Strongly minimal sets definable in expansions of RCF

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June 20, 2006
Peterzil’s question

Question (Kobi Peterzil, Norwich Conference 2005)

Let $\mathcal{M}$ be a strongly minimal structure definable in an o-minimal structure. Assume $\mathcal{M}$ is not locally modular. Is an algebraically closed field interpretable in $\mathcal{M}$?
Let $\mathcal{M} = (M, f_i, R_j)$ and $\mathcal{N} = (N, \ldots)$ be structures.

**Definition**

- $\mathcal{M}$ is **definable** in $\mathcal{N}$ if $M$, $f_i$’s and $R_j$’s are definable in $\mathcal{N}$.
- $\mathcal{M}$ is **inter-definable** with $\mathcal{N}$ if $\mathcal{M}$ is definable in $\mathcal{N}$ and $\mathcal{N}$ is definable in $\mathcal{M}$.
- We get **interpretable** or **bi-interpretable**, if we replace “definable” with “definable as the quotient by a definable equivalence relation”.
- If $\mathcal{M}$ is definable in $\mathcal{N}$ and $M = N$, then $\mathcal{M}$ is a **reduct** of $\mathcal{N}$ or $\mathcal{N}$ is an **expansion** of $\mathcal{M}$.
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A structure $M$ is **strongly minimal** if any $b$-definable set $X_b \subseteq M$ is either finite or cofinite uniformly in $b$.

**Example**

- $(\mathbb{C}, +, \cdot)$ is strongly minimal (as any algebraically closed field).
- $(\mathbb{C}, +, \cdot)$ is definable in $(\mathbb{R}, +, \cdot)$ which is o-minimal.
- If $(K, +, \cdot)$ is a field, then the vector space $(K, +, \cdot \lambda)_{\lambda \in K}$ is strongly minimal and it is a reduct of $(K, +, \cdot)$.
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A question of Kobi Peterzil
Proof of our theorem
Motivations
Other cases
Accessible form of Kobi’s question
Our answer to Kobi’s question

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For us a **locally-modular structure** is a strongly minimal structure which is inter-definable with a vector space or has no structure.

Two equivalent “formal” definitions of local-modularity

- No 2-dimensional family of plane curves through a point.
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Zilber’s conjecture

Zilber’s dichotomy conjecture
A strongly minimal set is either locally-modular or interprets a field.

Theorem (Hrushovski)
- There is a strongly minimal set which is not locally-modular and does not interpret even a group.
- There is a strongly minimal group which is not locally modular and does not interpret a field.
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Positive results

Zilber’s conjecture holds in:

- Zariski Geometries (Hrushovski-Zilber).
- Differentially closed fields (Hrushovski-Sokolovic).
- Separably closed fields (Hrushovski).
- Algebraically closed fields with a generic automorphism (Chatzidakis-Hrushovski-Peterzil).

Applications

Zilber’s Dichotomy for the structures above yields diophantine consequences – Mordell-Lang, Manin-Mumford.
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Informal Zilber’s conjecture

Zilber’s Dichotomy holds in structures with “geometric flavor”.

Informal statement

O-minimal structures and their reducts have geometric flavor.

Reformulation of Peterzil’s question

Does Zilber’s Dichotomy hold in reducts of o-minimal structures?
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Hasson, Kowalski
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Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, +, \cdot)$.

An accessible version of Peterzil’s question

Let $\mathcal{M}$ be a strongly minimal expansion of $(\mathbb{C}, +)$. Assume $\mathcal{M}$ is definable in $\mathcal{R}$. Does $\mathcal{M}$ satisfy Zilber’s Dichotomy?

This version reduces to:

A reformulation

Assume $X \subset \mathbb{C}^2$ is definable in $\mathcal{R}$ and $\mathbb{C}_X := (\mathbb{C}, +, X)$ is strongly minimal and not locally modular. Does $\mathbb{C}_X$ interpret a field?

We give the positive answer when $X$ is the graph of a function.
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Assume $f : \mathbb{C} \to \mathbb{C}$ is definable in $\mathcal{R}$ and $\mathbb{C}_f := (\mathbb{C}, +, f)$ is strongly minimal and not locally-modular. Then, there is $A \in \text{GL}_2(\mathbb{R})$ such that $\mathbb{C}_{AfA^{-1}}$ is bi-interpretable with $(\mathbb{C}, +, \cdot)$.

Although our assumptions are much stronger than Kobi’s, the conclusions are also stronger, since:

- We identify a definable field – complex field twisted by $A$.
- There is nothing more than the field structure on $\mathbb{C}_f$.
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The idea of the proof

1. Using topological arguments show that $f$ extends to a continuous ramified covering of the Riemann sphere.

2. Prove that for some special $a \in \mathbb{C}$

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   \det f'(a) = 0 \Rightarrow f'(a) = 0
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   (a weak version of Cauchy-Riemann).

3. Using the theory of Lie groups, find an open $U \subseteq \mathbb{C}$ such that $f|_U$ is holomorphic.

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Frontier of a strongly minimal set is finite

The first step (based on a paper of Peterzil-Starchenko) is:

**Fact**

\[ X \subset \mathbb{C}^2 \text{ be } \mathbb{C}_f\text{-definable and strongly minimal.} \]
\[ \text{Then } \text{cl}(X) \setminus X, \text{ called the frontier of } X, \text{ is finite.} \]

A few words about the proof.

Peterzil-Starchenko look how complex lines intersect with \( X \). We do not have enough lines, so we use the sets

\[ l^b_a = \text{graph}(f(x + a) + b). \]

The main problem is to show that enough of these \( l^b_a \) meet \( X \) transversally, and in particular that enough of the curves \( l^b_a \) are smooth at all the intersection points with \( X \).
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Without loss $f : S^2 \rightarrow S^2$ is continuous and open

Let $S^2 = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere.  
Using finiteness of the frontier (mostly with $\text{graph}(f)$), we show that $f$ has all the topological properties of rational functions:

**Fact**

- $f$ is continuous outside a finite set $F$.
- Resetting, if needed, the values of $f$ on $F$ (to possibly $\infty \in S^2$), we can assume that $f : \mathbb{C} \rightarrow S^2$ is continuous.
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$f : S^2 \to S^2$ is a ramified covering

We use the following topological theorem:

**Theorem**

If $f$ is as in our case, then $f$ is a ramified covering, i.e. it is locally topologically equivalent to $z \mapsto z^k$ on $|z| \leq 1$ ($k$ may vary).

**Definition**

1. If $k > 1$ at $c$, then $c$ is a branch point of degree $k$ (of $f$), e.g. $0$ is a branch point of degree $3$ of $g(z) = z^3 + 7$.
2. If $c$ is a branch point, then $f(c)$ is a ramification point.
\[ f : S^2 \rightarrow S^2 \] is a ramified covering

We use the following topological theorem:

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If \( f \) is as in our case, then \( f \) is a *ramified covering*, i.e. it is locally topologically equivalent to \( z \mapsto z^k \) on \(|z| \leq 1\) (\(k\) may vary).

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Hasson, Kowalski

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Since $f$ is definable in an o-minimal structure, $f$ is $C^1$ on a codimension 1 subset of $\mathbb{C}$.

Definition

Let $f'(c)$ denote the Jacobian matrix of $f$ at $c$ (if defined). It is an element of $M_2(\mathbb{R})$.

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We want to show that $f$ is holomorphic on some open $U \subseteq \mathbb{C}$, i.e. for each $c \in U$, $f'(c) \in M_1(\mathbb{C})$ ($M_1(\mathbb{C}) \hookrightarrow M_2(\mathbb{R})$).
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Jacobian matrix vanishes at branch points

Fact (weak Cauchy-Riemann)

If $f$ is $C^1$ at $c$ and $c$ is a branch point, then $f'(c) = 0$.

Idea of the proof.

Let $f = (f_1, f_2)$. We can assume $f(c) = 0$. It is enough to show that for almost all directions $\alpha \in S^1$,

$$\frac{\partial f_i}{\partial \alpha}(c) = 0, \quad i = 1, 2.$$ 

Since $f$ is equivalent locally at $c$ to $z \mapsto z^k$ and $k > 1$, $f^{-1}([-1, 1]) \setminus \{c\}$ has $2k$ connected components $X_j$. Since $f_2(X_j) = 0$, it is enough (for $f_2$) to take $\alpha \neq \alpha_j$, where

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Fact

1. **We can assume** \( f \) **is not 1-to-1.**
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1. If \( f \) is 1-1 (e.g. when \( f(x) = 1/x \)), we replace \( f \) with \( f(x + 1) - f(x) \).
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There is a $C_f$-definable $g : S^2 \to S^2$ having a $C^1$ branch point.

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- By the theory of local degrees (winding numbers), we can control the way branch points move in families.

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We want to show that for some open $U \subseteq \mathbb{C}$, we have:

$$f'(U) \subseteq \text{GL}_1(\mathbb{C}).$$

So, $f'(U)$ is a subset of a 2-dim. Lie subgroup of $\text{GL}_2(\mathbb{R})$.

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  We show this assertion first.
Some control on dimension

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Consider $f_a^b(x) = f(a + f(x)) - f(x + b)$. Then, for small enough $|a|, |b|$, $f_a^b$ has a $C^1$ branch point $c_a^b$. Hence $(f_a^b)'(c_a^b) = 0$, so:

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A question of Kobi Peterzil
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Other cases

Topology
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There is a local Lie subgroup of $\text{GL}_2(\mathbb{R})$ around

**Definition**

For a Lie group $G$, $A \subset G$ is a local Lie subgroup, if there is a relatively open $B \subset A$ such that $1 \in B$, $B = B^{-1}$ and $B \cdot B \subseteq A$.

Taking $n = 9$ in the last fact we obtain:

**Fact**

There is an open $U \subseteq \mathbb{C}$ such that $f'(U)$ is a subset of a local Lie subgroup $A \subset \text{GL}_2(\mathbb{R})$ and $\dim A \leq 2$. 

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For a Lie group $G$, a virtual Lie subgroup of $G$ is a smooth injective homomorphism of Lie groups $\phi : H \rightarrow G$.

Virtual Lie subgroups of $G$ correspond exactly to Lie subalgebras of $\text{Lie}(G)$ and the following is well-known:

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If $A$ is a local Lie subgroup of $G$, then there is a virtual Lie subgroup $\phi : H \rightarrow G$ such that $\dim H = \dim A$ and $\phi(H) \cap A$ is open in $A$. 
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A virtual Lie subgroup need not be Lie

The image of a virtual Lie subgroup need not be a Lie subgroup as the “non-commutative torus” example shows.

**Example**

Let $a$ be an irrational number, $T = S^1 \times S^1$ a 2-dimensional torus and take:

$$\mathbb{R} \ni r \mapsto \phi(r) = (r, ar) + \mathbb{Z}^2 \in \mathbb{R}^2/\mathbb{Z}^2$$

Then $\phi(\mathbb{R})$ is dense in $T$, so it is not a Lie subgroup. The quotient $T/\phi(\mathbb{R})$ is called a non-commutative torus.
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But in our case we still obtain:

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*There is a solvable Lie subgroup $\bar{H} < \text{GL}_2(\mathbb{R})$ containing $f'(U)$.***

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- We have $f : H \to \text{GL}_2(\mathbb{R})$ and $\dim H \leq 2$, hence $H$ is solvable.
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\( f'(U) \) is contained in a conjugate of \( \text{GL}_1(\mathbb{C}) \).

Proof.

- \( f'(U) \) is contained in a solvable Lie subgroup \( \bar{H} \).
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- Triangular group contradicts strong minimality of \( \mathbb{C}_f \) (one partial derivative of \( f_1 \) vanishes on \( U \)).

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This how we find the matrix \( A \) from the statement of our theorem. It is the conjugation matrix above.
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- $f'(U)$ is contained in a solvable Lie subgroup $\bar{H}$.
- By classification of such, $\bar{H}$ (possibly after conjugation) is a subgroup of the triangular group or $\text{GL}_1(\mathbb{C})$.
- Triangular group contradicts strong minimality of $\mathbb{C}_f$ (one partial derivative of $f_1$ vanishes on $U$).

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This how we find the matrix $A$ from the statement of our theorem. It is the conjugation matrix above.
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This how we find the matrix $A$ from the statement of our theorem. It is the conjugation matrix above.
**Fact**

There is an open $U \subseteq \mathbb{C}$ such that $f$ is holomorphic on $U$.

**Proof.**

The fact that for $a \in U$, $f'(a) \in \text{GL}_1(\mathbb{C})$ means exactly that $f$ satisfies Cauchy-Riemann at $a$, so $f$ is holomorphic at $a$.

**Remark**

If $U$ is dense in $\mathbb{C}$, we can easily show that $f$ is rational and we are done. But we do not know it at this stage.
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Proof of our theorem
Other cases

Topology
Differential Geometry
Lie groups
Analytic Geometry and Algebraic Geometry

\( f \) is holomorphic on an open set

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Fact

There is a field interpretable in $\mathbb{C}_f$.

Proof.

- Take $U$ such that $f$ is holomorphic on $U$.
- Then, for $c \in U$, $f'(c) = 0$ implies $c$ is a branch point.
- This allows us to pull-back by $f'|_U$ the group configuration of $\mathbb{G}_a(\mathbb{C}) \ltimes \mathbb{G}_m(\mathbb{C})$ acting on $\mathbb{G}_a(\mathbb{C})$ to get a $\mathbb{C}_f$-interpretable field.
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Conclusion of the proof.

- Let $K$ be a field interpretable in $C_f$. By a result of Peterzil-Starchenko and Hrushovski’s internality theory, $K$ is bi-interpretable with $C_f$.
- Hence, $(C, +)$ is a 1-dimensional $K$-algebraic group.
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- Using the above isomorphism, we get a $C_f$-definable operation $\star : C^2 \rightarrow C$ such that $(C, +, \star)$ is a field.
- Then, it is easy to find $A \in GL_2(R)$ such that

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Let $\mathcal{R}$ be an o-minimal expansion of an arbitrary real closed field.

**Remark**

The proof of our theorem generalizes from $\mathbb{C}$ to any $\mathcal{K} = \mathcal{R}[i]$.

**About the proof**

- The theory of winding numbers, differentiable/analityc manifolds etc. was developed in this context by Berarducci, Otero, Peterzil, Pillay, Starchenko and others.

- The only place in the proof where we left the o-minimal context was when a virtual Lie group showed up. But another argument using Lie algebras holds in this context too.
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Weakening conditions on $\mathcal{R}$

**Remark**

Not much of o-minimality was used in the proof.

**Question**

- Can we assume that $\mathcal{R}$ is e.g. just weakly o-minimal?
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Question

Can we replace \((\mathbb{C}, +, f)\) with any strongly minimal expansion of \((\mathbb{C}, +)\) definable in \(\mathcal{R}\)?

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We know it is enough to consider \(\mathbb{C}_X = (\mathbb{C}, +, X)\) with a relation (so “multi-function”) \(X\) replacing \(f\). We do not know if our proof still works. It should be still possible to prove the finiteness of frontier of strongly minimal \(\mathbb{C}_X\)-definable subsets of \(\mathbb{C}^2\).
Any expansion of \((\mathbb{C}, +)\)

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Open Question

Let \((A, +)\) be a strongly minimal group which is not locally modular. Is there a definable function \(f : A \rightarrow A\) such that \((A, +, f)\) is not locally-modular?

Remarks

- Positive answer to the above question extends our theorem to any strongly minimal expansion of \((\mathbb{C}, +)\).
- The answer is positive if \(A\) has elimination of imaginaries, i.e. “definable” = “interpretable”.
- Can elimination of imaginaries for \(\mathcal{R}\) be used?
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- Can elimination of imaginaries for \(R\) be used?
Remark

Most likely our argument still works when we replace $\left( \mathbb{C}, + \right)$ with another one-dimensional algebraic group, i.e. the multiplicative group or an elliptic curve.