, Section 4].

The result below corresponds to (B) from Section 3.

Lemma 4.4. *Let $(K, \mathbf{d}, \sigma) \models S_{\mathbf{d}}$, $b \in K$, and $n \geq 1$. If $\sigma^n(b) = b$ and $\gcd(n, q) = 1$, then $(K, \mathbf{d}, \sigma) \models \neg \Phi_{\text{PAC}}(b)$.*

Proof. Suppose, towards a contradiction, that

$$(K, \mathbf{d}, \sigma) \models \Phi_{\text{PAC}}(b),$$

and let $x, y \in K$ be corresponding witnesses. Let us define

$$f := (x - b)/(\zeta x - b) = cy^{\ell}.$$

Since we have

$$\sigma^n(b) = b, \quad \sigma^n(\zeta) = \zeta, \quad \sigma^n(c) = c, \quad \sigma(x) = \zeta x,$$

we obtain

$$\prod_{j=0}^{q-1} \sigma^{jn}(f) = \prod_{j=0}^{q-1} \frac{\zeta^{jn}x - b}{\zeta^{jn+1}x - b} = 1.$$

The last equality holds because multiplication by n permutes the residue classes modulo q , hence we have

$$\{\zeta^{jn} \mid j = 0, 1, \dots, q-1\} = \{\zeta^{jn+1} \mid j = 0, 1, \dots, q-1\}.$$

On the other hand, we obtain

$$1 = \prod_{j=0}^{q-1} \sigma^{jn}(f) = \prod_{j=0}^{q-1} c \sigma^{jn}(y^{\ell}) = c^q \left(\prod_{j=0}^{q-1} \sigma^{jn}(y) \right)^{\ell}.$$

Therefore, $c^q \in (K^{\times})^{\ell}$. Since $\gcd(q, \ell) = 1$, we get $c \in (K^{\times})^{\ell}$, which is a contradiction. $\square$

The following construction is the main additional ingredient. It originates from the idea to regard a pseudofinite field as the field of fixed points of a model of ACFA and then to try to use Hrushovski's proof for two commuting automorphisms in the case of pseudofinite fields.

Lemma 4.5. *Let I be nonempty and finite or countable. The theory of fields with an I -indexed family of automorphisms has a model companion ACFA_I . Its underlying fields are algebraically closed. An algebraically closed invariant subfield is an amalgamation base, and its automorphisms commuting with all the distinguished automorphisms are partial elementary maps of models of ACFA_I .*

If $(C, (\tau_i)_{i \in I}) \models \text{ACFA}_I$ is sufficiently saturated, then $N = \bigcap_{i \in I} \text{Fix}_C(\tau_i)$ is PAC.

Proof. For finite I , existence of the model companion in characteristic zero follows from [15, Corollary 4.7]. The resulting operators are automorphisms with no imposed relations, and the underlying fields of existentially closed models are algebraically closed by [15, Theorem 4.6]. For the corresponding result in positive characteristic, see [2, Section 4.2].

An existentially closed field with finitely many distinguished free generators remains existentially closed on deleting some generators. Indeed, first take an algebraically closed extension witnessing a finite-reduct existential condition. Each omitted automorphism of the algebraically closed base extends independently to this larger algebraically closed field. Existential closedness in the full language then gives the required witness.

For countable I , let ACFA_I be the union of the finite-reduct companions. Every finite subset of these axioms and the quantifier-free diagram of an I -field is realized by embedding the appropriate finite reduct into its companion. Compactness gives an embedding of the full I -field into a model of ACFA_I . An embedding between two such models is elementary, since each formula uses only finitely many distinguished automorphisms. Thus ACFA_I is the required companion.

Over an algebraically closed invariant field Ω , two extensions can be amalgamated using the fraction field of their tensor product over Ω . The tensor product is a domain, and each distinguished automorphism acts on both factors. There are no relations to check. Amalgamation and model-completeness show that ACFA_I together with the quantifier-free diagram of Ω is complete. Hence an automorphism of this I -field Ω is partial elementary in a model of ACFA_I .

Finally, let V/N be an absolutely irreducible affine variety. For every finite nonempty $J \subseteq I$, the J -reduct of C is existentially closed as a field with an action of the free group on J . By [4, Proposition 2.4], its common fixed field

$$N_J := \bigcap_{j \in J} \text{Fix}_C(\tau_j)$$

is PAC. Since $N \subseteq N_J$, the variety V has an N_J -rational point. Thus the type asserting membership in V and fixation by every τ_i , $i \in I$, is finitely satisfiable in C . As I is finite or countable, saturation gives a realization of this type. This is an N -rational point of V , proving that N is PAC. $\square$

The result below corresponds to (D) from Section 3.

Lemma 4.6. *Let K be a PAC field of characteristic zero, and let $(\gamma_i)_{i \in I}$ generate a dense subgroup of $\text{Gal}(K)$, where I is nonempty and finite or countable. Let Ω be an algebraically closed field containing a fixed algebraic closure K^{alg} of K . Suppose that $s, \tau_i \in \text{Aut}(\Omega)$ satisfy*

$$s(K) = K, \quad s\tau_i = \tau_i s, \quad \tau_i|_{K^{\text{alg}}} = \gamma_i \quad (i \in I).$$

Put

$$F_0 := \bigcap_{i \in I} \text{Fix}_\Omega(\tau_i).$$

Then there are a field extension $F_0 \subseteq L$ and an automorphism $\tilde{\sigma} \in \text{Aut}(L)$ such that

$$K \preceq L \quad \text{and} \quad \tilde{\sigma}|_{F_0} = s|_{F_0}.$$

In particular, $\tilde{\sigma}|_K = s|_K$.

Proof. Each τ_i fixes K pointwise, so $K \subseteq F_0$. The commutation relations give $s(F_0) = F_0$. We embed $(\Omega, (\tau_i)_{i \in I})$ into a sufficiently saturated and strongly homogeneous model

$$(C, (\tau_i)_{i \in I}) \models \text{ACFA}_I.$$

By Lemma 4.5, the automorphism s of the algebraically closed I -field Ω is partial elementary. It therefore extends to an automorphism S of this entire structure. In particular,

$$S\tau_i = \tau_i S \quad (i \in I).$$

Set

$$N := \bigcap_{i \in I} \text{Fix}_C(\tau_i).$$

By Lemma 4.5, N is PAC. Moreover, $F_0 \subseteq N$ and $S(N) = N$. Since the γ_i topologically generate $\text{Gal}(K)$, we have

$$N \cap K^{\text{alg}} = \bigcap_{i \in I} \text{Fix}_{K^{\text{alg}}}(\gamma_i) = K.$$

Thus N/K is regular. Let N^{alg} denote the relative algebraic closure of N in C ; it contains K^{alg} . The restriction map is consequently surjective:

$$r : \text{Gal}(N) \twoheadrightarrow \text{Gal}(K).$$

Since K is PAC, $\text{Gal}(K)$ is projective. Choose a continuous section $j : \text{Gal}(K) \rightarrow \text{Gal}(N)$ of r , and put

$$H := j(\text{Gal}(K)), \quad L := (N^{\text{alg}})^H.$$

The subgroup H is closed. The extension L/N is algebraic, so L is PAC.

Each τ_i fixes N pointwise and preserves N^{alg} . Their restrictions to N^{alg} topologically generate $\text{Gal}(N)$, since their common fixed field there is exactly N . Also, $S(N) = N$ implies that S preserves N^{alg} and induces a continuous conjugation automorphism of $\text{Gal}(N)$. As S commutes with every τ_i , it centralizes their closed generated subgroup:

$$SgS^{-1} = g \quad \text{for } g \in \text{Gal}(N).$$

In particular, $S(L) = L$. Define $\tilde{\sigma} := S|_L$.

We now verify that $K \subseteq L$ is elementary. Since $r(H) = \text{Gal}(K)$,

$$L \cap K^{\text{alg}} = (K^{\text{alg}})^{r(H)} = (K^{\text{alg}})^{\text{Gal}(K)} = K.$$

Thus L/K is regular. Moreover, N^{alg} is also an algebraic closure of L , and restriction gives an isomorphism

$$\text{Gal}(L) = H \xrightarrow{\sim} \text{Gal}(K).$$

The elementary-embedding theorem for perfect PAC fields therefore yields $K \preceq L$.

Finally, $F_0 \subseteq N \subseteq L$, and S extends s on Ω . Hence $\tilde{\sigma}|_{F_0} = s|_{F_0}$, as required. $\square$

The next result about increasing finite periods corresponds to (C) from Section 3.

Lemma 4.7. *There is $(K_0, \mathbf{d}, \sigma) \models S_{\mathbf{d}}$ such that*

- $M \preceq K_0$;
- $\sigma|_M = \text{id}$;
- for all $m \geq 1$, there is $b_m \in K_0$ such that

$$\left| \left\{ \sigma^i(b_m) \mid i \in \mathbb{Z} \right\} \right| = \ell^m,$$

that is b_m has exact σ -period ℓ^m .

Proof. Choose a nonempty finite or countable family $(\gamma_i)_{i \in I}$ of topological generators of $\text{Gal}(M)$. Since $\mu_\infty \subseteq M$, every γ_i fixes all roots of unity in M^{alg} . Apply Lemma 2.1(ii) with $k = M^{\text{alg}}$ and $\rho = \text{id}$. This gives an algebraically closed field Ω containing M^{alg} , commuting lifts s, d_i with

$$s|_{M^{\text{alg}}} = \text{id}, \quad d_i|_{M^{\text{alg}}} = \gamma_i,$$

and elements b_m in

$$F_0 := \bigcap_{i \in I} \text{Fix}_\Omega(d_i)$$

having exact s -period ℓ^m for every $m \geq 1$. Lemma 4.6, applied with $K = M$, gives $M \preccurlyeq K_0$ and an automorphism σ of K_0 extending $s|_{F_0}$. Thus σ fixes M pointwise and preserves the exact periods of all the b_m . Since $\mathbf{d} \subseteq M$, we have $(K_0, \mathbf{d}, \sigma) \models S_{\mathbf{d}}$. $\square$

Below is the crucial result about realizing the coset at an infinite orbit which corresponds to (A) from Section 3. The main difficulty still comes from combining with the corresponding (D) which is Lemma 4.6.

Lemma 4.8. *Suppose $(M, \mathbf{d}) \preccurlyeq (K, \mathbf{d}) \models T_{\mathbf{d}}$, $\sigma \in \text{Aut}(K/M)$, and $b \in K$ has infinite σ -orbit. There is an extension $(L, \mathbf{d}, \tilde{\sigma})$ of $(K, \mathbf{d}, \sigma)$ in $S_{\mathbf{d}}$ such that*

$$(K, \mathbf{d}) \preccurlyeq (L, \mathbf{d}) \quad \text{and} \quad (L, \mathbf{d}, \tilde{\sigma}) \models \Phi_{\text{PAC}}(b).$$

Proof. By Lemma 4.1, we have $K^{\text{alg}} = KM^{\text{alg}}$ and σ extends to $\rho \in \text{Aut}(K^{\text{alg}})$ fixing M^{alg} pointwise and centralizing $\text{Gal}(K)$. Let us choose $\beta \in M^{\text{alg}}$ with $\beta^\ell = c$ and $\omega \in \mu_\ell \setminus \{1\}$.

Since K is bounded, $\text{Gal}(K)$ is small and has a finite or countable family $(\gamma_i)_{i \in I}$ of topological generators. We first construct suitable commuting lifts.

Claim 1. There are an element x transcendental over K , an algebraic closure Ω of $K(x)$ containing K^{alg} , an element $\alpha \in \Omega$, and automorphisms $s, d_i \in \text{Aut}(\Omega)$ such that

$$\alpha^\ell = \frac{x - b}{\zeta x - b},$$

and, for every $i \in I$,

$$s|_{K^{\text{alg}}} = \rho, \quad d_i|_{K^{\text{alg}}} = \gamma_i, \quad s(x) = \zeta x, \quad d_i(x) = x, \quad sd_i = d_i s, \quad d_i(\alpha) = \alpha.$$

Proof of Claim 1. Since b has infinite σ -orbit, we have $b \neq 0$. Every γ_i fixes all roots of unity in K^{alg} , since these belong to $M \subseteq K$. Apply Lemma 2.1(i) with $k = K^{\text{alg}}$ and series variable x , choosing $\epsilon(1) = \zeta$. We obtain an algebraic closure Ω of $K(x)$ containing K^{alg} inside

$$A := \bigcup_{m \geq 1} K^{\text{alg}}((x^{1/m})),$$

together with automorphisms s, d_i of A restricting to Ω and satisfying

$$s|_{K^{\text{alg}}} = \rho, \quad d_i|_{K^{\text{alg}}} = \gamma_i, \quad s(x) = \zeta x, \quad d_i(x) = x, \quad sd_i = d_i s.$$

Here each d_i is induced by the coefficientwise action of γ_i on A .

Since $b \neq 0$, we have

$$R(x) := \frac{x - b}{\zeta x - b} = \frac{1 - x/b}{1 - \zeta x/b} \in 1 + xK[[x]] \subset A.$$

The reduction modulo x of $Y^\ell - R(x)$ is $Y^\ell - 1$, which has the simple root 1 because $\text{char}(K) = 0$. Hensel's lemma therefore gives a unique $\alpha \in 1 + xK[[x]]$ such that

$$\alpha^\ell = R(x) = \frac{x - b}{\zeta x - b}.$$

This α is algebraic over $K(x)$ and hence belongs to Ω . Finally, every d_i fixes $K[[x]]$ pointwise, so $d_i(\alpha) = \alpha$ for all $i \in I$. $\square$ Claim 1

To realize $\Phi_{\text{PAC}}(b)$, we need the corrected lifts to fix $y := \alpha/\beta$. Write

$$d_i(\beta) = \omega^{a_i}\beta, \quad \text{where } a_i \in \{0, \dots, \ell - 1\}.$$

Since $d_i(\alpha) = \alpha$, we have $d_i(y) = \omega^{-a_i}y$. Claim 2 provides an automorphism $\eta \in \text{Gal}(\Omega/K^{\text{alg}}(x))$ commuting with s and satisfying $\eta(\alpha) = \omega\alpha$. The corrected lifts $d_i\eta^{a_i}$ will then fix y and still commute with s .

Choose $\lambda \in K^{\text{alg}}$ with $\lambda^\ell = \zeta$. Then

$$(\lambda\alpha)^\ell = \frac{x-b}{x-\zeta^{-1}b}, \quad K^{\text{alg}}(x, \lambda\alpha) = K^{\text{alg}}(x, \alpha).$$

Since $b \neq 0$ and $\zeta \neq 1$, the points b and $\zeta^{-1}b$ are distinct. Lemma 2.3(iii), applied to $\lambda\alpha$, therefore shows that $K^{\text{alg}}(x, \alpha)/K^{\text{alg}}(x)$ is cyclic of degree ℓ , totally ramified at $b, \zeta^{-1}b$, and unramified elsewhere.

Claim 2. There is $\eta \in \text{Gal}(\Omega/K^{\text{alg}}(x))$ such that

$$s\eta s^{-1} = \eta \quad \text{and} \quad \eta(\alpha) = \omega\alpha.$$

Proof of Claim 2. Let us set $G := \text{Gal}(\Omega/K^{\text{alg}}(x))$. Since b is a point of total ramification, by the “Moreover” part of Lemma 2.3(iii), there is an inertia element $g_0 \in G$ above b with $g_0(\alpha) = \omega\alpha$. For $k \in \mathbb{Z}$, set

$$g_k := s^k g_0 s^{-k}, \quad b'_k := \zeta^{-k} \sigma^k(b).$$

By Lemma 2.3(iv), g_k is an inertia element above b'_k .

Subclaim. The points b'_k are pairwise distinct, and $b'_k \notin \{b, \zeta^{-1}b\}$ whenever $k \neq 0$.

Proof of the Subclaim. Since $\zeta \in M$ and $\sigma|_M = \text{id}$, we have $\sigma(\zeta) = \zeta$. For $r \in \mathbb{Z} \setminus \{0\}$ and $h \in \mathbb{Z}$, an equality $\sigma^r(b) = \zeta^h b$ would therefore imply

$$\sigma^{qr}(b) = \zeta^{qh} b = b,$$

contrary to the infinite σ -orbit of b .

If $b'_i = b'_j$ with $i \neq j$, then

$$\sigma^{i-j}(b) = \zeta^{i-j} b,$$

contradicting the preceding observation. Likewise, if $b'_k = \zeta^{-\delta} b$ for $k \neq 0$ and $\delta \in \{0, 1\}$, then

$$\sigma^k(b) = \zeta^{k-\delta} b,$$

which is again impossible. □_{Subclaim}

By Lemma 2.3(i), every finite Galois extension $E/K^{\text{alg}}(x)$ in Ω ramifies at only finitely many points. Since K^{alg} is algebraically closed, inertia at an unramified point restricts trivially to E by Lemma 2.3(ii). Since the points b'_k are pairwise distinct (see Subclaim), it follows that

$$g_k|_E = \text{id} \quad \text{for all but finitely many } k.$$

Apply Lemma 2.4 to the continuous automorphism

$$G \longrightarrow G, \quad g \longmapsto sgs^{-1}.$$

Its iterates send g_0 to $g_k = s^k g_0 s^{-k}$. We obtain the convergent ordered product

$$\eta = \lim_{r \rightarrow \infty} g_{-r} g_{-r+1} \cdots g_r \in G$$

such that $s\eta s^{-1} = \eta$. The condition

$$b'_k \notin \{b, \zeta^{-1}b\} \quad (k \neq 0)$$

from Subclaim ensures that every g_k with $k \neq 0$ fixes α . Indeed, the finite Galois extension $K^{\text{alg}}(x) \subset K^{\text{alg}}(x, \alpha)$ is ramified only at b and $\zeta^{-1}b$. Since g_k is an inertia element above b'_k , Lemma 2.3(ii) gives

$$g_k|_{K^{\text{alg}}(x, \alpha)} = \text{id} \quad (k \neq 0).$$

Consequently, every finite product defining η acts on α exactly as g_0 :

$$(g_{-r} \cdots g_r)(\alpha) = g_0(\alpha) = \omega\alpha.$$

Passing to the limit yields $\eta(\alpha) = \omega\alpha$ by Lemma 2.4 again. □Claim 2

Let us set

$$\tau_i := d_i \eta^{a_i}.$$

Then $\tau_i|_{K^{\text{alg}}} = \gamma_i$, each τ_i fixes x , and $s\tau_i = \tau_i s$. Moreover, we obtain

$$\tau_i(\alpha) = \omega^{a_i} \alpha, \quad \tau_i(\beta) = \gamma_i(\beta) = \omega^{a_i} \beta \quad \implies \quad \tau_i(\alpha/\beta) = \alpha/\beta.$$

We finally define $y := \alpha/\beta$. By Lemma 4.6, we get $K \preceq L$ and $\tilde{\sigma} \in \text{Aut}(L)$ such that

$$x, y \in L, \quad \tilde{\sigma}(x) = \zeta x, \quad \frac{x-b}{\zeta x-b} = \alpha^\ell = \beta^\ell \frac{\alpha^\ell}{\beta^\ell} = cy^\ell, \quad \tilde{\sigma}|_K = \sigma.$$

By Remark 4.3, we obtain

$$(L, \mathbf{d}, \tilde{\sigma}) \models \Phi_{\text{PAC}}(b).$$

Elementarity on the field reduct preserves the full type of $\mathbf{d}$, hence $(K, \mathbf{d}) \preceq (L, \mathbf{d})$. □

We can finish now the proof of our first main result.

Proof of Theorem 1.1. Proposition 4.2 gives model-completeness of $T_{\mathbf{d}}$ and inductivity of $S_{\mathbf{d}}$. Suppose $S_{\mathbf{d}}$ had a model companion $S_{\mathbf{d}}^*$. Because $S_{\mathbf{d}}$ is inductive, the models of $S_{\mathbf{d}}^*$ would be precisely its existentially closed models, and would satisfy $S_{\mathbf{d}}$.

We embed the model from Lemma 4.7 into $(K, \mathbf{d}, \sigma) \models S_{\mathbf{d}}^*$. Model-completeness of $T_{\mathbf{d}}$ makes this embedding elementary on the field reduct. The fixed base M and the exact periods are preserved. Lemma 4.4 gives $\neg\Phi_{\text{PAC}}(b_m)$ for every m .

Let us take a nonprincipal ultrapower, with

$$(K^*, \mathbf{d}, \sigma) = (K, \mathbf{d}, \sigma)^{\mathbb{N}_{>0}}/\mathcal{U}, \quad b^* = [(b_m)_m]_{\mathcal{U}}.$$

It is a model of $S_{\mathbf{d}}^*$ satisfying $\neg\Phi_{\text{PAC}}(b^*)$. For each fixed $k \geq 1$, the inequality $\sigma^k(b_m) \neq b_m$ holds eventually, since $\ell^m > k$. Thus b^* has infinite orbit. The diagonal copy of M remains elementary and pointwise fixed. Lemma 4.8 produces an extension in $S_{\mathbf{d}}$ realizing the existential formula $\Phi_{\text{PAC}}(b^*)$, contradicting existential closedness. □

Corollary 4.9. *Let F be a PAC field of characteristic zero containing μ_∞ . If $\text{Gal}(F)$ is free profinite of finite rank $e \geq 1$, or free pro- p of finite rank $e \geq 1$, then the conclusions of Theorem 1.1 hold.*

Proof. A finitely generated profinite group is small, hence F is bounded. A nontrivial finite-rank free profinite group has a quotient C_ℓ for every prime ℓ ; a nontrivial free pro- p group has a quotient C_p . It is enough now to apply Theorem 1.1. □

Remark 4.10. Let us note that boundedness has the following three roles in our arguments.

- (i) It allows us to obtain a model-complete theory by naming countably many constants, as in Proposition 4.2; this also makes the associated automorphism theory inductive.
- (ii) It gives the elementary base-change identity $K^{\text{alg}} = KM^{\text{alg}}$ for $M \preceq K$, which permits the simultaneous commuting coefficient lifts.
- (iii) It ensures that the absolute Galois group has a countable dense generating set, allowing the countable auxiliary construction. Countability is only a convenience here: the same auxiliary construction works for arbitrary index sets I , using a sufficiently saturated and homogeneous model of ACFA_I .

Remark 4.11. The PAC argument does not directly extend to arbitrary PAC fields of characteristic $p > 0$. For an imperfect PAC field K , the principal difficulty is the completion construction: a common fixed field of automorphisms of an algebraically closed field is necessarily perfect, since the unique p th root of a fixed element is also fixed. Consequently, the fields produced by this construction cannot be elementary extensions of K ; indeed,

$$\left(K^{\text{alg}}\right)^{\text{Gal}(K)} = K^{\text{perf}} \quad (\text{the perfect closure of } K),$$

rather than K .

There is a separate difficulty when the available cyclic-prime quotient has order p : Kummer theory no longer applies, there are no nontrivial p th roots of unity, and a perfect field has no nontrivial multiplicative cosets modulo p th powers. Treating this case would require an Artin–Schreier replacement for the multiplicative obstruction and a corresponding modification of the period construction.

Finally, Puiseux series are not algebraically closed in positive characteristic, although this particular issue can be resolved using the Hahn-field construction of Remark 2.2. For perfect bounded PAC fields containing $\mathbb{F}_p^{\text{alg}}$ and admitting a cyclic quotient of prime order $\ell \neq p$, the specific obstacles above can be avoided by replacing Puiseux series with Hahn series and choosing the rotation order to be a prime

$$q \notin \{p, \ell\}.$$

We finish this section with the following table summarizing a possible usage of the choices of primes ℓ and q .

Parameter choices for the characteristic-zero PAC theorem.

Absolute Galois group	Kummer prime ℓ	Rotation order q
$\widehat{\mathbb{Z}}$ (pseudofinite case)	3	2
Free profinite, finite rank $e \geq 1$	3	2
Free pro- p , finite rank, p odd	p	2
Free pro-2, finite rank	2	3

5. A SECOND AUTOMORPHISM OF ACFA

In this section, we show that the road map from Section 3 still works if we consider models of ACFA with automorphisms. We fix a completion T of ACFA, write σ for the distinguished automorphism in the language of T , and define

$$S_T := T \cup \{\tau \text{ is a field automorphism and } \tau\sigma = \sigma\tau\}.$$

We first establish the preservation of the prescribed completion, corresponding to property (D) from Section 3.

Lemma 5.1. *Let $(A, \sigma_A) \models T$ and B be an algebraically closed field with commuting automorphisms σ_B, τ_B such that $(A, \sigma_A) \subseteq (B, \sigma_B)$. Then (B, σ_B, τ_B) embeds into (C, σ_C, τ_C) , where $\sigma_C\tau_C = \tau_C\sigma_C$ and $(C, \sigma_C) \models T$.*

Proof. Embed (B, σ_B) into a sufficiently saturated and strongly homogeneous model (C, σ_C) of ACFA. Here and below, $\text{Diag}_{\text{qf}}(N)$ denotes the set of all quantifier-free sentences in the language expanded by constants for the elements of N that hold in N . Since B is algebraically closed and σ_B is an automorphism of B , [6, Theorem 1.3, p. 3008] implies that the theory

$$\text{ACFA} \cup \text{Diag}_{\text{qf}}(B, \sigma_B)$$

is complete in the difference-field language with constants for all elements of B . Thus the difference-field automorphism τ_B is a partial elementary map of (C, σ_C) , and extends to an automorphism τ_C of that structure. In particular, τ_C commutes with σ_C . Since

$(A, \sigma_A) \models T$ and $(A, \sigma_A) \subseteq (C, \sigma_C)$, model-completeness of ACFA gives $(A, \sigma_A) \preceq (C, \sigma_C)$. Hence $(C, \sigma_C) \models T$. $\square$

Let us fix now an odd prime $\ell \neq \text{char}(T)$ and a primitive ℓ -th root ζ in a model $(A_0, \sigma) \models T$. Choose $c_0, c_1 \in A_0$ and $\varepsilon \in \{1, -1\}$ by the following rule:

- (a) If $\sigma(\zeta) \neq \zeta$, set $c_0 = 0, c_1 = 1$ and $\varepsilon = 1$.
 - (b) If $\sigma(\zeta) = \zeta$, choose a two-cycle $c_0 \neq c_1$ with $\sigma(c_0) = c_1$ and $\sigma(c_1) = c_0$, and set $\varepsilon = -1$.
- The two-cycle in (b) exists by existential closedness. Since ℓ is odd, both cases give

$$(3) \quad \sigma(\zeta) \neq \zeta^\varepsilon.$$

With these parameters, we define the following existential formula

$$\Phi_{\text{ACFA}}(z) := \exists x \exists y \left[\sigma(x) = \tau(x) = x \wedge c_0 \neq x + z \neq c_1 \wedge y^\ell = \frac{x + z - c_0}{x + z - c_1} \wedge \tau(y) = \zeta y \right].$$

Retain the parameters chosen above throughout this section. For the next two results, let (k, σ, τ) be a field with commuting automorphisms containing the difference subfield of (A_0, σ) generated by c_0, c_1, ζ , and suppose that τ fixes these parameters. Let $b \in k$, and put

$$a := c_0 - b, \quad d := c_1 - b.$$

The next proposition is the construction needed to realize an existential formula in an extension, which corresponds to property (A) from Section 3. The proof follows the lines of the proof of Lemma 4.8, so we will be brief.

Proposition 5.2. *Assume that $(k, \sigma, \tau) \models S_T$ and $\tau(b) = b$. If a has infinite σ -orbit not containing d , then there is an S_T -extension*

$$(k, \sigma, \tau) \subseteq (K, \sigma, \tau)$$

such that $(K, \sigma, \tau) \models \Phi_{\text{ACFA}}(b)$.

Proof. Let $H := k((x^\mathbb{Q}))$ and Ω be the relative algebraic closure of $k(x)$ in H . By Remark 2.2, Ω is an algebraic closure of $k(x)$, and the coefficientwise extensions of σ and τ to H restrict to commuting automorphisms σ_0, τ_0 of Ω fixing x . Let us choose $y \in \Omega$ such that

$$y^\ell = \frac{x + b - c_0}{x + b - c_1} = \frac{x - a}{x - d}.$$

Since τ fixes b, c_0, c_1 , τ_0 fixes y^ℓ . Thus, for some $j \in \{0, \dots, \ell - 1\}$, we have

$$(4) \quad \tau_0(y) = \zeta^j y.$$

By Lemma 2.3(iii), the extension $k(x, y)/k(x)$ is cyclic of degree ℓ , totally ramified at a, d , and unramified elsewhere. Let us choose an inertia element

$$\rho \in G := \text{Gal}(\Omega/k(x))$$

above a such that $\rho(y) = \zeta y$. For $i \in \mathbb{Z}$, we set $\rho_i := \sigma_0^i \rho \sigma_0^{-i}$. Since $\sigma_0(x) = x$, Lemma 2.3(iv) shows that ρ_i is an inertia element above $\sigma^i(a)$.

Every finite Galois extension $E/k(x)$ inside Ω ramifies at only finitely many points. Since the points $\sigma^i(a)$ are pairwise distinct, Lemma 2.3(i)–(ii) gives

$$\rho_i|_E = \text{id}_E \quad \text{for all but finitely many } i.$$

Apply Lemma 2.4 to the continuous automorphism

$$G \longrightarrow G, \quad g \longmapsto \sigma_0 g \sigma_0^{-1},$$

whose iterates send ρ to ρ_i . We obtain

$$\eta := \lim_{m \rightarrow \infty} \rho_{-m} \rho_{-m+1} \cdots \rho_m \in G, \quad \sigma_0 \eta \sigma_0^{-1} = \eta.$$

The orbit hypotheses imply that $\sigma^i(a) \notin \{a, d\}$ whenever $i \neq 0$. Thus every ρ_i with $i \neq 0$ fixes $k(x, y)$ pointwise, and the final assertion of Lemma 2.4 gives

$$(5) \quad \eta(y) = \zeta y.$$

Let us set $\tau_1 := \eta^{1-j}\tau_0$. Since η fixes $k(x)$ pointwise, τ_1 extends τ on k and fixes x . Both η and τ_0 commute with σ_0 , so τ_1 also commutes with σ_0 . Moreover, (4) and (5) give

$$\tau_1(y) = \eta^{1-j}(\zeta^j y) = \zeta^j \zeta^{1-j} y = \zeta y.$$

Since x is transcendental over k , we also have $c_0 \neq x + b \neq c_1$. Hence

$$(\Omega, \sigma_0, \tau_1) \models \Phi_{\text{ACFA}}(b).$$

Finally, Lemma 5.1 embeds this algebraically closed pair into a model of S_T , preserving the formula $\Phi_{\text{ACFA}}(b)$, which finishes the proof. $\square$

For the remainder of this section, fix $r \geq 1$ such that σ^r fixes the tuple (c_0, c_1, ζ) . The result below corresponds to (B) from Section 3.

Lemma 5.3. *For every positive integer $n \equiv 1 \pmod{r}$, we have*

$$\sigma^n(b) = b \quad \implies \quad (k, \sigma, \tau) \models \neg \Phi_{\text{ACFA}}(b).$$

Proof. Suppose x, y witness $\Phi_{\text{ACFA}}(b)$. In particular, $y \neq 0$. Applying σ^n to its Kummer equation fixes x, b and acts on c_0, c_1 as σ does. The fraction from the Φ_{ACFA} -formula is therefore unchanged if $\varepsilon = 1$ and inverted if $\varepsilon = -1$. Hence

$$\sigma^n(y) = \xi y^\varepsilon \quad \text{for some } \xi \in \mu_\ell(k).$$

Because $\tau(\zeta) = \zeta$, the automorphism τ fixes every element of $\mu_\ell(k)$, including ξ . Consequently,

$$\tau \sigma^n(y) = \xi \zeta^\varepsilon y^\varepsilon, \quad \sigma^n \tau(y) = \sigma^n(\zeta) \xi y^\varepsilon = \sigma(\zeta) \xi y^\varepsilon.$$

Since $\sigma\tau = \tau\sigma$ and $y \neq 0$, we get $\zeta^\varepsilon = \sigma(\zeta)$, contradicting (3). $\square$

The next result corresponds to (C) from Section 3.

Lemma 5.4. *There is $(K, \sigma, \tau) \models S_T$ extending $(A_0, \sigma, \text{id}_{A_0})$, with $\tau = \text{id}_K$, such that for every $n > 0$ there is $b_n \in K$ satisfying $\sigma^n(b_n) = b_n$ and such that $b_n, \sigma(b_n), \dots, \sigma^{n-1}(b_n)$ are algebraically independent over A_0 .*

Proof. Let us define

$$B := A_0(b_{n,j} : n \geq 1, 0 \leq j < n),$$

where the $b_{n,j}$ are algebraically independent over A_0 . We extend σ from A_0 to an automorphism of B by setting

$$\sigma(b_{n,j}) = \begin{cases} b_{n,j+1}, & j < n-1, \\ b_{n,0}, & j = n-1 \end{cases}$$

and embed (B, σ) into a model $(K, \sigma) \models \text{ACFA}$. Since $(A_0, \sigma) \models T$, model-completeness of ACFA gives $(K, \sigma) \models T$.

Let us set $\tau = \text{id}_K$ and $b_n = b_{n,0}$ for every $n \geq 1$. Then τ commutes with σ , so $(K, \sigma, \tau) \models S_T$. Moreover,

$$\sigma^n(b_n) = b_n, \quad \sigma^j(b_n) = b_{n,j} \quad (0 \leq j < n).$$

The latter elements remain algebraically independent over A_0 , as required. In addition, τ fixes every b_n and all the chosen parameters from A_0 . $\square$

Proof of Theorem 1.2. We follow the ultrapower argument from Section 3 and the proof of Theorem 1.1. Suppose that S_T has a model companion T_A . Since T is model-complete, S_T is inductive, so models of T_A are existentially closed models of S_T .

Take (K, σ, id_K) and the elements b_n from the construction in Lemma 5.4, which supplies (C), and embed this structure into $(M, \sigma, \tau) \models T_A$. Put

$$J := \{n \geq 2 : n \equiv 1 \pmod{r}\}.$$

By (B), Lemma 5.3, we have $M \models \neg \Phi_{\text{ACFA}}(b_n)$ for every $n \in J$.

For a nonprincipal ultrafilter $\mathcal{U}$ on J , set

$$(N, \sigma, \tau) := (M, \sigma, \tau)^J / \mathcal{U}, \quad b := [b_n]_{\mathcal{U}}.$$

Identifying A_0 with its diagonal image, Łoś's theorem gives

$$(6) \quad N \models T_A, \quad \tau(b) = b, \quad N \models \neg \Phi_{\text{ACFA}}(b).$$

For each fixed $m \in \mathbb{Z} \setminus \{0\}$ and every $n \in J$ with $n > |m|$, algebraic independence of the n -cycle over A_0 gives

$$\sigma^m(c_0 - b_n) \notin \{c_0 - b_n, c_1 - b_n\}.$$

Indeed, either equality would give a nontrivial linear relation over A_0 between b_n and $\sigma^m(b_n)$. Together with $c_0 \neq c_1$, Łoś's theorem therefore shows that $a := c_0 - b$ has infinite σ -orbit and $c_1 - b$ lies outside that orbit.

Proposition 5.2, supplying (A) together with the preservation requirement (D), now gives an S_T -extension of (N, σ, τ) satisfying $\Phi_{\text{ACFA}}(b)$. Existential closedness forces the same formula to hold in N , contradicting (6). $\square$

Remark 5.5. We finish this section with some comments on the scope of the argument

- (i) For some completions of ACFA, σ fixes every root of unity. A proof requiring $\sigma(\zeta) \neq \zeta$ would miss these completions. Interchanging c_0, c_1 makes σ invert the radicand; this replaces the required mismatch by $\sigma(\zeta) \neq \zeta^{-1}$, which holds when ℓ is odd and $\sigma(\zeta) = \zeta$.
- (ii) Finite ramification and trivial restriction of inertia at unramified points are valid more generally. In this proof, however, the coefficient field is algebraically closed because it is the field reduct of a model of ACFA. Algebraic closedness is used for the Hahn construction and for Lemma 5.1. The proof does not assume that an arbitrary pair of commuting automorphisms extends to an algebraic closure. It uses a particular algebraically closed extension and modifies one lift by a power of η .
- (iii) The theorem concerns a fixed completion of ACFA, so the nonexistence theorem for arbitrary fields with two commuting automorphisms alone is not enough. The distinction and the algebraic-lifting difficulty are discussed in [12, Section 3] and [5, Remark 1.12].

6. MODEL-COMPLETE REDUCTS AND EXISTENTIALLY CLOSED MODELS

The arguments above concern automorphism expansions of prescribed complete theories. We conclude with some general observations about completions and existentially closed models. First, we prove a transfer result for model companions over model-complete reducts and show that its converse fails even for automorphism expansions. We then characterize model-completeness of a complete theory by the existence of a weakly saturated existentially closed model. Finally, we show that an incomplete universal theory can have a model-complete completion without having such a model.

We use the usual general definition: a model companion of a theory S is a model-complete theory S^* such that every model of either theory embeds into a model of the other. In particular, we do not assume $S \subseteq S^*$ in this definition.

Proposition 6.1. *Let $L_0 \subseteq L$ be languages, R be a model-complete L_0 -theory, and S be an L -theory extending R . Suppose that S has a model companion S^* . If Γ is a set of L_0 -sentences such that $S \cup \Gamma$ is consistent, then $S^* \cup \Gamma$ is a model companion of $S \cup \Gamma$.*

Proof. We first show that $S^* \models R$. Given $A \models S^*$, we use the companion property to obtain the following L -embeddings:

$$A \subseteq B \subseteq C, \quad B \models S, \quad C \models S^*.$$

Since S^* is model-complete, $A \preceq C$. The model-complete theory R admits an $\forall\exists$ -axiomatization. Consider such an axiom

$$\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}),$$

where φ is quantifier-free, and let $\bar{a}$ be an arbitrary tuple from A of length $|\bar{x}|$. Since $B \models R$, there is a tuple $\bar{b}$ from B such that $B \models \varphi(\bar{a}, \bar{b})$. The inclusion $B \subseteq C$ preserves quantifier-free formulas, so $C \models \varphi(\bar{a}, \bar{b})$ and consequently $C \models \exists \bar{y} \varphi(\bar{a}, \bar{y})$. As $A \preceq C$ and $\bar{a}$ lies in A , we obtain $A \models \exists \bar{y} \varphi(\bar{a}, \bar{y})$. Thus A satisfies every axiom of R , and hence $A \models R$.

Now embed $M \models S \cup \Gamma$ into $N \models S^*$. Both L_0 -reducts satisfy R (since $R \subseteq S$ and $S^* \models R$), so the embedding is elementary on these reducts. Hence $N \models \Gamma$. Conversely, if $N \models S^* \cup \Gamma$, embed N into $M \models S$; the same argument gives $M \models \Gamma$. This proves the mutual embedding property. Finally, $S^* \cup \Gamma$ is model-complete, being an extension of a model-complete theory in the same language. $\square$

Corollary 6.2. *Let R be model-complete, possibly incomplete. If R_τ has a model companion C , then $C \cup R'$ is a model companion of $(R')_\tau$ for every completion R' of R . Consequently, if $(R')_\tau$ has no model companion for some completion R' of R , then R_τ has no model companion.*

Proof. Apply Proposition 6.1 with $S = R_\tau$ and $\Gamma = R'$. Consistency holds because every model of R' admits the identity automorphism. $\square$

In the setting of Theorem 1.2, take $R = \text{ACFA}$ in the language containing σ . An automorphism τ of this reduct is precisely a field automorphism commuting with σ . Thus the failure of a model companion over a single completion of ACFA would already imply its failure over ACFA itself; Theorem 1.2 establishes the stronger assertion for every completion. In Theorem 1.1, the requirement that the added automorphism fix $\mathbf{d}$ is likewise incorporated by including the named constants in the reduct language.

We show that the implication from Corollary 6.2 cannot be reversed. We need a result from [9] first.

Lemma 6.3. *The theory of infinite sets with a bijection τ has a model companion, axiomatized by requiring infinitely many τ -cycles of each finite positive length.*

Proof. Apply [9, Example 2.8] with $G = \mathbb{Z}$. A $\mathbb{Z}$ -action is equivalent to a single bijection τ , and the axiom scheme given there is equivalent to requiring infinitely many τ -cycles of every finite positive length. $\square$

Example 6.4. There is a model-complete theory T such that $(T')_\tau$ has a model companion for every completion T' of T , whereas T_τ has no model companion.

Proof. Let $L = \{P_n \mid n \geq 2\}$ be the language consisting of unary predicate symbols, and let θ_n be an L -sentence saying that the universe has at least n elements. Let us define

$$T := \{\forall x (P_n(x) \leftrightarrow \theta_n) \mid n \geq 2\}.$$

Thus if $M \models T$, then $P_n(M)$ is full or empty according to the cardinality of the universe. Hence for an extension $M \subseteq N$ of models of T we have:

- if M is finite, then $M = N$;

- if M is infinite, then for all $n \geq 2$ we have $P_n(M) = M$ and $P_n(N) = N$, so $M \preceq N$ by the theory of infinite pure sets.

Therefore T is model-complete.

The completions of T are exactly

$$T_m = T \cup \{\theta_m, \neg\theta_{m+1}\} \quad \text{for } m \geq 1, \quad T_\infty = T \cup \{\theta_n : n \geq 2\}.$$

Every embedding between models of $(T_m)_\tau$ is an isomorphism, so $(T_m)_\tau$ is already model-complete. The theory $(T_\infty)_\tau$ has the companion from Lemma 6.3, together with the axioms making every P_n full.

Suppose that TA were a model companion of T_τ . For any $m \geq 1$, let $A_m \models T_\tau$ be the m -element model of T with $\tau = \text{id}$. There are embeddings

$$A_m \hookrightarrow B \hookrightarrow D, \quad B \models TA, \quad D \models T_\tau.$$

The composite preserves all cardinality thresholds, so $|D| = m$. It follows that the first embedding is an isomorphism, and hence $A_m \models TA$.

For a nonprincipal ultrafilter $\mathcal{V}$ on the positive integers, the ultraproduct

$$A = \prod_{m \geq 1} A_m / \mathcal{V}$$

is an infinite model of TA with every P_n full and $\tau = \text{id}$. Adjoining a disjoint two-element τ -cycle, and keeping all predicates full, gives an extension $E \models T_\tau$. Embed E into $B' \models TA$. The induced embedding $A \hookrightarrow B'$ is not elementary, since

$$A \models \forall x \tau(x) = x, \quad B' \models \exists x \tau(x) \neq x.$$

This contradicts model-completeness of TA . $\square$

We now consider existential closedness relative to an arbitrary theory T . The following result was pointed out to us by Udi Hrushovski.

Proposition 6.5. *Let T be a theory and M be an existentially closed model of T . Suppose that M is “weakly saturated”, that is it realizes every complete type over $\emptyset$ in finitely many variables consistent with $\text{Th}(M)$. Then $\text{Th}(M)$ is model-complete.*

Proof. Put $S = \text{Th}(M)$. Let $A \models S$, let $A \subseteq B \models T$, and suppose that $B \models \varphi(\bar{a})$, where φ is existential and $\bar{a}$ is a finite tuple from A . Let us choose $\bar{a}'$ from M such that

$$\text{tp}_M(\bar{a}'/\emptyset) = \text{tp}_A(\bar{a}/\emptyset).$$

We claim that the theory

$$T' := T \cup \text{Diag}_{\text{qf}}(M) \cup \{\varphi(\bar{a}')\}$$

is consistent.

Let us take $\delta(\bar{a}', \bar{m}) \in \text{Diag}_{\text{qf}}(M)$. Since $M \models \exists \bar{y} \delta(\bar{a}', \bar{y})$, equality of types gives $A \models \exists \bar{y} \delta(\bar{a}, \bar{y})$. Witnesses in A still satisfy δ in B . Interpreting the finitely many diagram constants by these witnesses and $\bar{a}$ therefore realizes the finite diagram conditions together with T and $\varphi(\bar{a}')$ in B , proving the claim.

A model of T' yields an extension $N \supseteq M$ satisfying T and $\varphi(\bar{a}')$. Existential closedness gives $M \models \varphi(\bar{a}')$, and equality of types gives $A \models \varphi(\bar{a})$. Thus every $A \models S$ is existentially closed among models of T . In particular, every embedding between models of S is existential, so Robinson’s test implies that S is model-complete. $\square$

Corollary 6.6. *If T has a weakly saturated model M which is existentially closed among models of T , then $\text{Th}(M)$ is a model-complete completion of T .*

For complete theories, the preceding implication yields an exact characterization of model-completeness.

Corollary 6.7. *If T is a complete theory, then T is model-complete if and only if T has a weakly saturated model which is existentially closed.*

Proof. Suppose first that T is model-complete. Every embedding between models of T is elementary, so every model of T is existentially closed among models of T . Choosing an $\aleph_0$ -saturated model gives the required model.

Conversely, suppose that $M \models T$ is weakly saturated and existentially closed among models of T . By Corollary 6.6, $\text{Th}(M)$ is model-complete. Since T is complete, it is logically equivalent to $\text{Th}(M)$, and hence is model-complete. $\square$

Remark 6.8. For every infinite cardinal κ , Corollary 6.7 remains true with “weakly saturated” replaced by “ κ -saturated”. Indeed, every consistent theory has a κ -saturated model, and if T is model-complete, every model of T is existentially closed among models of T . Conversely, κ -saturation implies weak saturation.

Finally, we observe that the converse of Corollary 6.6 fails, even when T is universal.

Example 6.9. There is a universal theory T in a countable language, with no finite models, which has a model-complete completion but no weakly saturated existentially closed model.

Proof. Let $L = \{E\} \cup \{c_n \mid n < \omega\}$, where E is a binary predicate symbol and the c_n are constant symbols. Let T assert that E is an equivalence relation, that the constants are pairwise distinct, and that

$$\forall x (E(x, c_n) \longrightarrow x = c_n) \quad \text{for } n < \omega.$$

These are universal axioms, and every model is infinite. The theory

$$S = T \cup \{\forall x \forall y (E(x, y) \leftrightarrow x = y)\}$$

is a model-complete completion of T : it is the theory of an infinite pure set with distinct named constants and E defined as equality. Quantifier elimination for infinite pure sets gives quantifier elimination here, and the prescribed inequalities between constants decide all quantifier-free sentences.

Now let $M \models T$ be weakly saturated. Consider the partial type over the empty set

$$p(x) = \{x \neq c_n : n < \omega\} \cup \{\forall y (E(x, y) \longrightarrow y = x)\}.$$

Every finite subset of $p(x)$ is realized in M by some c_m whose index is not among those excluded. By compactness, $p(x)$ extends to a complete type over $\emptyset$ consistent with $\text{Th}(M)$. Weak saturation therefore gives a realization $a \in M$ of $p(x)$.

We adjoin a new element b to the class of a , leaving all other classes unchanged. The resulting extension N still satisfies T , but

$$N \models \exists y (E(a, y) \wedge y \neq a), \quad M \models \neg \exists y (E(a, y) \wedge y \neq a).$$

Hence M is not an existentially closed model of T . $\square$

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