OF MODEL COMPLETENESS AND ALGEBRAIC GROUPS

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Abstract. We show that if G is a simply connected semi-simple algebraic group and K is a model complete field, then the theory of the group $G(K)$ is model complete as well.

1. Introduction

The goal of this project is to show model completeness of some groups with geometric origin. Recall that model completeness is a weaker and, hence, more flexible variant of quantifier elimination - we can reduce formulas to the form, where only a tuple of existential quantifiers remains. Many structures from algebra, geometry, and number theory are model complete but do not have quantifier elimination in their natural languages. Examples include fields $\mathbb{R}$ of real numbers [10, Theorem 2.7.3], the field $\mathbb{Q}_p$ of p-adic numbers [16], the exponential field $(\mathbb{R}, \exp)$ of real numbers [31], etc. Another favorite source of examples for model theorists comes from the model companion construction, which serves as an analogue to the universal domain à la Weil for more delicate theories other than fields. Note that model companions are automatically model complete but usually do not have quantifier elimination. Examples include real closed fields (RCF) [10, Theorem 2.7.3], p-adically closed fields (pCF) [16], algebraically closed valued fields (ACVF) [29], algebraically closed fields with a generic automorphism ACFA [3], or more exotic examples like G-TCF [1, 11], ACFO and interpolative fusions [28, 15], ACFG [5], CXFs [14].

An *algebraic group*, for us, is a group scheme of finite type over a field K. An algebraic group G is *semi-simple* if it is infinite and any normal commutative subgroup of $G(K^{\mathrm{alg}})$ is finite. Furthermore, G is *connected* if it is connected with respect to the Zariski topology. A connected algebraic group G is *simply connected* if every isogeny from a connected algebraic group to G is an isomorphism. Here, an *isogeny* is defined as an algebraic group epimorphism with a finite kernel. Our main result (see Theorem 3.14) is as follows:

Main Theorem. *Let G be a simply connected semi-simple algebraic group over $\mathbb{Z}$ and K be a model complete field (in the language of rings). Then $G(K)$ (in pure group language) is model complete.*

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In particular, our result implies both $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ are model complete. Initially, the project started without P.K., and we concentrated on these examples to demonstrate the feasibility of such results. P.K. joined the project following D.M.H.'s talk at the Antalya Algebra Days conference in May 2023, contributing new ideas to establish model completeness for a broader class of groups.

There are several further motivations for the main theorem. For the field $\mathbb{R}$ and exponential field $(\mathbb{R}, \exp)$, model completeness can be used to show the geometric properties of these structures (more precisely, these structure are o-minimal, so definable sets and definable groups are very close to classical geometric objects like smooth manifolds and Lie groups). Our heuristic is that the implication also often goes the other way in the presence of groups, in particular, non-abelian groups of geometric significance such as $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ should be model-complete. This view is also informed by a separated project on locally compact groups and Lie groups by D.M.H., where a related first-order structure is shown to have well behavior provided the theory of the corresponding pure group is model complete.

Let us briefly shift the focus to the related problem of describing canonical topologies on natural examples in model theory (e.g., the Euclidean topology on the field of real numbers, the valuation topology on ACVF, etc). In the case of pure fields, the natural topology can be recovered by considering étale images, as outlined in [12], with subsequent studies further expanding on this concept [13, 7, 30]. This approach has limitations; for instance, in a model of ACVF, it incorrectly yields the Zariski topology, where we do not have constructibility of definable sets. A viewpoint by C.M.T., also one of the authors of [12], is that one would ultimately like to recover topology from group structures instead of fields. For $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$, this should come from suitable variant of the notion of étale images. Establishing the model completeness of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ is the first step in this direction.

Last but not least, the interpretation of a field within an algebraic group continues to be an interesting research direction in model theory. For instance, Ali Nesin achieved this for $\mathrm{SO}_3(\mathbb{R})$ in [20]. in [20]. A similar result is obtained in [21], in Theorem 1.1 from [22], and more recently in [25]. In our approach, interpreting the field inside the group is one of the steps in proving model completeness of a given algebraic group. More precisely, if $H \equiv G(K)$ for some algebraic group G, field K and group H, then we want to find a field M such that $H \cong G(M)$. The last thing follows from earlier results ([25], [27]) for simply connected simple algebraic groups (Theorem 2.17). In our study, we are more interested in the complexity of such bi-interpretations. Particularly, we want this interpretation to be simple in the sense of logical quantifiers so it remains geometric to some extent. To highlight our result, in Corollary 2.12, we provide a fact saying that homomorphisms between simply connected simple algebraic groups decompose into a field homomorphism part and an algebraic group isomorphism part. Having that on board, we can prove Theorem 3.1, stating model completeness of simply connected simple algebraic groups over model complete fields. Then, we provide a variant of the Borel-Tits theorem (Theorem 2.14), develop a criterion for model completeness of products (Theorem 3.10) and use the Feferman-Vaught theorem to generalize our Theorem 3.1 to the class of simply connected semi-simple algebraic groups over model complete fields. This is one of the main parts of the paper, finalized with Theorem 3.14.

The main text is divided into 3 sections: in Section 2 we recall definitions and classical facts from model theory and algebraic groups and in Section 3 we obtain the main result, i.e. model completeness of (simply connected) semi-simple algebraic groups.

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2. Homomorphism between rational points of algebraic groups

2.1. **Classical results on algebraic groups.** By a *simple algebraic group* over a field K, we mean an infinite algebraic group G over K such that any proper normal subgroup of $G(K^{\mathrm{alg}})$ is finite. By a *simple group scheme* over $\mathbb{Z}$, we mean a group scheme G over $\mathbb{Z}$ such that for each field K, the base change group scheme G_K is a simple algebraic group G over K. **From now on, when we simply refer to an algebraic group G, it is assumed to be over $\mathbb{Z}$.**

By the radical of an algebraic group G over K, denoted $\mathrm{Rad}(G)$, we mean its maximal connected normal algebraic solvable subgroup over K. By a *semi-simple algebraic group* G over a field K, we mean an infinite algebraic group over K such that $\mathrm{Rad}(G)$ is trivial (equivalently, any normal commutative subgroup of $G(K^{\mathrm{alg}})$ is finite). The definition of a *semi-simple group scheme* over $\mathbb{Z}$ is analogous to the one above.

A connected algebraic group G is *simply connected* if any isogeny from a connected algebraic group to G is an isomorphism, where an *isogeny* is an algebraic group epimorphism with a finite kernel.

Remark 2.1. For other possible choices of the terminology above, the reader is advised to consult [19, Definition 19.8]. General definitions are given e.g. here [6, Def XIX, 2.7].

We state below several classical results.

Theorem 2.2 (Chapter 24a in [19])**.** *Let G be a semi-simple algebraic group over a field K.*

(1) *There is an "almost-decomposition" of G into the product of simple algebraic groups $G_1, \ldots, G_l$ over K that is there is an epimorphism*

$$G_1 \times \ldots G_l \longrightarrow G$$

with a finite kernel.

(2) *If G is simply connected, then the simple algebraic groups $G_1, \ldots, G_l$ are simply connected as well and the above epimorphism is an isomorphism.*

The second item in the above theorem might be found as well in [18], see Prop. 1.4.10 there. It is worth mentioning that the decomposition from the second item above is unique up to isomorphism.

Theorem 2.3. *Simple group schemes over $\mathbb{Z}$ are fully classified: each of them is given (up to an isomorphism over $\mathbb{Z}$) by a Dynkin diagram as in Table 9.2 on* [17, page 72] *(this table also lists the simply connected ones).*

Theorem 2.4 ([4])**.** *Let H be a (semi-)simple algebraic group over an algebraically closed field K. Then, there is a (semi-)simple algebraic group scheme G over $\mathbb{Z}$ such that $H \cong G_K$.*

The result below was proved by Rosenlicht in the case of a perfect field M and by Grothendieck in the arbitrary case. We will need it for model complete fields which are necessarily perfect.

Theorem 2.5 ([24] and [6])**.** *If G is a reductive algebraic group defined over a field K (in particular: it applies to semi-simple algebraic groups), then $G(K)$ is Zariski dense in $G(K^{\mathrm{alg}})$.*

Theorem 2.6. *Suppose that G is a simply-connected simple algebraic group scheme over $\mathbb{Z}$ and K is an infinite field. Then we have the following.*

1. *Any proper normal subgroup of $G(K)$ is contained in $Z(G(K))$ and $Z(G(K))$ is finite.*
2. *Each element of $G(K)$ is a commutator.*

Proof. Item (1) follows by [23, Theorem 7.1] and the paragraph above [23, Proposition 7.5], since, by Theorem 2.5, we have $Z(G(K)) \subseteq Z(G(K^{\mathrm{alg}}))$.

For Item (2), by [8, Theorem 1] and the paragraph below it, we obtain that $G(K)$ satisfies the Thompson conjecture, that is each element of $G(K)$ is a commutator (see also the second paragraph of the Introduction to [8]). □

Remark 2.7. We will actually need only that for each G as above, there is $n > 0$ such that for each infinite field K, each element of $G(K)$ is the product of n commutators. This follows by the Compactness Theorem for any elementary class of perfect groups and it applies to our case by Theorem 2.17(1).

For a homomorphism of algebraic groups $f : H \to G$ over K, we denote by $f_K : H(K) \to G(K)$ the corresponding group homomorphism between the rational points. If $\varphi : K \to K'$ is a field homomorphism, then $\varphi_G : G(K) \to G(K')$ denotes the corresponding group homomorphism between the rational points and by ${}^{\varphi}H$ we denote the corresponding (change of basis) algebraic group over K' (we have $H(K') = {}^{\varphi}H(K')$). We will often use the following crucial result [26, Theorem 1.3] originating from [2, (A)], which we state in a slightly simplified form.

Theorem 2.8 (Theorem 1.3 in [26])**.** *Let H, G be simple algebraic groups defined over infinite fields L, M respectively. Assume that H is simply connected. Let $\alpha : H(L) \to G(M)$ be a group homomorphism such that $\alpha(H(L))$ is Zariski dense (see Remark 2.9(1)). Then there exist:*

- *a field homomorphism $\varphi : L \to M$,*
- *an isogeny $\beta : {}^{\varphi}H \to G$,*
- *a homomorphism $\gamma : H(L) \to Z(G(M))$*

such that for all $h \in H(L)$, we have:

$$\alpha(h) = \gamma(h) \cdot \beta_M(\varphi_H(h)),$$

where $\cdot$ is the group operation in $G(M)$.

Remark 2.9. We collect here some observations regarding the last theorem.

(1) By Theorem 2.5 the assumption that $\alpha(H(L))$ is Zariski dense (in the statement of Theorem 2.8) is unambiguous since being Zariski dense in $G(M)$ is the same as being Zariski dense in $G(M^{\mathrm{alg}})$.
(2) By Theorem 2.6, $H(K)$ is perfect, hence γ is trivial and we get that $\alpha = \beta_M \circ \varphi_H$ in the situation from Theorem 2.8. (It is conjectured in [2] that γ is always trivial.)
(3) We will usually apply Theorem 2.8 in the situation when G, H come from simple group schemes over $\mathbb{Z}$ (as in Theorem 2.4). In such a case, there will be no need for the base change ${}^{\varphi}H$, which appears in the statement of Theorem 2.8.

2.2. **Homomorphisms.** In this subsection, we collect some consequences of Theorem 2.8.

Proposition 2.10. *Let H be a simply connected simple algebraic group over an infinite field L, G be a linear algebraic group over a field M, and $\alpha : H(L) \to G(M)$ be a non-trivial homomorphism. Let G_0 denote the algebraic subgroup of G being the Zariski closure of $\alpha(H(L))$ in $G(M^{\mathrm{alg}})$ (we identify here G_0 with $G_0(M^{\mathrm{alg}})$). Then, there is a field homomorphism $\varphi : L \to M^{\mathrm{alg}}$ and an isogeny between ${}^{\varphi}H$ and a quotient of G_0.*

Proof. By Theorem 2.5 and Theorem 2.6(1), α has infinite image. By Theorem 2.6(1) again, we have:

$$\mathrm{Rad}(G_0)(M) \cap \alpha(H(L)) \subseteq \alpha(Z(H(L))).$$

Let us define:

$$G_1 := (G_0)_{\mathrm{ss}} := G_0/\mathrm{Rad}(G_0).$$

Then, the following composition map:

$$H(L) \longrightarrow G_0\left(M^{\mathrm{alg}}\right) \longrightarrow G_1\left(M^{\mathrm{alg}}\right)$$

has infinite image that is dense. Since G_1 is semi-simple, by Theorem 2.2(1) and the same argument as above, we can assume that G_1 is simple. We apply now Theorem 2.8 for the induced homomorphism $H(L) \to G_1(M^{\mathrm{alg}})$, hence we obtain a field homomorphism $\varphi : L \to M^{\mathrm{alg}}$ and an isogeny ${}^{\varphi}H \to G_1$. □

Remark 2.11. One observation and two minor generalizations.

(1) The algebraic group G_0 appearing in the statement of Proposition 2.10 need not be simple, for example let us consider:
$$H = \mathrm{SL}_2,\ G = \mathrm{SL}_2 \times \mathrm{SL}_2,\ L = M = \mathbb{C},\ \alpha(g) = (g, \bar{g}),$$
where $g \mapsto \bar{g}$ is the complex conjugation.
(2) The group G_0 is definable over M, so it is defined (as an algebraic group) over the perfect hull of M.
(3) By the Chevalley's structure theorem, we could have dropped the linearity assumption on G.

We immediately obtain the following.

Corollary 2.12. *Suppose that H, G are simply connected simple algebraic groups defined over infinite fields L, M respectively. Let $\alpha : H(L) \to G(M)$ be a non-trivial homomorphism. Then we have the following.*

(1) $\dim(H) \leqslant \dim(G)$.
(2) *If* $\dim(H) = \dim(G)$, *then there is a field homomorphism* $\varphi : L \to M$ *and an algebraic group isomorphism* $\beta : {}^{\varphi}H \to G$ *such that* $\alpha = \beta_M \circ \varphi_H$.

Proof. Item (1) follows directly from Proposition 2.10. For Item (2), Proposition 2.10 implies that the image of α is Zariski dense, so we can apply Theorem 2.8 and Remark 2.9(2). Since G is simply connected as well, the isogeny β is an isomorphism. □

We will need some results about homomorphisms between rational points of products of simple group schemes.

Theorem 2.13. *Let* G_1, G_2 *be simply connected simple group schemes defined over* $\mathbb{Z}$ *and* L_1, L_2, M_1, M_2 *be infinite fields. Assume that*

$$\Psi : G_1(L_1) \times G_2(L_2) \longrightarrow G_1(M_1) \times G_2(M_2)$$

is a group monomorphism. Then one of the following happens:

- *there are monomorphisms*
$$\Psi_1 : G_1(L_1) \longrightarrow G_1(M_1), \quad \Psi_2 : G_2(L_2) \longrightarrow G_2(M_2)$$
such that $\Psi = \Psi_1 \times \Psi_2$

or

- *there are monomorphisms*
$$\Psi_1 : G_1(L_1) \longrightarrow G_2(M_2), \quad \Psi_2 : G_2(L_2) \longrightarrow G_1(M_1)$$
such that $\Psi = \mathrm{twist} \circ (\Psi_1 \times \Psi_2)$ *and* $G_1 \cong G_2$, *where "twist" is the obvious permutation of coordinates.*

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Proof. We have four homomorphisms $\Psi^i_j : G_i(L_i) \to G_j(M_j)$ for $i, j \in \{1, 2\}$. For example, Ψ^1_1 is defined as the following composition:

$$G_1(L_1) \hookrightarrow G_1(L_1) \times G_2(L_2) \xrightarrow{\Psi} G_1(M_1) \times G_2(M_2) \twoheadrightarrow G_1(M_1).$$

We need to show that Ψ^1_2, Ψ^2_1 are trivial OR Ψ^1_1, Ψ^2_2 are trivial and $G_1 \cong G_2$.

Claim
If Ψ^1_1 is non-trivial, then Ψ^2_1 is trivial (similarly for Ψ^2_2 and Ψ^1_2).

Proof of Claim. By Proposition 2.10, the image of Ψ^1_1 is Zariski dense. Therefore, we have:

$$C_{G_1(M_1)}\left(\Psi^1_1\left(G_1(L_1)\right)\right) = Z\left(G_1(M_1)\right).$$

Since

$$\Psi^2_1\left(G_2(L_2)\right) \subseteq C_{G_1(M_1)}\left(\Psi^1_1\left(G_1(L_1)\right)\right),$$

we obtain that the image of Ψ^2_1 is a commutative group. Since $G_2(L_2)$ is perfect (by Theorem 2.6(2)), Ψ^2_1 is trivial. □

We consider two cases.

Case 1 Ψ^1_1 is non-trivial.
By Claim, Ψ^2_1 is trivial. Since Ψ is one-to-one, we obtain that Ψ^2_2 is non-trivial. Using Claim again, we get that Ψ^1_2 is trivial.

Case 2 Ψ_1^1 is trivial.
Since Ψ is one-to-one, we obtain that Ψ_2^1 is non-trivial. By Claim, we get that Ψ_2^2 is trivial. It is enough to show now that $G_1 \cong G_2$.

By Corollary 2.12(1), we get that $\dim(G_1) \leqslant \dim(G_2)$. Since Ψ is one-to-one, we obtain as above that Ψ_1^2 is non-trivial. By Corollary 2.12(1), we obtain that $\dim(G_2) \leqslant \dim(G_1)$, so $\dim(G_1) = \dim(G_2)$. By Corollary 2.12(2), we get $\beta : G_1 \cong G_2$. □

Using the same arguments as in the proof of Theorem 2.13, one can show the following generalization.

Theorem 2.14. *Let $G_1, \ldots, G_n$ be simply connected simple algebraic groups defined over $\mathbb{Z}$ and $L_1, M_1, \ldots, L_n, M_n$ be infinite fields. Assume that*

$$\Psi : \prod_{i=1}^{n} G_i(L_i) \longrightarrow \prod_{i=1}^{n} G_i(M_i)$$

is a group monomorphism. Then, there is $\sigma \in \mathrm{Sym}(n)$ and monomorphisms

$$\Psi_i : G_i(L_i) \longrightarrow G_{\sigma(i)}\left(M_{\sigma(i)}\right) \qquad (i = 1, \ldots, n)$$

such that

$$\Psi = \widetilde{\sigma^{-1}} \circ (\Psi_1 \times \ldots \times \Psi_n),$$

where $\widetilde{\sigma^{-1}}$ is the obvious coordinate permutation and for each i, we have $G_i \cong G_{\sigma(i)}$.

2.3. **Miscellaneous model theory.** Here, very shortly, we provide some easy standard notions and facts from model theory.

Definition 2.15. Let $\mathcal{L}$ be a language and M be an $\mathcal{L}$-structure. We say that:

(1) M is *model complete* if $\mathrm{Th}(M)$ is model complete;
(2) M is *model complete with parameters* if $\mathrm{Th}_M(M)$ is model complete.

The next result is folklore.

Theorem 2.16. *Let G be an algebraic group over a field K. Then the group $G(K)$ has* finite centralizer dimension*, that is there is $n > 0$ such that there is no strictly decreasing chain of centralizers in $G(K)$ of length n.*

We need one more result, which was proved by Dan Segal and Katrin Tent, and also by Simon Thomas.

Theorem 2.17. *Let G be a simply connected simple group scheme defined over $\mathbb{Z}$ and K be a field. Then we have the following.*

(1) *If N is a group and $N \equiv G(K)$, then there is a field M such that $N \cong G(M)$.*
(2) *If M is a field such that $G(K) \equiv G(M)$, then $K \equiv M$.*

Proof. If $G \ncong \mathrm{SL}_2$, then the result follows from [25, Corollary 1.2]. If $G \cong \mathrm{SL}_2$, then the result follows from Theorem 2 and the discussion below it on page 55 of [27]. □

Lemma 2.18. *Assume that G_1, G_2 are simply connected simple algebraic groups over $\mathbb{Z}$ and K is a field. Then the subgroups $G_1(K) \times \{1\}$ and $\{1\} \times G_2(K)$ are existentially definable (with parameters) in the pure group $G_1(K) \times G_2(K)$.*

Proof. By Theorem 2.16, there are $u_1, \dots, u_n \in G_2(K)$ such that $C_{G_2(K)}(u_1, \dots, u_n) = Z(G_2(K))$. Let $\gamma(x)$ be the following formula:

$$[x, (1, u_1)] = \dots = [x, (1, u_n)] = (1, 1).$$

Then we have:

$$\gamma(G_1(K) \times G_2(K)) = C_{G_1(K)\times G_2(K)}((1, u_1), \dots, (1, u_n)) = G_1(K) \times Z(G_2(K)).$$

Let $\varphi(x)$ be the following formula:

$$\exists y, z \quad \gamma(y) \ \wedge \ \gamma(z) \ \wedge \ x = [y, z].$$

By Theorem 2.6(2), we obtain that $\varphi(G_1(K) \times G_2(K)) = G_1(K) \times \{1\}$. The argument for $\{1\} \times G_2(K)$ is analogous. □

In similar way, we can obtain the following, more general, variant:

Lemma 2.19. *Assume that $G_1, \dots, G_l$ are simply connected simple algebraic groups over $\mathbb{Z}$ and K is a field. Then there are parameters $\bar{u}$ in $G(K) := G_1(K)\times \dots G_l(K)$ and existential formulas $\psi_1(\bar{y}, x), \dots, \psi_l(\bar{y}, x)$ in the language of groups such that for every $i \leqslant l$ we have*

$$\psi_i(\bar{u}, G) = \{1\} \times \dots \times \{1\} \times G_i(K) \times \{1\} \times \dots \times \{1\}.$$

3. Proof of the main theorem

In this section, we start with proving the main result of our paper in the simple case, then we consider the semi-simple case with parameters and finally we consider the general semi-simple case without parameters.

3.1. Simple case.

Theorem 3.1. *Let G be a simply connected simple algebraic group and K be a model complete field. Then the structure $(G(K), \cdot)$ is model complete (in the language of groups).*

Proof. Let H, N be models of $\mathrm{Th}(G(K), \cdot)$ and $f : H \to N$ be a monomorphism. We need to show that f is elementary. Since any isomorphism is elementary and the composition of two elementary monomorphisms is again elementary, we can (and will) often replace f with $h \circ f \circ g$, where h and g are group isomorphisms.

By Theorem 2.17, there are fields L, M such that:

$$H \cong G(L), \quad N \cong G(M), \quad L \equiv K \equiv M.$$

Therefore (by the "we can often replace" observation above), we can assume that $f : G(L) \to G(M)$.

By Corollary 2.12, there there is a field homomorphism $\varphi : L \to M$ and an algebraic group automorphism $\beta : G \to G$ (defined over M) such that

$$\alpha = \beta_M \circ \varphi_G.$$

Therefore, we can assume that $f = G(\varphi)$, which is elementary, since the field K is model complete and $L \equiv K \equiv M$. □

3.2. **Model completeness with parameters.** We decided to prove two variants of model completeness. The easier one involves parameters in the language, the more difficult one, does not and thus implies the easier variant. The reason behind this decision is that we need to introduce the same tools for both variants, and after introducing these tools there is only one step (Lemma 3.8) before achieving the easier variant, so the model completeness with parameters (Proposition 3.9).

Definition 3.2. Consider a language $\mathcal{L}$, an $\mathcal{L}$-theory T, a model M of T and a definable subset $D \in \mathrm{Def}_x(M)$. We define a *D-restriction* of any formula $\varphi(\bar{x}) \in \mathcal{L}$ (where $\bar{x}$ is a tuple of variables, each from the same sort as the variable x) as follows. Let $\varphi(\bar{x})$ be $(Q_1\, y_1)\dots(Q_k\, y_k)\varphi_0(y_1,\dots,y_k,\bar{x})$, where $\varphi_0(y_1,\dots,y_k,\bar{x})$ is quantifier-free and $Q_1,\dots,Q_k$ is a sequence of quantifiers. Then $\varphi^D(\bar{x})$ is defined as

$$(Q_1\, y_1 \in D)\dots(Q_k\, y_k \in D)\big(\bigwedge_{x\in\bar{x}} x \in D \ \wedge\ \varphi_0(y_1,\dots,y_k,\bar{x})\big).$$

If D is given by a formula $\psi(x)$, then we also use the convention $\varphi^{\psi}(\bar{x})$ to denote $\varphi^D(\bar{x})$.

Remark 3.3. Consider the situation from Lemma 2.19 and define $D_j := \psi_j(\bar{u}, G(K))$ for some $j \leqslant l$. If $\varphi(x)$ is a formula in the group language, $\chi(x)$ is the formula

$$(\exists y_1,\dots,y_l)\big(x = y_1\cdot\ldots\cdot y_l \ \wedge\ \bigwedge_{k\leqslant l}\psi_k(\bar{u},y_k)\ \wedge\ \varphi^{D_j}(y_j)\big),$$

and $g = (g_1,\dots,g_l) \in G(K)$, then

$$G_j(K) \models \varphi(g_j) \quad\iff\quad G(K) \models \chi(g).$$

The proof is left to the reader.

Fact 3.4 (Cor 9.6.4 in [9])**.** *Let $\mathcal{L}$ be a language, let A and B be $\mathcal{L}$-structures and let $\varphi(\bar{x})$ be an $\mathcal{L}$-formula. Then there is a sequence $\big((\theta_i(\bar{x}),\chi_i(\bar{x}))\big)_{i\leqslant n}$ of pairs of $\mathcal{L}$-formulas, such that for all tuples $\bar{a} = (a_0,a_1,\dots) \in A^{\bar{x}}$ and $\bar{b} = (b_0,b_1,\dots) \in B^{\bar{x}}$, we have*

$$A\times B \models \varphi\big((a_0,b_0),(a,b_1),\dots\big) \quad\iff\quad \bigvee_{i\leqslant n}(A\models\theta_i(\bar{a})\ \wedge\ B\models\chi_i(\bar{b})).$$

In a straightforward manner, we conclude the below corollary.

Corollary 3.5. *Let $\mathcal{L}$ be a language, let $A_1,\dots,A_l$ be $\mathcal{L}$-structures and let $\varphi(\bar{x})$ be an $\mathcal{L}$-formula. Then there is a sequence $\big((\theta_i^1(\bar{x}),\dots,\theta_i^l(\bar{x}))\big)_{i\leqslant n}$ of l-tuples of $\mathcal{L}$-formulas, such that for all tuples $\bar{a}^1 = (a_0^1,a_1^1,\dots) \in A_1^{\bar{x}},\dots,\bar{a}^l = (a_0^l,a_1^l,\dots) \in A_l^{\bar{x}}$, we have*

$$\begin{aligned} A_1\times\ldots\times A_l \ &\models\ \varphi\big((a_0^1,\dots,a_0^l),(a_1^1,\dots,a_1^l),\dots\big)\\ &\iff \bigvee_{i\leqslant n}(A_1\models\theta_i^1(\bar{a}^1)\ \wedge\ldots\wedge\ A_l\models\theta_i^l(\bar{a}^l)).\end{aligned}$$

Corollary 3.6. *There is no harm to improve the above corollary, so it will work for sets definable over parameters: let $\mathcal{L}$ be a language, let $A_1,\dots,A_l$ be $\mathcal{L}$-structures, let $\bar{u}_0 = (u_0^1,\dots,u_0^l),\dots,\bar{u}_k = (u_k^1,\dots,u_k^l) \in A_1\times\ldots\times A_l$, and let $\varphi(\bar{x})$ be an $\mathcal{L}(\bar{u}_0,\dots,\bar{u}_k)$-formula. Then there is a sequence $\big((\theta_i^1(\bar{x}),\dots,\theta_i^l(\bar{x}))\big)_{i\leqslant n}$ of l-tuples* *of formulas, each* $\theta_i^j(\bar{x})$ *being an* $\mathcal{L}(u_0^j,\ldots,u_k^j)$*-formula, such that for all tuples* $\bar{a}^1=(a_0^1,a_1^1,\ldots)\in A_1^{\bar{x}},\ldots,\ \bar{a}^l=(a_0^l,a_1^l,\ldots)\in A_l^{\bar{x}}$*, we have*

$$\begin{aligned} A_1\times\ldots\times A_l \quad &\models \quad \varphi\big((a_0^1,\ldots,a_0^l),(a_1^1,\ldots,a_1^l),\ldots\big) \\ &\iff \bigvee_{i\leqslant n}(A_1\models\theta_i^1(\bar{a}^1)\ \wedge\ldots\wedge\ A_l\models\theta_i^l(\bar{a}^l)). \end{aligned}$$

Fact 3.7 (Cor 9.6.5(b) in [9])**.** *If* $M_1\preccurlyeq N_1, M_2\preccurlyeq N_2$ *then* $M_1\times M_2\preccurlyeq N_1\times N_2$*.*

Lemma 3.8. *Assume that* $G_1(K),\ldots,G_l(K)$ *are simply connected simple algebraic groups over a model complete field* K*, and set* $G(K):=G_1(K)\times\ldots\times G_l(K)$*. There exist parameters* $\bar{u}$ *in* $G(K)$ *such that, if* $\varphi(\bar{x})$ *is a formula in the language of groups, extended by parameters* $\bar{u}$*, with* $|\bar{x}|=m$*, then there is an existential formula* $\varphi_\exists(\bar{u},\bar{x})$ *in the language of groups such that*

$$G(K)\models(\forall\bar{x})\big(\varphi(\bar{x})\ \leftrightarrow\ \varphi_\exists(\bar{u},\bar{x})\big).$$

Proof. Let $\bar{u}$ and $\psi_1,\ldots,\psi_l$ be as in Lemma 2.19, and define $D_j:=\psi_j(\bar{u},G(K))$.

Consider formulas $\big((\theta_i^1(\bar{x}),\ldots,\theta_i^l(\bar{x}))\big)_{i\leqslant n}$ given by Corollary 3.6. By Theorem 3.1, without loss of generality we may assume that every $\theta_i^j(\bar{x})$ is an existential formula. Fix tuples $\bar{g}^1=(g_0^1,g_1^1,\ldots,g_{m-1}^1)\in G_1(K)^{\bar{x}},\ldots,\ \bar{g}^l=(g_0^l,g_1^l,\ldots,g_{m-1}^l)\in G_l(K)^{\bar{x}}$. Set

$$\bar{g}:=\big((g_0^1,\ldots,g_0^l),\ldots,(g_{m-1}^1,\ldots,g_{m-1}^l)\big).$$

Then

$$G(K)\models\varphi(\bar{g})\qquad\iff\qquad\bigvee_{i\leqslant n}\bigwedge_{j\leqslant l}G_j(K)\models\theta_i^j(\bar{g}^j).$$

To finish the proof, we need to be able to express "$G_j(K)\models\theta_i^j(\bar{g}^j)$" in terms of the satisfiability in $G(K)$ of some existential formulas with the tuple $\bar{g}$.

We fix $i\leqslant n$ and $j\leqslant l$ and work with the case of "$G_j(K)\models\theta_i^j(\bar{g}^j)$". Consider the formula $\chi_i^j(\bar{u},x_0,\ldots,x_{m-1})$ given by

$$\begin{gathered}(\exists y_0^1,\ldots,y_0^l,\ldots,y_{m-1}^1,\ldots,y_{m-1}^l))\\ \Big(\bigwedge_{k<m}(x_k=y_k^1\cdot\ldots\cdot y_k^l\ \wedge\ \psi_1(\bar{u},y_k^1)\ \wedge\ldots\wedge\ \psi_l(\bar{u},y_k^l))\ \wedge\\ (\theta_i^j)^{D_j}(y_0^j,y_1^j,\ldots,y_{m-1}^j)\Big),\end{gathered}$$

where $(\theta_i^j)^{D_j}$ is the D_j-restriction of formula θ_i^j. Similarly as in Remark 3.3, it follows that

$$G_j(K)\models\theta_i^j(\bar{g}^j)\qquad\iff\qquad G(K)\models\chi_i^j(\bar{u},\bar{g}).$$

Finally, we set $\varphi_\exists(\bar{u},\bar{x})$ to be

$$\bigvee_{i\leqslant n}\bigwedge_{j\leqslant l}\chi_i^j(\bar{u},\bar{x}).$$

□

Proposition 3.9. *Let* $G(K)$ *be a simply connected semi-simple algebraic group and* K *be a model complete field. Then there exists a finite tuple of parameters* $\bar{u}$ *in* $G(K)$ *such that the structure* $(G(K),\cdot,\bar{u})$ *(pure group language with parameters* $\bar{u}$*) is model complete.*

Proof. By Theorem 2.2, there are simply connected simple algebraic groups $G_1, \dots, G_l$ such that:

$$G(K) \cong G_1(K) \times \dots G_l(K).$$

Now, we use Lemma 3.8 to show that every formula is equivalent to an existential one, which is equivalent to model completeness. □

3.3. **Model completeness without parameters.** We start this subsection with proving a general result, which will be used in the proof of the main theorem. This general result is a criterion for model completeness of a product of model complete structures. As usual, we avoid heavy involved notation in the proof by proving the statement for the case of pair of structures.

Theorem 3.10. *Let $M_1, \dots, M_n$ be $\mathcal{L}$-structures in a common language $\mathcal{L}$ such that the following holds.*

(1) *$M_1, \dots, M_n$ are model complete.*
(2) *For any $\aleph_0$-saturated $S \equiv M_1 \times \dots \times M_n$, there are $M_1' \equiv M_1, \dots, M_n' \equiv M_n$ such that $S \cong M_1' \times \dots \times M_n'$.*
(3) *For any $M_1' \equiv M_1 \equiv M_1'', \dots, M_n' \equiv M_n \equiv M_n''$ and any embedding*

$$\Psi : M_1' \times \dots \times M_n' \to M_1'' \times \dots \times M_n'',$$

there is $\sigma \in \mathrm{Sym}(n)$ and embeddings

$$\Psi_i : M_i' \longrightarrow M''_{\sigma(i)} \qquad (i = 1, \dots, n)$$

such that

$$\Psi = \widetilde{\sigma^{-1}} \circ (\Psi_1 \times \dots \times \Psi_n)$$

(see Theorem 2.14 for the corresponding notation) and for each i, we have $M_i' \equiv M''_{\sigma(i)}$.

Then, $M_1 \times \dots \times M_n$ is model complete.

Proof. We provide the proof for $n = 2$, the general case is similar. Take $S' \subseteq S''$ being models of $\mathrm{Th}(M_1 \times M_2)$. Without loss of generality, we can pass to ultrapowers and assume that both S' and S'' are $\aleph_0$-saturated. By the second assumption, we replace S' and S'' with products $M_1' \times M_2'$ and $M_1'' \times M_2''$ respectively. Then, the first and the third assumption allow us to reduce the problem to Fact 3.7.

□

Remark 3.11.
- Clearly, we could have replaced $\aleph_0$ in the statement of Theorem 3.10 with any other cardinal.
- Martin Ziegler kindly provided to us two proofs of a stronger version of Theorem 3.10, where the assumption (2) is removed. One proof uses special models together with expandable models, and the other proof uses Robinson's joint consistency lemma.

Roughly speaking, if G is a finite product of simply connected simple algebraic groups $G_1, \dots, G_l$, then we are almost in the situation of Theorem 3.10 (the first point follows by Theorem 3.1, the third point by Theorem 2.14) - we need to satisfy the second point and let us achieve that now.

Lemma 3.12. *Let G_1, G_2 be simply connected simple algebraic groups over an infinite model complete field K and $G := G_1 \times G_2$. Let H be an $\aleph_0$-saturated group such that $H \equiv G(K)$. Then there exist $H_1 \equiv G_1(K)$ and $H_2 \equiv G_2(K)$ such that $H \cong H_1 \times H_2$.*

Proof. If $\varphi(x, \bar{y})$ is a formula (over $\emptyset$) such that $\varphi(G(K), \bar{g}) \leqslant G(K)$ for some $\bar{g} \subset G(K)$, then for any sentence α, we define the formula $\alpha_\varphi(\bar{y})$ such that for all groups N and all $\bar{n} \subset N$ of the appropriate length, we have:

$$N \models \alpha_\varphi(\bar{n}) \qquad \Leftrightarrow \qquad \varphi(N, \bar{n}) \leqslant N \ \wedge \ \varphi(N, \bar{n}) \models \alpha$$

(one may check the formalism of Definition 3.2 and Remark 3.3 to see that the formula $\alpha_\varphi(\bar{y})$ exists). By Lemma 2.19, there exists a tuple $\bar{u}$ in G, and formulas $\varphi_1(x, \bar{y})$ and $\varphi_2(x, \bar{y})$ such that

$$\varphi_1(G(K), \bar{u}) = G_1(K) \times \{1\}, \qquad \varphi_2(G(K), \bar{u}) = \{1\} \times G_2(K).$$

We define the following type (with respect to $\mathrm{Th}(G(K))$)

$$\Sigma(\bar{y}) := \{\alpha_{\varphi_1}(\bar{y}) \wedge \beta_{\varphi_2}(\bar{y}) \mid \alpha \in \mathrm{Th}(G_1(K)), \beta \in \mathrm{Th}(G_2(K))\} \cup \{\psi(\bar{y})\},$$

where for all groups N and all $\bar{n} \subset N$, we have:

$$N \models \psi(\bar{n}) \qquad \Leftrightarrow \qquad N \text{ is the internal direct product of } \varphi_1(N, \bar{n}) \text{ and } \varphi_2(N, \bar{n}).$$

The set $\Sigma(\bar{y})$ is a consistent type (with respect to $\mathrm{Th}(G(K))$), since $G(K) \models \Sigma(\bar{u})$. Since H is $\aleph_0$-saturated and $H \equiv G(K)$, there is $\bar{h} \subset H$ such that $H \models \Sigma(\bar{h})$ and we set $H_i := \varphi_i(H, \bar{h})$ for $i = 1, 2$. □

The same idea of the proof yields a more notationally involved:

Lemma 3.13. *Let $G_1(K), \ldots, G_l(K)$ be simply connected simple algebraic groups over a model complete field K (K infinite). Set $G := G_1(K) \times \ldots \times G_l(K)$ and consider an $\aleph_0$-saturated $H \equiv G$. Then there exists $H_1 \equiv G_1(K), \ldots, H_l \equiv G_l(K)$ such that $H \cong H_1 \times \ldots \times H_l$.*

Theorem 3.14. *Let $G(K)$ be a simply connected semi-simple algebraic group and K be a model complete field. Then the structure $(G(K), \cdot)$ (pure group language) is model complete.*

Proof. By Theorem 2.2, there are simply connected simple algebraic groups $G_1, \ldots, G_l$ such that:

$$G(K) \cong G_1(K) \times \ldots G_l(K).$$

We need to verify the three assumptions of Theorem 3.10. By Theorem 3.1, each group $G_i(K)$ is model complete and so the first assumption holds. The second assumption follows by Lemma 3.13. The last point of Theorem 3.10 is implied in our situation by Theorem 2.14. More precisely, if

$$H_1' \equiv G_1(K) \equiv H_1'', \ldots, H_l' \equiv G_l(K) \equiv H_l''$$

then, by Theorem 2.17, there exist fields $K_1', K_1'', \ldots, K_l', K_l'' \equiv K$ such that for every $i \leqslant l$

$$H_i' \cong G_i(K_i'), \ H_i'' \cong G_i(K_i'').$$

Theorem 2.14 states that for each $i \leqslant l$, the group schemes G_i and $G_{\sigma(i)}$ are isomorphic. Thus

$$G_i(K_i') \equiv G_i(K) \cong G_{\sigma(i)}(K) \equiv G_{\sigma(i)}(K''_{\sigma(i)})$$

and so $H_i' \equiv H''_{\sigma(i)}$ for every $i \leqslant l$ as in the third assumption of Theorem 3.10. □

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