

# A MINIMAL TYPE OF MORLEY RANK $\omega$ IN A PARTIAL DIFFERENTIAL FIELD

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ABSTRACT. We consider the Kolchin-generic complete 1-type of the inviscid Burgers equation  $Dx = x\partial x$  in a differentially closed field of characteristic zero with two commuting derivations. We prove that this type has Lascar rank 1 and Morley rank  $\omega$ .

## 1. INTRODUCTION AND MAIN RESULT

Model-theoretic ranks provide notions of dimension for definable sets and complete types. In differentially closed fields, these ranks can also be compared with differential-algebraic dimensions. The Lascar rank of a complete type measures the complexity of its forking extensions, whereas its Morley rank is the minimum of the Morley ranks of its “definable neighbourhoods”. These ranks can therefore behave differently when a type avoids exceptional definable sets that nevertheless contribute to the rank of every neighbourhood.

We are interested in the differential case, where rank discrepancies were established by Hrushovski and Scanlon [6]: Lascar rank and Morley rank need not coincide, even on definable sets of finite Morley rank in  $\text{DCF}_0$ . With one derivation, however, finite  $U$ -rank, finite Morley rank, and finite-dimensionality are equivalent; see [12]. Here a type is finite-dimensional if the differential field generated by a realization has finite transcendence degree over the differential base field.

For several commuting derivations, the situation is different. McGrail [10] claimed the inequalities

$$\omega^\tau \cdot a_\tau \leq U(q) \leq \text{RM}(q) < \omega^\tau \cdot (a_\tau + 1),$$

where  $\tau$  is the degree of the Kolchin polynomial of  $q$  and  $a_\tau$  is its leading coefficient when written in the binomial basis. These inequalities would imply the same equivalence of the three finiteness conditions as in the ordinary differential case. However, Süer [15] showed that the proposed lower bound for  $U$ -rank fails. For the set of solutions of the *heat equation*

$$H = \{x \mid \partial_1 x = \partial_2^2 x\}$$

in  $\text{DCF}_{0,2}$ , the Kolchin polynomial of a generic solution is  $2t + 1$ , but  $U(H) = \omega$ , rather than being at least  $\omega \cdot 2$ . This example disproves the lower bound, but does not settle the asserted equivalence of the finiteness conditions.

The present paper was motivated by ongoing joint work with James Freitag and Omar León Sánchez on these questions; see also the author’s slides from a recent workshop in Chania [8]. In [9], León Sánchez asked whether the equation

$$\partial_1(x) = x^3 + c, \quad c \text{ differentially generic,}$$

defines a set of finite  $U$ -rank in  $\text{DCF}_{0,2}$ . In our joint work, we have shown that this set, and more general families, are strongly minimal in  $\text{DCF}_{0,2}$ , despite the infinite transcendence degree of the differential field generated by a generic solution. These examples therefore separate finiteness of model-theoretic rank from finite-dimensionality, but their  $U$ -rank and

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Morley rank are both 1. This leaves the question whether finite  $U$ -rank can coexist with infinite Morley rank. We answer this question by considering the Kolchin-generic type of the inviscid Burgers equation.

The *inviscid Burgers equation*  $u_t + u u_x = 0$  belongs to the classical theory of nonlinear wave propagation originating from Riemann's study of finite-amplitude plane waves [13]. Its modern name reflects Burgers's investigation of the *viscous equation*  $u_t + u u_x = \nu u_{xx}$  as a simplified model of turbulence, notably in [2], although this viscous equation had already appeared in Bateman's work in [1]. Hopf and Cole developed its reduction to the heat equation, now known as the Cole–Hopf transformation, and investigated its solutions in [5, 3]. The inviscid equation is obtained by setting  $\nu = 0$ .

We describe now our main result. Let us fix a sufficiently saturated model

$$(\mathbb{U}; D, \partial) \models \text{DCF}_{0,2}, \quad D\partial = \partial D,$$

and write

$$\Delta = \{D, \partial\}, \quad \mathcal{C} = \ker D \cap \ker \partial$$

for the set of derivations and the field of total constants, respectively. Choose  $s, t \in \mathbb{U}$  such that

$$Ds = 0, \quad \partial s = 1, \quad Dt = 1, \quad \partial t = 0,$$

and fix a small algebraically closed differential subfield  $M \subseteq \mathbb{U}$  containing  $s$  and  $t$ . Such  $s$  and  $t$  exist by differential closedness. These parameters will be used in the Morley-rank calculation.

Consider the solution set of the inviscid Burgers equation (up to reversing the time variable)

$$X = \{x \in \mathbb{U} : Dx = x\partial x\}.$$

We show in Lemma 2.4 that  $X$  is absolutely Kolchin irreducible, so its Kolchin-generic type over  $M$  is well defined (see its description above the statement of Theorem 2.7).

**Theorem 1.1.** *Let  $p \in S_1(M)$  be the Kolchin-generic type of  $X$ . Equivalently,  $a \models p$  if and only if*

$$Da = a\partial a \quad \text{and} \quad a, \partial a, \partial^2 a, \dots \text{ are algebraically independent over } M.$$

*Then*

$$U(p) = 1, \quad \text{RM}(p) = \omega, \quad \text{RM}(X) = \omega, \quad \text{MD}(X) = 1, \quad U(X) = \omega.$$

*Moreover, the type  $p$  is strictly disintegrated and hence  $p$  is geometrically trivial.*

The argument has two parts. The minimality argument rests on a differential-algebraic independence statement: distinct solutions that are individually  $\partial$ -differentially transcendental over a base field have jointly algebraically independent  $\partial$ -jets. Its proof uses standard properties of Kähler differentials and the operators induced by  $D$  and  $\partial$ . Quantifier elimination then shows that any two ordered tuples of distinct realizations of  $p$ , of the same length, have the same type over  $M$ . Stability gives minimality.

For the Morley-rank calculation, every proper Kolchin-closed subset of  $X$  is finite-dimensional, while  $X$  contains finite-dimensional definable sets  $Y_n$  of rank  $n + 1$  for every  $n \geq 1$ . These sets are given by the following polynomial relations

$$s + tx = \sum_{i=0}^n c_i x^i, \quad c_i \in \mathcal{C}, \quad c_n \neq 0.$$

The rank-one assertion concerns the *complete type*. The whole solution set satisfies

$$U(X) = \text{RM}(X) = \omega,$$

since the sets  $Y_n$  also have  $U$ -rank  $n + 1$ . In particular,  $p$  is nonisolated. The distinction between ranks of complete types and ranks of definable sets leads to the question discussed

in Section 3: whether finite  $U$ -rank and finite Morley rank are equivalent for definable sets in  $\text{DCF}_{0,m}$ .

The paper is organized as follows. In Section 2, we recall the necessary model-theoretic and differential-algebraic preliminaries, describe the generic type of  $X$ , and prove the independence theorem. In Section 3, we compute the ranks and complete the proof of Theorem 1.1. We finish Section 3 with some further observations and a question concerning definable sets.

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## 2. PRELIMINARIES AND THE INDEPENDENCE THEOREM

In this section we provide model-theoretic and algebraic preliminaries. We start with two general stability results and then continue towards our partial differential context.

Let  $T$  be a complete stable theory,  $\mathbb{U}$  be a sufficiently saturated monster model,  $A \subset B \subset \mathbb{U}$  be small, and  $p \in S_1(A)$  be a nonalgebraic stationary type. We write  $a \downarrow_A B$  for nonforking independence over  $A$  and set

$$P := p(\mathbb{U}) \subset \mathbb{U}.$$

We start with the standard minimality characterization (see e.g. [11, Remark 5.6]).

**Lemma 2.1.** *If any two finite ordered tuples (of the same length) of distinct realizations of  $p$  have the same type over  $A$ , then  $U(p) = 1$ .*

*Proof.* Suppose a formula  $\varphi(x; b)$  cuts  $P$  into two infinite pieces. Fix distinct  $a_1, \dots, a_n \in P$ . For each  $I \subseteq \{1, \dots, n\}$ , choose distinct  $a'_1, \dots, a'_n \in P$  with  $\varphi(a'_i; b)$  holding exactly when  $i \in I$ . By the hypothesis, there is  $\sigma_I \in \text{Aut}(\mathbb{U}/A)$  sending  $(a'_1, \dots, a'_n)$  to  $(a_1, \dots, a_n)$ . Applying it to  $b$ , we obtain

$$\varphi(a_i; \sigma_I(b)) \iff \varphi(a'_i; b) \iff i \in I.$$

Hence, for our fixed formula  $\varphi$ , we have

$$\forall n \geq 1 \exists a_1, \dots, a_n \forall I \subseteq \{1, \dots, n\} \exists b_I \forall i \leq n (\varphi(a_i; b_I) \iff i \in I),$$

which contradicts stability.

Thus every externally definable subset of  $P$  is finite or cofinite. Therefore every nonalgebraic extension of  $p$  must choose the cofinite side of every formula. Such an extension is unique over every larger parameter set. By stationarity it is the nonforking extension, proving  $U(p) = 1$ .  $\square$

**Remark 2.2.** Assume in addition that  $U(p) = 1$ . We collect here some well-known facts regarding geometric triviality.

- (P) *Pure induced structure over  $A$ .* For every  $n \geq 1$ , the restriction to  $P^n$  of every  $A$ -definable relation is definable without parameters in the language of equality. Equivalently, every such restriction is a Boolean combination of equalities  $x_i = x_j$ .

Another equivalent formulation is that for all  $n$  all ordered  $n$ -tuples of pairwise distinct elements of  $P$  have the same type over  $A$ .

- (D) *Strict disintegration over  $A$ .* Every finite tuple  $(a_1, \dots, a_n)$  of pairwise distinct realizations of  $p$  is independent over  $A$ :

$$a_i \downarrow_A a_1 \dots a_{i-1} \quad (1 \leq i \leq n).$$

(C) *The algebraic-closure condition.* For every  $B \subseteq P$ , we have the following

$$\text{acl}(A \cup B) \cap P = B.$$

By finite character of algebraic closure, it suffices to require this for finite  $B$ .

(T) For every  $B \subseteq P$ , let us write

$$\text{cl}_p(B) := \text{acl}(A \cup B) \cap P \quad (B \subseteq P).$$

The type  $p$  is *geometrically trivial*, or *disintegrated*, over  $A$  if for every  $B \subseteq P$ , we have

$$\text{cl}_p(B) = \bigcup_{b \in B} \text{cl}_p(\{b\}).$$

Then, we have the following:

$$\boxed{(P) \iff (D) \iff (C) \implies (T).}$$

Indeed, stationarity implies that all independent ordered  $n$ -tuples of realizations of  $p$  have the same type over  $A$ , so (D) implies (P). Conversely, comparing any tuple of distinct realizations with a Morley sequence in  $p$  gives (P) implies (D). Since  $U(p) = 1$ , an extension of  $p$  forks over  $A$  if and only if it is algebraic. This gives the equivalence of (D) and (C). Finally, (C) immediately implies (T). We can describe more precisely:

$$p \text{ is strictly disintegrated} \iff p \text{ is disintegrated and } (\forall a \in P) (\text{cl}_p(\{a\}) = \{a\}).$$

We move now to our case of interest: the theory  $\text{DCF}_{0,2}$ . We have the set-up from the introduction, that is  $(\mathbb{U}; D, \partial)$  is a monster model and  $X$  is defined by the inviscid Burgers equation. We use the standard facts that  $\text{DCF}_{0,2}$  is  $\omega$ -stable and has quantifier elimination, that algebraic closure is field-theoretic algebraic closure of the generated differential field, and that types over algebraically closed differential fields are stationary. Definable sets are Boolean combinations of Kolchin-closed sets, and complete types are determined by their differential polynomial equations; see McGrail [10] and the summary in León Sánchez [9].

Let  $\Delta = \{D, \partial\}$  and  $K \subseteq \mathbb{U}$  be a  $\Delta$ -differential subfield. For any  $a \in \mathbb{U}$ , we denote by  $K\langle a \rangle_\Delta$  the smallest  $\Delta$ -differential subfield of  $\mathbb{U}$  containing  $K$  and  $a$ . A tuple  $a$  is *finite-dimensional over  $K$*  if  $\text{trdeg}(K\langle a \rangle_\Delta/K) < \infty$ . We also use the standard rank bound stated below and include its proof for completeness.

**Lemma 2.3.** *Let  $K \subset \mathbb{U}$  be a small algebraically closed  $\Delta$ -differential field, and let  $Z$  be a  $K$ -definable set. Then, for any  $d \geq 0$ , we have the following*

$$(\forall a \in Z \text{ trdeg}(K\langle a \rangle_\Delta/K) \leq d) \implies \text{RM}(Z) \leq d.$$

*Proof.* We may assume that  $Z$  is nonempty. Choose  $a \in Z$  such that

$$\text{RM}(a/K) = \text{RM}(Z),$$

and put

$$e = \text{trdeg}(K\langle a \rangle_\Delta/K) \leq d.$$

If  $e = 0$ , then  $a$  is algebraic over  $K$ , so  $\text{RM}(Z) = 0$ . Otherwise, the Kolchin polynomial of  $q = \text{tp}(a/K)$  is the constant polynomial  $e$ . Thus  $\tau(q) = 0$  and  $a_0(q) = e$ .

The upper-bound argument in [10, Lemma 5.2.6 and Proposition 4.2.9], applied to the constant Kolchin polynomial  $e$ , gives

$$\text{RM}(q) < \omega^0(e + 1) = e + 1.$$

Hence  $\text{RM}(q) \leq e$ . □

The next result gives Kolchin irreducibility.

**Lemma 2.4.** *For any  $\Delta$ -differential subfield  $K$  of  $\mathbb{U}$ , let  $K\{Y\}_\Delta$  denote the  $\Delta$ -differential ring of  $\Delta$ -differential polynomials over  $K$ . Let  $I$  be the  $\Delta$ -differential ideal of  $K\{Y\}_\Delta$  which is  $\Delta$ -differentially generated by  $DY - Y\partial Y$ . Then we have the following  $\Delta$ -differential isomorphism over  $K$*

$$K\{Y\}_\Delta/I \cong K[y_0, y_1, y_2, \dots],$$

where  $y_0, y_1, y_2, \dots$  are algebraically independent over  $K$  and

$$\partial y_j = y_{j+1}, \quad Dy_j = \partial^j(y_0 y_1).$$

In particular,  $X$  is absolutely Kolchin irreducible.

*Proof.* We put  $R := K[y_0, y_1, y_2, \dots]$ , where the  $y_0, y_1, y_2, \dots$  are algebraically independent over  $K$ . Extend the derivations of  $K$  to  $R$  by

$$\partial y_j = y_{j+1}, \quad Dy_j = \partial^j(y_0 y_1) = \sum_{k=0}^j \binom{j}{k} y_k y_{j-k+1}.$$

These derivations commute on  $K$  and on every generator, since

$$D(\partial y_j) = Dy_{j+1} = \partial^{j+1}(y_0 y_1) = \partial(Dy_j).$$

Their commutator is a derivation, so they commute on all of  $R$ . Moreover, we have  $Dy_0 = y_0 \partial y_0$ . Hence the  $\Delta$ -differential evaluation map

$$\text{ev}_{y_0} : K\{Y\}_\Delta \longrightarrow R$$

induces a  $\Delta$ -differential  $K$ -algebra homomorphism

$$\Phi : K\{Y\}_\Delta/I \longrightarrow R.$$

It is enough to show that  $\Phi$  is a bijection. We define a  $K$ -algebra homomorphism

$$\Psi : R \longrightarrow K\{Y\}_\Delta/I, \quad y_j \longmapsto \overline{\partial^j Y},$$

where for  $f \in K\{Y\}_\Delta$ , we set  $\bar{f} := f + I$ . The relation defining  $I$  gives

$$D\overline{\partial^j Y} = \overline{\partial^j(Y\partial Y)} = \sum_{k=0}^j \binom{j}{k} \overline{\partial^k Y} \overline{\partial^{j-k+1} Y}.$$

Thus the image of  $\Psi$  is closed under both derivations. Since this image contains  $K$  and  $\bar{Y}$ , the map  $\Psi$  is onto. Since we also have  $\Phi \circ \Psi = \text{id}_R$ , both maps  $\Phi$  and  $\Psi$  are bijections.

For the “In particular” part, we have shown that the ring  $R$  is an integral domain, so  $I$  is a prime differential ideal. By the differential Nullstellensatz [7, Chapter IV, §3, Theorem 2, p. 147],  $X$  is Kolchin irreducible. The same construction works after replacing  $K$  by any differential field extension; therefore  $X$  is absolutely Kolchin irreducible.  $\square$

We need the following.

**Proposition 2.5.** *Let  $(\mathcal{U}, \Delta) \models \text{DCF}_{0,m}$ , let  $K \subseteq \mathcal{U}$  be a differential subfield, and let  $V \subseteq \mathcal{U}^n$  be a nonempty Kolchin-closed set defined over  $K$  and irreducible over  $K$ . Then Kolchin genericity in  $V$  over  $K$  determines a unique complete type over  $K$ .*

This proposition follows from a more general result below (see the paragraph immediately preceding [4, Proposition 4.7]).

**Remark 2.6.** Let  $\mathfrak{C} \models T$  be a monster model of a Noetherian theory and let  $A \subseteq \mathfrak{C}$  be small. Thus the  $A$ -closed sets form a Noetherian topology, and the  $A$ -definable sets are precisely the Boolean combinations of  $A$ -closed sets.

If  $V \subseteq \mathfrak{C}^n$  is nonempty,  $A$ -closed, and  $A$ -irreducible, then the partial type

$$\Sigma_V(x) = \{x \in V\} \cup \{x \notin W : W \subsetneq V \text{ is } A\text{-closed}\}$$

has a unique complete extension  $p_V \in S_n(A)$ . Moreover, the assignment  $V \mapsto p_V$  is a bijection between nonempty  $A$ -closed  $A$ -irreducible subsets of  $\mathfrak{C}^n$  and  $S_n(A)$ .

By Lemma 2.4, the Kolchin-generic type  $p(x)$  of  $X$  over  $M$  is determined by:

$$Dx = x\partial x \quad \text{and} \quad x \text{ is } \partial\text{-differentially transcendental over } M.$$

We prove now the main result of this section.

**Theorem 2.7.** *Let  $K \subseteq \mathbb{U}$  be a  $\Delta$ -differential subfield and  $a_1, \dots, a_n \in \mathbb{U}$  be pairwise distinct such that for each  $1 \leq i \leq n$  we have:*

- $Da_i = a_i\partial a_i$ ,
- $a_i$  is  $\partial$ -differentially transcendental over  $K$ .

Then the sequence

$$(\partial^j a_i : 1 \leq i \leq n, j \geq 0)$$

is algebraically independent over  $K$ .

*Proof.* We will use the module of Kähler differentials and its standard properties (see e.g. [14]). Most importantly, we use the fact that, for a field extension  $K \subseteq F$  in characteristic zero, a family of elements of  $F$  is algebraically independent over  $K$  if and only if its differentials are  $F$ -linearly independent in  $\Omega_{F/K}$ . Put

$$F := K(\partial^j a_i : 1 \leq i \leq n, j \geq 0), \quad \Omega := \Omega_{F/K}.$$

Then  $F$  is stable under both derivations and they induce the following commuting Lie derivatives on  $\Omega$

$$\delta_\Omega(f dg) = \delta(f) dg + f d(\delta g), \quad \delta \in \{D, \partial\}.$$

These operators are well defined because both derivations preserve  $K$ . Let us write

$$e_{i,j} = d(\partial^j a_i), \quad E_i^{\leq m} = e_{i,0}F + \dots + e_{i,m}F, \quad E_i = \bigcup_{m \geq 0} E_i^{\leq m}.$$

For each fixed  $i$ , the family  $(e_{i,j})_{j \geq 0}$  is  $F$ -linearly independent by our transcendence assumption. Thus every nonzero  $v \in E_i$  has a unique expression

$$v = c_m e_{i,m} + \dots + c_0 e_{i,0}, \quad c_m \neq 0.$$

Let us define  $\text{ord}_i(v) = m$  and write

$$v = c_m e_{i,m} + \text{l.o.t.},$$

where l.o.t. stands for lower-order terms. We have the following.

$$\begin{aligned} D_\Omega(e_{i,0}) &= d(D(a_i)) \\ &= d(a_i \partial(a_i)) \\ &= a_i d(\partial(a_i)) + \partial(a_i) d(a_i) \\ &= a_i e_{i,1} + \partial(a_i) e_{i,0}. \end{aligned}$$

Applying  $\partial_\Omega^j$ , using commutativity and the Leibniz rule, we obtain

$$(1) \quad \partial_\Omega e_{i,j} = e_{i,j+1}, \quad D_\Omega e_{i,j} - a_i e_{i,j+1} \in E_i^{\leq j} \quad (j \geq 0).$$

### Claim

The sum of the spaces  $E_i$  is direct.

*Proof of Claim.* Suppose not and choose a relation involving the fewest possible spaces, and renumber:

$$(2) \quad v_1 + \dots + v_r = 0, \quad 0 \neq v_i \in E_i, \quad r \geq 2.$$

The key point is that  $D_\Omega$  has leading coefficient  $a_i$  on the  $i$ th family. The distinctness of the  $a_i$  therefore allows us to cancel the leading term on one family while retaining it on all the others. Set

$$T := D_\Omega - a_1 \partial_\Omega, \quad T : \Omega \longrightarrow \Omega.$$

Then  $T$  need not be  $F$ -linear, but it is additive and satisfies

$$(3) \quad T(fv) = (Df - a_1 \partial f)v + fT(v), \quad f \in F, v \in \Omega.$$

Equations (1) and (3) imply

$$(4) \quad T(E_1^{\leq m}) \subseteq E_1^{\leq m} \quad \text{for every } m \geq 0.$$

On the other hand, for  $i \neq 1$  and  $v = c_m e_{i,m} + \text{lower-order terms with } c_m \neq 0$ , they give

$$Tv = c_m(a_i - a_1)e_{i,m+1} + \text{lower-order terms}.$$

Since  $a_i \neq a_1$ , it follows that

$$(5) \quad \text{ord}_i(Tv) = \text{ord}_i(v) + 1 \quad (i \neq 1, 0 \neq v \in E_i).$$

Let  $m = \text{ord}_1(v_1)$ . By (4), all the vectors

$$v_1, Tv_1, \dots, T^{m+1}v_1$$

belong to the  $(m+1)$ -dimensional space  $E_1^{\leq m}$ . Hence there exist  $c_0, \dots, c_q \in F$ , with  $1 \leq q \leq m+1$  and  $c_q \neq 0$ , such that

$$(6) \quad \sum_{k=0}^q c_k T^k v_1 = 0.$$

For  $i = 2, \dots, r$ , put

$$w_i = \sum_{k=0}^q c_k T^k v_i \in E_i.$$

By (5), the vectors  $T^k v_i$  have strictly increasing orders. Therefore the summand  $c_q T^q v_i$  has strictly higher order than every other nonzero summand, and  $w_i \neq 0$ .

Apply  $T^k$  to (2), multiply by  $c_k$ , and sum over  $k$ . Additivity of  $T$  and (6) yield

$$w_2 + \dots + w_r = 0, \quad 0 \neq w_i \in E_i.$$

This contradicts the minimality of  $r$ . □

By the claim, the sum of the  $E_i$  is direct. Since each individual family  $(e_{i,j})_{j \geq 0}$  is linearly independent, all the  $e_{i,j}$  are jointly  $F$ -linearly independent, which gives the desired algebraic independence. □

### 3. RANK COMPUTATIONS

Throughout this section,  $p \in S_1(M)$  denotes the Kolchin-generic type from Theorem 1.1. Having the preliminary results from Section 2, we put them together in this section to obtain a proof of Theorem 1.1. We start from the U-rank computations.

**Proposition 3.1.** *We have  $U(p) = 1$ .*

*Proof.* We will check the condition from Lemma 2.1. Let us take

$$a_1, b_1, \dots, a_n, b_n \models p, \quad |\{a_1, \dots, a_n\}| = n = |\{b_1, \dots, b_n\}|.$$

By Theorem 2.7, we get

$$\text{qftp}(a_1, \dots, a_n/M) = \text{qftp}(b_1, \dots, b_n/M).$$

By quantifier elimination, the complete types coincide as well, so the result holds by Lemma 2.1. □

We can show now geometric triviality.

**Corollary 3.2.** *For every  $A \subseteq p(\mathbb{U})$ , we have*

$$\text{acl}(M \cup A) \cap p(\mathbb{U}) = A.$$

*In particular, the geometry of  $p$  is strictly disintegrated (hence trivial, see Remark 2.2).*

*Proof.* The inclusion  $A \subseteq \text{acl}(M \cup A) \cap p(\mathbb{U})$  is clear. Conversely, let  $b \in \text{acl}(M \cup A) \cap p(\mathbb{U})$ . By finite character, choose distinct  $a_1, \dots, a_n \in A$  such that  $b \in \text{acl}(M \cup \{a_1, \dots, a_n\})$ . Suppose that  $b \notin A$ . By Theorem 2.7, the  $\partial$ -jets of  $a_1, \dots, a_n, b$  are jointly algebraically independent over  $M$ . In particular,  $b$  is transcendental over

$$F := M(\partial^j a_i : 1 \leq i \leq n, j \geq 0) = M\langle a_1, \dots, a_n \rangle_\Delta.$$

This contradicts

$$b \in \text{acl}(M \cup \{a_1, \dots, a_n\}) = F^{\text{alg}} \cap \mathbb{U}.$$

Thus  $b \in A$ . □

We move now to Morley rank computations.

**Lemma 3.3.** *Every proper Kolchin-closed subset of  $X$ , over any differential parameter field, has finite Morley rank.*

*Proof.* Let  $Z \subsetneq X$  be Kolchin closed, and choose a small algebraically closed differential field  $K$  containing its parameters of definition. We may assume that  $Z$  is nonempty.

By Lemma 2.4, there is  $r \geq 0$  and  $Q \in K[X_0, \dots, X_r] \setminus \{0\}$  such that

$$\forall x \in Z \quad Q(x, \partial x, \dots, \partial^r x) = 0 \quad \text{and} \quad \exists x \in X \quad Q(x, \partial x, \dots, \partial^r x) \neq 0.$$

By Lemma 2.3, it is enough to show the following

$$(*) \quad \forall a \in Z \quad \text{trdeg}(K\langle a \rangle_\Delta / K) \leq r.$$

Fix  $a \in Z$  and write  $a_j = \partial^j a$ . Let  $h \leq r$  be least such that  $a_0, \dots, a_h$  are algebraically dependent over  $K$ . Then  $a_h$  is algebraic over  $K(a_0, \dots, a_{h-1})$ , so  $a_{h+1} \in K(a_0, \dots, a_h)$ . Hence  $K(a_0, \dots, a_h)$  is closed under  $\partial$ . Moreover, the Burgers equation gives

$$Da_j = \partial^j(a_0 a_1) \in K(a_0, \dots, a_h),$$

so  $K(a_0, \dots, a_h)$  is also closed under  $D$ . Therefore, we obtain

$$K\langle a \rangle_\Delta = K(a_0, \dots, a_h), \quad \text{trdeg}(K\langle a \rangle_\Delta / K) = h \leq r,$$

which gives  $(*)$ . □

**Proposition 3.4.** *Any definable subset of  $X$  either has finite Morley rank or has complement of finite Morley rank. In particular,  $\text{RM}(X) \leq \omega$ .*

*Proof.* Let  $V$  be a definable subset of  $X$ . By Lemma 2.4, we have  $\overline{V} = X$  or  $\overline{X \setminus V} = X$ , where  $\overline{\phantom{x}}$  is the Kolchin closure of  $T$ . Therefore, by quantifier elimination, we get that  $V$  has non-empty Kolchin interior or  $X \setminus V$  has non-empty Kolchin interior. Assume that  $V$  has non-empty Kolchin interior. Then,  $X \setminus V$  is contained in a proper Kolchin closed subset of  $X$ , so  $\text{RM}(X \setminus V) < \omega$  by Lemma 3.3.

For the “In particular” part, if  $\text{RM}(X) \geq \omega + 1$ , then there are two disjoint definable subsets of  $X$  of Morley rank at least  $\omega$ . By the preceding argument, the complement of the first has finite Morley rank, contradicting the rank of the second. Hence  $\text{RM}(X) \leq \omega$ . □

For  $n \geq 1$ , let

$$T_n = \{(c_0, \dots, c_n) \in \mathcal{C}^{n+1} \mid c_n \neq 0\}$$

and define (recall the fixed  $s, t \in \mathbb{U}$  from the Introduction):

$$Y_n = \left\{ x \mid \exists \bar{c} \in T_n, \quad s + tx = \sum_{i=0}^n c_i x^i \right\}.$$

**Proposition 3.5.** *For every  $n \geq 1$ , we have*

$$Y_n \subseteq X \quad \text{and} \quad U(Y_n) = \text{RM}(Y_n) = n + 1.$$

*Proof.* Let  $x \in Y_n$ , and choose  $\bar{c} = (c_0, \dots, c_n) \in T_n$  such that

$$s + tx = P(x), \quad P(T) = \sum_{i=0}^n c_i T^i \in \mathcal{C}[T].$$

Differentiation gives

$$(P'(x) - t)\partial x = 1, \quad (P'(x) - t)Dx = x.$$

Thus we obtain  $\partial(x) \neq 0$  and  $D(x) = x\partial(x)$ , so  $Y_n \subseteq X$ .

Since  $\mathcal{C}$  is algebraically closed and  $x \notin \mathcal{C}$ , the element  $x$  is transcendental over  $\mathcal{C}$ . Consequently,

$$\forall \bar{c}, \bar{d} \in T_n \quad \left( \sum_{i=0}^n c_i x^i = s + tx = \sum_{i=0}^n d_i x^i \implies \bar{c} = \bar{d} \right).$$

Hence the relation defining  $Y_n$  determines a definable function

$$f : Y_n \longrightarrow T_n, \quad x \longmapsto \bar{c}.$$

For every  $\bar{c} \in T_n$ , the polynomial  $P(T) - tT - s$  has degree  $n$ . For  $n = 1$ , this uses  $c_1 - t \neq 0$ , which follows from  $Dt = 1$ . Since  $\mathbb{U}$  is algebraically closed, this polynomial has a root. Thus  $f$  is surjective and has finite fibres.

The field  $\mathcal{C}$  is stably embedded with its induced structure that of a pure algebraically closed field. Therefore

$$\text{RM}(T_n) = U(T_n) = n + 1.$$

Since  $f$  is a finite-to-one definable surjection,

$$\text{RM}(Y_n) = U(Y_n) = n + 1,$$

which we needed to show.  $\square$

*Completion of the proof of Theorem 1.1.* Propositions 3.4 and 3.5 imply  $\text{RM}(X) = \omega$ . The finite-rank-complement assertion in Proposition 3.4 also gives  $\text{MD}(X) = 1$ .

If  $\varphi(x) \in p$ , then  $X \cap \varphi(\mathbb{U})$  contains a Kolchin-open subset of  $X$  over  $M$ . Its complement in  $X$  is contained in a proper Kolchin-closed subset and has finite Morley rank. Therefore  $\text{RM}(X \cap \varphi(\mathbb{U})) = \omega$ . Since  $X \in p$ , this proves  $\text{RM}(p) = \omega$ .

Since for each  $n \geq 1$ , Proposition 3.5 gives:

$$n + 1 = U(Y_n) \leq U(X) \leq \text{RM}(X) = \omega,$$

we also have  $U(X) = \omega$ . The remaining assertions are Proposition 3.1 and Corollary 3.2.  $\square$

**Remark 3.6.** We collect here some final observations.

(i) For  $a \models p$ , we have the following

$$M(D^i \partial^j a : i + j \leq r) = M(a, \partial a, \dots, \partial^r a).$$

Consequently its Kolchin polynomial is  $r + 1$ , the differential type is one, and the typical differential dimension is one.

(ii) Each  $Y_n$  from the proof of Proposition 3.5 is disjoint from  $p(\mathbb{U})$ . Indeed,  $Y_n$  is  $M$ -definable, so if it contained a realization of  $p$ , the formula defining  $Y_n$  would belong to  $p$ . This would imply

$$\text{RM}(p) \leq \text{RM}(Y_n) = n + 1,$$

contrary to  $\text{RM}(p) = \omega$ .

(iii) The type  $p$  is nonisolated. Indeed, if it were isolated, its isolating set would equal  $p(\mathbb{U})$  and would be strongly minimal by Proposition 3.1. Its Morley rank would then be one, contrary to Theorem 1.1.

I have not found an example of a definable set  $V$  in  $\text{DCF}_{0,2}$  such that

$$U(V) = 1 \quad \text{and} \quad \text{RM}(V) = \omega.$$

Actually, one can show that for our set  $X$  from the introduction and any definable  $V \subseteq X$ , we have

$$U(V) < \omega \quad \Longleftrightarrow \quad \text{RM}(V) < \omega.$$

This leads to the following.

**Question 3.7.** Does every definable set of finite  $U$ -rank in  $\text{DCF}_{0,m}$  have finite Morley rank?

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