Pro-algebraic and differential algebraic group structures on affine spaces

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Abstract

We prove the following result, solving a problem raised in [1]: If G is a differential algebraic group whose underlying set is some affine n-space, then G is unipotent. A key result, possibly of independent interest, concerns infinite-dimensional group schemes: a group scheme whose underlying scheme is $Spec(K[X_1, X_2, ...])$ (K an algebraically closed field of characteristic 0) is a projective limit of unipotent algebraic groups.

1 Introduction

The work in this paper is motivated by a problem discussed by Buium and Cassidy in [1] concerning the category of differential algebraic groups. Suppose (K, D) is a differentially closed field (of characteristic 0), and G is a group whose underlying set is K^n for some n and whose group operation and inverse are given by (sequences of) differential polynomials. (This is what we mean by a differential algebraic group structure on K^n.) They ask whether G has to be a linear unipotent differential algebraic group. (A differential algebraic group G is defined there to be unipotent if G has a finite descending normal sequence of differential algebraic subgroups, whose factor groups are D-embeddable in the algebraic group G_a. Owing to a result in [9] stating that any differential algebraic group has a D-embedding into some algebraic group, the above definition of unipotence is equivalent to there being

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a D-embedding of G into some unipotent linear algebraic group.) This is a differential algebraic generalization of what Lazard ([6]) proved for algebraic groups. In any case, Cassidy ([4]) gave a positive answer to the question, in the case that $n \leq 2$. Here we give a positive answer in general.

The proof in [9] that any differential algebraic group G D-embeds in some algebraic group, depended on constructing a certain canonical pro-algebraic group G^∞ from G. If G lives on some affine n-space, then G^∞ will live on infinite-dimensional affine space. So the main work involves studying such proalgebraic groups. This will be done in section 2. The differential algebraic conclusions will be given in section 3. We give another fast proof of Lazard's Theorem in section 4.

The reader can look at [2] and [3] for more background on the category of differential algebraic groups. By a "D-homomorphism" we mean a homomorphism between differential algebraic groups in the category of differential algebraic varieties (so given locally by differential rational functions, or equivalently definable in the language of differential fields). Background concerning the model theory of differential fields is provided in [9], and any facts needed concerning linear algebraic groups can be found in [8].

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2 Pro-algebraic varieties and groups over C

To begin we work inside a large algebraically closed field K of characteristic 0. The reader can assume this to be $\mathbf{C}$ if he or she wishes, and in any case we will assume this later in the section.
For the purposes of this paper we will work with rather restrictive notions of pro-algebraic variety and the category of pro-algebraic varieties.

Definition 2.1 *(i) By a pro-algebraic variety V we will mean a system $(V_i, \pi_{i,j} : i, j \in \Lambda, i \leq j)$ where Λ is a countable directed set, V_i is a variety for $i \in \Lambda$, for $i \leq j \in \Lambda$, $\pi_{i,j}$ is a surjective morphism from V_j onto V_i,*

and $i \leq j \leq k$ *implies that* $\pi_{j,k}.\pi_{i,j} = \pi_{i,k}$.
(ii) By a point of the pro-algebraic variety V *we mean a sequence* $(a_i)_{i\in\Lambda}$ *such that* $a_i \in V_i$ *for all* i *and* $\pi_{i,j}(a_j) = a_i$ *whenever* $i \leq j$.
(iii) Let $V = (V_i, \pi_{i,j})_{i\in\Lambda}$, $W = (W_i, \nu_{i,j})_{i\in\Gamma}$ *be pro-algebraic varieties. By a morphism* f *from* V *to* W *we mean a mapping from the set of points of* V *to the set of points of* W *which is induced by a family of morphisms* $f_j : V_{g(j)} \to W_j$ $(j \in \Gamma)$. *What this means is that there should be an* $\leq$*-preserving map* g *from* Γ *to* Λ, *and for each* $j \in \Gamma$ *a morphism* f_j *from* $V_{g(j)}$ *to* W_j, *such that* $j \leq j'$ *implies* $\nu_{j,j'}.f_{j'} = f_j.\pi_{g(j),g(j')}$, *and such that* $f((a_i)_i) = (f_j(a_{g(j)}))_j$
(iv) So an isomorphism between pro-algebraic varieties V *and* W *is a bijection* f *between the sets of points, such that both* f *and* f^{-1} *are morphisms.*
(v). If $V = (V_i, \pi_{i,j})_{i\in\Lambda}$ *and* $W = (W_i, \nu_{i,j})_{i\in\Gamma}$ *are pro-algebraic varieties, then* $V \times W$ *is the pro-algebraic variety* $(V_i \times W_j, \pi_{i,i'} \times \nu_{j,j'})_{(i,j)\in\Lambda\times\Gamma}$ *where* $\Lambda \times \Gamma$ *is given the product ordering.*
(vi). By a group object in the category of pro-algebraic varieties, we mean a pro-algebraic variety V *with morphisms* $\mu : V \times V \to V$ *(for multiplication) and* $\tau : V \to V$ *(for inversion), which equip the set of points of* V *with a group structure.*
(vii) By a pro-algebraic group we mean (as usual) a pro-algebraic variety $(V_i, \pi_{i,j})_i$ *where the* V_i *are algebraic groups and the* $\pi_{i,j}$ *are (surjective) homomorphisms.*

Remark 2.2 *(i) Note the surjectivity (rather than generic surjectivity) requirement in (i).*
(ii) We are assuming the index set Λ *to be countable, so it is easy to see that any proalgebraic variety* V *will be isomorphic to one whose index set is* $(\omega, \leq)$ *(or a single point).*
(iii) Note the definitional distinction between a group object in the category of pro-algebraic varieties and a pro-algebraic group. However the latter is clealyr among the former.
(iv) We will call a pro-algebraic variety V *irreducible if all the* V_i *are irreducible varieties.*
(v) We will say that a pro-algebraic variety $(V_i, \pi_{i,j})$ *is defined over* k *iff all the* V_i *and* $\pi_{i,j}$ *are. Similarly for a morphism between pro-algebraic varieties to be defined over* k. *Assume that* $(V_i)_i$ *is defined over* k. *By a generic point of* V *over* k *we mean a point* $(a_i)_i$ *of* V *such that each* a_i *is a generic point*

of V_i over k.
(vi) If the morphism $f : V \to W$ is surjective on points, then it follows from the definition that each $f_j : V_{g(j)} \to W_j$ is surjective.
(vii) We may omit mention of the surjective morphisms $\pi_{i,j}$ if the context allows.

The next result was almost proved in [9]. However in that paper we were working birationally, without the formalism of Definition 2.1, which is needed even to state the result.

Proposition 2.3 *Any irreducible group object in the category of pro-algebraic varieties is isomorphic (in this category) to an irreducible pro-algebraic group.*

Proof. Let $V = (V_i, \pi_{i,j})_{i\in\Lambda}$, $\mu : V \times V \to V$ be the group object in the category of pro-algebraic varieties, which we assume to be defined over k. We write $a.b$ for $\mu(a,b)$. By a rational map f defined over k from V to a pro-algebraic variety W (also defined over k) we mean data a, b with $b \in k(a)$ where a is a generic point in V over k and b some point in W. We write $b = f(a)$ and call f generically surjective if b is a generic point in W over k. We call f birational if $k(a) = k(b)$. It is easy to see that if a, b are generic independent (over k) points of V then so is $a.b$. By [9], there is an irreducible pro-algebraic group $G = (G_j : j \in \Gamma)$ defined over k and a birational map f from V to G such that for generic independent (over k) points $a, b \in V$, $f(a.b) = f(a).f(b)$. For any $c \in V$, let a, b be generic in V over $k(c)$ such that $c = a.b$, and define $f^*(c) = f(a).f(b)$. It is not difficult to see that f^* is a well-defined abstract group isomorphism between the points of V and those of G, which agrees with f on generic points of V over k. We must show that f^* is an isomorphism in the sense of Definition 2.1 (iv). We first show that f^* is a morphism. Let us fix $j \in \Gamma$. We must show that for arbitrarily large $i \in \Lambda$, for all points a of V the value of $f^*_j(a)$ depends only on a_i and that the induced map from V_i onto G_j is a morphism of algebraic varieties. Our description of f^* implies that f^*_j is type-definable (in the structure $(K, +, .)$). The compactness theorem yields arbitrarily large i such that $f^*_j(a)$ depends only on a_i for all points a of V. For any such i we will write the induced map from V_i to G_j also as f^*_j. Let us fix some such i. We can easily find some Zariski-open subset U of V_i defined over k such that the restriction of f^*_j to U is a morphism. On the other hand, by the definition of V as a group object in the category of pro-algebraic varieties, there is $i' \geq i$, and a morphism P from $V_{i'} \times V_{i'} \rightarrow V_i$ defined over k such that for all points a, b of V, $(a.b)_i = P(a_{i'}, b_{i'})$. Choose any $c \in V_{i'}$, and choose $d \in V_{i'}$ generic over $k(c)$. Then $P(d, c)$ is a generic point of V_i over k so is in U. Thus there is a Zariski open subset U_1 of $V_{i'}$ containing c such that for all $c_1 \in U_1$, $P(d, c_1) \in U$. Let $e = f_j^*(d)$. Note now that for any $c_1 \in U_1$, $f_j^*(c_1) = e^{-1}.f_j^*(P(d, c_1)$, and thus the restriction of f_j^* to U_1 is a morphism. We have shown that the map f_j^* from $V_{i'}$ to G_j is a morphism, as required. As j was arbitrary, it follows that $f^* : V \rightarrow G$ is a morphism. A similar (but easier) argument shows the inverse of f^* to be a morphism. This completes the proof.

Definition 2.4 *By* $\mathbf{A}^\infty$ *we mean the pro-algebraic variety* $(\mathbf{A}^n : n = 1, 2, ...)$ *with the natural projections from* $\mathbf{A}^n$ *to* $\mathbf{A}^m$ *for* $m \leq n$.

Proposition 2.5 *Suppose* $G = (G_i)_i$ *is an irreducible pro-solvable pro-algebraic group, whose underlying pro-algebraic variety is isomorphic to* $\mathbf{A}^\infty$*. Then* G *is pro-unipotent, namely each* G_i *is linear unipotent.*

Proof. Fix i. By Remark 2.2 (vi), there is a surjective morphism from some $\mathbf{A}^n$ onto G_i. The structure of algebraic groups yields a connected linear normal algebraic subgroup H_1 of G_i such that G/H_1 is an abelian variety. As there are no surjective morphisms from $\mathbf{A}^n$ onto an abelian variety, $G = H_1$ is a (solvable) linear algebraic group. Let H_2 be the unipotent radical of G (maximal normal unipotent algebraic subgroup). Then G/H_2 is an algebraic torus (namely isomorphic to G_m^N for some N, where G_m denotes here the multiplicative group of the field). So we obtain a surjective morphism from $\mathbf{A}^n$ onto G_m^N, and thus also a surjective morphism onto one factor G_m. From the latter we obtain a polynomial map $f : K^n \rightarrow K$ whose image is $K \setminus \{0\}$, which is impossible. Thus $G = H_2$ is linear unipotent.

The remainder of this section is devoted to showing:

Proposition 2.6 *Suppose* $G = (G_i)_i$ *is an irreducible pro-algebraic group whose underlying pro-algebraic variety is isomorphic to* $\mathbf{A}^\infty$*. Then each* G_i *is solvable (and linear).*

We now assume our universal domain K to be the field of complex numbers $\mathbf{C}$. (This can clearly be done as our index sets for pro-algebraic varieties are countable). The advantage is that we can view our varieties (and even pro-algebraic varieties) as topological spaces, and use elementary algebraic topology. The set of complex points of any complex variety V is naturally a topological space. Any morphism f between complex varieties will be a continuous map between the corresponding topological spaces. In particular if V and W are isomorphic as varieties then they are homeomorphic.

We will use some homotopy-theoretic facts about complex algebraic groups to prove Proposition 2.6. We assume familiarity with the homotopy groups $\pi_n(X, x)$ of pointed topological spaces (X, x), for $n \geq 1$. We refer the reader to [10]. (Also $\pi_0(X, x)$ is the set of arc-wise connected components of X.) For G a real Lie group with identity element e, $\pi_n(G)$ is by definition $\pi_n(G, e)$ and for $n \geq 1$ equals $\pi_n(G^0)$ where G^0 is the connected component of G. We need:

Fact 2.7 *(i). Let $f : H \to G$ be a surjective continuous homomorphism of Lie groups, then f induces a surjective homomorphism from $\pi_3(H)$ to $\pi_3(G)$.*
(ii). Let G be a connected complex linear algebraic group (or even a connected complex linear Lie group), with $\pi_3(G) = 0$. Then G is solvable.

Proof. These are both well-known. We give a brief explanation. For (i): $f : H \to G$ is a fibre-bundle, so a fibration, and hence by 7.2.14 of [10], f induces a long-exact sequence
$....\pi_{i+1}(H) \to \pi_{i+1}(G) \to \pi_i(L)....\pi_0(H) \to \pi_0(G)$, where $L = ker(f)$.
But π_2 of any Lie group is trivial (this is mentioned on p.82 of [7] for example). So we get surjectivity for π_3 as required.
(ii): Let R be the solvable radical of G (maximal normal solvable connected algebraic subgroup). By (i) $\pi_3(G/R) = 0$. On the other hand G/R is, if nontrivial, a semisimple complex algebraic group and it is well-known that such a group has nontrivial π_3. (For any complex semisimple Lie group H there will be a nontrivial continuous homomorphism of $SU(2)$ (and thus S^3) into H, yielding a nontrivial element of $\pi_3(H)$.) Thus G/R is trivial, and $G = R$ is solvable.

Now we can give:
Proof of Proposition 2.6. We may assume that $G = (G_i : i = 1, 2, 3...)$ with surjective homomorphisms $G_{i+1} \to G_i$. As in the proof of 2.5, the G_i are linear. We may work over $\mathbf{C}$. Now the underlying pro-algebraic

variety of G is isomorphic (in the sense of Definition 2.1) to $\mathbf{A}^\infty$. Fix i. We can then find a surjective morphism f from some $\mathbf{A}^n$ onto G_i and a surjective morphism g from some G_j ($j > i$) onto $\mathbf{A}^n$ such that $f.g$ is the given surjective homomorphism from G_j onto G_i. By Fact 2.7(i) $f.g$ induces a surjective homomorphism $(f.g)_*$ say from $\pi_3(G_j)$ onto $\pi_3(G_i)$. But $(f.g)_*$ $= f_*.g_*$, where $f_* : \pi_3(G_j) \to \pi_3(\mathbf{A}^n)$ and $g_* : \pi_3(\mathbf{A}^n) \to \pi_3(G_i)$ are induced by f, g respectively. But $\mathbf{A}^n = \mathbf{C}^n$ has trivial π_3. Thus $\pi_3(G_i) = 0$ and by Fact 2.7(ii), G_i is solvable.

Corollary 2.8 *Let $(\mathbf{A}^\infty, .)$ be a group object in the category of pro-algebraic varieties. Then $(\mathbf{A}^\infty, .)$ is isomorphic, in this category to a projective limit of unipotent algebraic groups.*

Proof. By Propositions 2.3, 2.5 and 2.6.

Translated into the language of group schemes this says that a group scheme whose underlying scheme is $K[X_1, X_2, ...]$ (K an algebraically closed field of characteristic 0) is pro-unipotent.

Remark 2.9 *In the proof of 2.5 we concluded pro-unipotence of $G = (G_i)_i$ from the the existence of surjective morphisms from affine spaces to the G_i. The assumption that the G_i are solvable was necessary, for any semisimple algebraic group H will be the image of some affine space under a morphism: H is generated in finitely many steps by its unipotent subgroups, and this yields a morphism from some finite product of unipotent subgroups onto H.*

3 Differential algebraic groups

We will use the results of section 2 to prove:

Theorem 3.1 *Let (K, D) be a differentially closed field (of characteristic 0), and let G be a differential algebraic group whose underlying set is K^n for some n. Then G is unipotent, namely there is D-embedding of G into some unipotent algebraic group.*

Proof. Note that K is an algebraically closed field. We may assume (K, D) to be reasonably saturated, and we treat K also as a universal domain for algeraic geometry.

The meaning of the hypothesis of the theorem is that there are *differential* polynomials $P_1, .., P_n$ (over K) in *differential* indeterminates $X_1, .., X_n, Y_1, .., Y_n$ and differential polynomials $Q_1, .., Q_n$ (over K) in differential indeterminates $X_1, .., X_n$, such that for $a, b \in G$, $a.b = (P_1(a, b), .., P_n(a, b))$ and $a^{-1} = (Q_1(a), .., Q_n(a))$. We can identify any differential polynomial P in differential indeterminates $X_1, .., X_r$ with an ordinary polynomial P' over K in indeterminates
$X_1, .., X_r, D(X_1), .., D(X_r), ..., D^k(X_1), .., D^k(X_r), ...$ Note that D operates as a derivation on the ring of such polynomials: by $D : K \to K$ as given, and $D(D^s(X_i)) = D^{s+1}(X_i)$ and using the Leibniz rule. In particular, each of the given P_i "is" an ordinary polynomial P'_i, say, over K, in indeterminates
$X_1, ., X_n, D(X_1), ..., D(X_n), ..., D^m(X_1), .., D^m(X_n)$ and
$Y_1, .., Y_n, D(Y_1), .., D(Y_n), ..., D^m(Y_1), .., D^m(Y_n)$, and similarly for the Q_i. (We choose m large enough). Notationally, we introduce indeterminates X_i^j for $i, j < \omega$ to represent $D^j(X_i)$.
We construct a canonical group structure, which we call G^∞ on $\mathbf{A}^\infty(K)$. First we will enumerate the coordinates of a point $x \in \mathbf{A}^\infty$ as
$(x_1^0, .., x_n^0, x_1^1, .., x_n^1, ..., x_1^r, .., x_n^r, ...)$. We call x_i^j the $(i, j)th$ coordinate of x. Let the operation $\star$ on $\mathbf{A}^\infty$ be defined by: $(x \star y)_i^j = D^j(P'_i)(x, y)$.

Claim 1. $\star$ is a group operation on $\mathbf{A}^\infty$. Moreover the $(i, j)th$ coordinate of the inverse of x is $D^j(x_i)$.
Proof. We just point out associativity of $\star$, leaving the rest to the reader. Let k be a small differential subfield of K over which the P_i, Q_i are defined. Let $a = (a_1, .., a_n)$, $b = (b_1, .., b_n)$, $c = (c_1, .., c_n)$ be D-generic D-independent (over k) elements of G. This means that the set $\{D^j(a_i) : i = 1, .., n, j = 0, 1, 2, ...\} \cup \{D^j(b_i) : i = 1, .., n, j = 0, 1, 2, ..\} \cup \{D^j(c_i) : i = 1, .., n, j = 0, 1, 2, ...\}$ is algebraically independent over k. As $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, it follows that $((D^j(a_i))_{i,j}) \star (D^j(b_i))_{i,j}) \star (D^j(c_i))_{i,j} = (D^j(a_i))_{i,j} \star ((D^j(b_i))_{i,j} \star (D^j(c_i))_{i,j})$. As the latter is expressed by a system of polynomial equations over k, the algebraic independence implies that $(x \star y) \star z = x \star (y \star z)$ for *all* $x, y, z \in \mathbf{A}^\infty$.

Claim 2. $\star$ is a morphism from $\mathbf{A}^\infty \times \mathbf{A}^\infty$ to $\mathbf{A}^\infty$ (in the category of pro-algebraic varieties). Similarly for inversion.
Proof. By definition. (Note that $D^j(P'_i)$ is a polynomial in indeterminates $X_1^0, .., X_n^0, .., X_1^{m+j}, .., X_n^{m+j}, Y_1^0, .., Y_n^0, .., Y_1^{m+j}, .., Y_n^{m+j}$.)

Thus G^∞ is a group object in the category of pro-algebraic varieties.

Claim 3. The map taking $(a_1, .., a_n)$ to
$(a_1, ..., a_n, D(a_1), .., D(a_n), .., D^j(a_1), .., D^j(a_n), ...)$ is an (abstract) group embedding of G into G^∞
Proof. By construction of $\star$.

By Corollary 2.8, G^∞ is isomorphic to a projective limit $(G_i)_i$ of unipotent linear algebraic groups. The D-embeddability of G in some unipotent group follows as in [9], but we recall the argument. We have (homo)morphisms $h_i : G^\infty \to G_i$, such that the intersection of all the $ker(h_i)$ is trivial. Composing with the embedding given by Claim 3, gives us homomorphisms $f_i : G \to G_i$, with $\bigcap_i ker(f_i)$ trivial. Each f_i is easily seen to be a D-homomorphism, and so the $ker(f_i)$ are differential algebraic subgroups of G. By the DCC on differential algebraic subgroups, some f_i is already an embedding, and we finish.

4 Remark's on Lazard's Theorem

Lazard showed in [6] that an algebraic group G (in any characteristic) whose underlying variety is some affine n-space is nilpotent, answering a question of Cartier. In [1] Lazard is quoted as proving that G is unipotent. For characteristic 0 this latter result follows from our results in section 2. We will sketch another proof of this, valid in all characteristics, and with a model-theoretic flavour (reminiscent of Ax's proof that injectivity implies surjectivity for a morphism of a variety into itself). Like Lazard's proof, it depends on counting points over finite fields.
Let us fix our universal domain as K, an algebraically closed field of arbitrary characteristic.

Proposition 4.1 *Let G be an algebraic group whose underlying variety is $\mathbf{A}^n$ for some n. Then G is a unipotent linear algebraic group.*

Proof. Note that G is connected and linear. Let R be the solvable radical of G. So we have a surjective homomorphism $f : G \to H$ with kernel R, such that H is, if nontrivial, a semisimple linear algebraic group. We show that H is trivial. Suppose not. Then H contains a 1-dimensional algebraic torus T, namely an algebraic subgroup T isomorphic via some h to $(K^*, .)$. We can find a formula $\phi(x,y)$ in the language of fields, such that for some $b \in K$, $\phi(x,b)$ defines (G,H,f,T,h), and such that for any algebraically closed field K_1, and $b_1 \in K_1$, $\phi(x,b_1)$, if consistent, defines a surjective homomorphism $f_1 : G_1 \to H_1$ between connected linear algebraic groups G_1, H_1, and an isomorphism h_1 between the algebraic subgroup T_1 of G_1 and K_1^* such that $ker(f_1)$ is connected and solvable, and the underlying variety of G_1 is K_1^n. The sentence $\exists x \exists y \phi(x,y)$ is true in K so a consequence of $Th(K)$. Using compactness if necessary, this sentence is true in $\bar{\mathbf{F}}_p$ for some prime p. So there are G_1, H_1, f_1, T_1, h_1 as above, all defined over a finite field $\mathbf{F}_{p^m}$ say. For each finite field $\mathbf{F}_q$ containing $\mathbf{F}_{p^m}$, $G_1(\mathbf{F}_q)$ has cardinality q^n. On the other hand for each such $\mathbf{F}_q$ there is $q' > q$ such that $H_1(\mathbf{F}_q)$ is contained in $f_1(G_1(\mathbf{F}_{q'}))$. It follows that every element of $H_1(\bar{\mathbf{F}}_p)$ has order a power of p. Thus every element of $T_1(\bar{\mathbf{F}}_p)$ has order a power of p. But the latter group is isomorphic (via h_1) to the multiplicative group of $\bar{\mathbf{F}}_p$, and we have a contradiction. So H is trivial, and $G = R$. R is the semidirect product of its unipotent radical R_u and an algebraic torus T. As above, T is trivial. So G is unipotent. The proof is complete.

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