Model theory of Galois actions
(joint work with Özlem Beyarslan)

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G-fields as first-order structures

We fix a finitely generated (marked) group

$$G = \langle \rho \rangle, \quad \rho = (\rho_1, \ldots, \rho_m).$$

By a G-field, we mean a field together with a Galois action by G.
Similarly, we define G-field extensions, G-rings, etc.
We consider a G-field as a first-order structure in the following way

$$\mathbf{K} = (K, +, -, \cdot, \rho_1, \ldots, \rho_m).$$

Note that any ρ_i above denotes *three* things at the same time:

- an element of G,
- a function from K to K,
- a formal function symbol.

Existentially closed G-fields 1

Let us fix a G-field (K, ρ).

Systems of G-polynomial equations

Let $x = (x_1, \ldots, x_n)$ be a tuple of variables and $\varphi(x)$ be a system of G-polynomial equations over K

$$\varphi(x): \quad F_1(g_1(x_1), \ldots, g_n(x_n)) = 0, \ldots, F_n(g_1(x_1), \ldots, g_n(x_n)) = 0$$

for some $g_1, \ldots, g_n \in G$ and $F_1, \ldots, F_n \in K[X_1, \ldots, X_n]$.

Existentially closed G-fields

The G-field (K, ρ) is existentially closed (e.c.) if any system $\varphi(x)$ of G-polynomial equations over K which is solvable in a G-extension of (K, ρ) is already solvable in (K, ρ).

Existentially closed G-fields 2

- Any G-field has an e.c. G-extension.
- For $G = \{1\}$, e.c. G-fields coincide with algebraically closed fields.
- For $G = \mathbb{Z}$, e.c. G-fields coincide with *transformally* (or *difference*) *closed fields*.
- It often happens that (K, ρ) is e.c., but K is *not* algebraically closed. By a result of Sjögren, if (K, ρ) is an e.c. G-field, then

$$\mathrm{Gal}(K) \cong \ker\left(\widetilde{\hat{G}} \to \hat{G}\right),$$

where for a profinite group H, the map $\widetilde{H} \to H$ is the universal Frattini cover.

- Analogy: for "$G = \widehat{\mathbb{G}_{\mathrm{a}}}$", we get differentially closed (Hasse-Schmidt) fields.

The theory G-TCF

Definition

If the class of existentially closed G-fields is *elementary*, then we call the resulting theory G-TCF and say that G-TCF exists.

Example

- For $G = \{1\}$, we get G-TCF $=$ ACF.
- For $G = F_m$ (free group), we get G-TCF $=$ ACFA$_m$.
- If G is finite, then G-TCF exists (Sjögren, independently Hoffmann-K.)
- $(\mathbb{Z} \times \mathbb{Z})$-TCF does *not* exist (Hrushovski).

Axioms for ACFA

We fix now a difference field (K, σ), i.e. $(G, \rho) = (\mathbb{Z}, 1)$ (or, for technical reasons, $(G, \rho) = (\mathbb{Z}, 0, 1)$).

- By a variety, we always mean an affine K-variety which is K-irreducible and K-reduced (i.e. a prime ideal of $K[\bar{X}]$).
- For any variety V, we also have the variety ${}^{\sigma}V$ and the bijection (not a morphism!)

$$\sigma_V : V(K) \to {}^{\sigma}V(K).$$

- A pair of varieties (V, W) is called a $\mathbb{Z}$-pair, if $W \subseteq V \times {}^{\sigma}V$ and the projections $W \to V, W \to {}^{\sigma}V$ are dominant.

Axioms for ACFA (Chatzidakis-Hrushovski)

The difference field (K, σ) is e.c. if and only if for any $\mathbb{Z}$-pair (V, W), there is $a \in V(K)$ such that $(a, \sigma_V(a)) \in W(K)$.

Axioms for G-TCF, G-finite

Let $G = \{\rho_1 = 1, \ldots, \rho_e\}$ be a finite group and (K, ρ) be a G-field.

Definition of G-pair

A pair of varieties (V, W) is a G-pair, if:

- $W \subseteq {}^{\rho_1}V \times \ldots \times {}^{\rho_e}V$;
- all projections $W \to {}^{\rho_i}V$ are dominant;
- Iterativity Condition: for any i, we have ${}^{\rho_i}W = \pi_i(W)$, where

$$\pi_i : {}^{\rho_1}V \times \ldots \times {}^{\rho_e}V \to {}^{\rho_i\rho_1}V \times \ldots \times {}^{\rho_i\rho_e}V$$

  is the appropriate coordinate permutation.

Axioms for G-TCF, G finite (Hoffmann-K.)

The G-field (K, ρ) is e.c. if and only if for any G-pair (V, W), there is $a \in V(K)$ such that

$$((\rho_1)_V(a), \ldots, (\rho_e)_V(a)) \in W(K).$$

Our strategy 1

- Find a generalization of the known results (mentioned above) for free groups and finite groups.
- Natural class of groups for such a generalization: *virtually free* groups.
- For a fixed (G, ρ), the general scheme of axioms should be as follows: for any G-pair (V, W), there is $a \in V(K)$ such that

$$\rho_V(a) := ((\rho_1)_V(a), \ldots, (\rho_m)_V(a)) \in W(K).$$

Hence one needs to find the right notion of a G-pair.

G-pairs in general

A pair of varieties (V, W) will be called a G-pair, if:

- $W \subseteq {}^{\rho}V := {}^{\rho_1}V \times \ldots \times {}^{\rho_m}V$;
- all projections $W \to {}^{\rho_i}V$ are dominant;
- Iterativity Condition: to be found.

Our strategy 2

We need to find a good Iterativity Condition for a virtually free, finitely generated group (G, ρ).

- G free: trivial Iterativity Condition.
- G finite: Iterativity Condition as before.

We need a procedure to obtain virtually free groups from finite groups, luckily such a procedure exists and gives the right Iterativity Condition.

Theorem (Karrass, Pietrowski and Solitar)

Let H be a finitely generated group. TFAE:

- *H is virtually free,*
- *H is isomorphic to the fundamental group of a finite graph of finite groups.*

Bass-Serre theory

Graph of groups (slightly simplified)

A graph of groups $G(-)$ is a connected graph $(\mathcal{V}, \mathcal{E})$ together with:

- a group G_i for each vertex $i \in \mathcal{V}$;
- a group A_{ij} for each edge $(i, j) \in \mathcal{E}$ together with monomorphisms $A_{ij} \to G_i, A_{ij} \to G_j$.

Fundamental group

For a fixed maximal subtree $\mathcal{T}$ of $(\mathcal{V}, \mathcal{E})$, the fundamental group of $(G(-), \mathcal{T})$ (denoted by $\pi_1(G(-), \mathcal{T})$) can be obtained by successively performing:

- one free product with amalgamation for each edge in $\mathcal{T}$;
- and then one HNN extension for each edge not in $\mathcal{T}$.

$\pi_1(G(-), \mathcal{T})$ does not depend on the choice of $\mathcal{T}$ (up to $\cong$).

Iterativity Condition for amalgamated products

Let $G = G_1 * G_2$, where G_i are finite. We define $\rho = \rho_1 \cup \rho_2$, where $\rho_i = G_i$ and the neutral elements of G_i are identified in ρ. We also define the projection morphisms $p_i : {}^{\rho}V \to {}^{\rho_i}V$.

Iterativity Condition for $G_1 * G_2$

- $W \subseteq {}^{\rho}V$ and dominance conditions;
- $(V, p_i(W))$ is a G_i-pair for $i = 1, 2$ (up to Zariski closure).

Let $G = \pi_1(G(-))$, where $G(-)$ is a tree of groups. We take $\rho = \bigcup_{i \in \mathcal{V}} G_i$, where for $(i,j) \in \mathcal{E}$, G_i is identified with G_j along A_{ij}.

Iterativity Condition for fundamental group of tree of groups

- $W \subseteq {}^{\rho}V$ and dominance conditions;
- $(V, p_i(W))$ is a G_i-pair for all $i \in \mathcal{V}$ (up to Zariski closure).

Iterativity Condition for HNN extensions

Let $C_2 \times C_2 = \{1, \sigma, \tau, \gamma\}$ and consider the following:

$$\alpha : \{1, \sigma\} \cong \{1, \tau\}, \quad G := (C_2 \times C_2) *_{\alpha} .$$

Then the crucial relation defining G is $\sigma t = t\tau$. We take:

- $\rho := (1, \sigma, \tau, \gamma, t, t\sigma, t\tau, t\gamma)$;
- $\rho_0 := (1, \sigma, \tau, \gamma)$;
- $t\rho_0 := (t, t\sigma, t\tau, t\gamma)$.

Iterativity Condition for $(C_2 \times C_2)*_{\alpha}$

- ${}^{t}(p_{\rho_0}(W)) = p_{t\rho_0}(W)$.
- $(V, p_{\rho_0}(W))$ is a $(C_2 \times C_2)$-pair.

Main Theorem

We find a complicated Iterative Condition for virtually free groups using the conditions from the two previous slides as building blocks.

Theorem (Beyarslan-K.)

If G is finitely generated and virtually free, then G-TCF exists.

Properties of G-TCF

- If G is finite, then G-TCF is supersimple of finite rank($=|G|$).
- If G is infinite and free, then G-TCF is simple.
- Sjögren: for any G, if (K, ρ) is an e.c. G-field then K is PAC and K^G is PAC.
- Chatzidakis: for a PAC field K, the theory $\mathrm{Th}(K)$ is simple iff K is bounded (i.e. $\mathrm{Gal}(K)$ is small).

New theories are not simple

Theorem (Beyarslan-K.)

Assume that G is finitely generated, virtually free, infinite and not free. Then the following profinite group

$$\ker\left(\widetilde{\hat{G}} \to \hat{G}\right)$$

is not small.

Corollary

Putting everything together, we get the following.

- If G is finitely generated virtually free, then the theory G-TCF is simple if and only if G is finite or G is free.
- If G is finitely generated, virtually free, infinite and not free, then the theory G-TCF is not even NTP_2.

Maybe these theories are NSOP...

Negative results

- Hrushovski showed that $(\mathbb{Z} \times \mathbb{Z})$-TCF does not exists.
- Modifying his proof, we show that $(\mathbb{Z} \rtimes \mathbb{Z})$-TCF does not exists.
- Actually, in both cases the proof gives that there is no $\aleph_0$-saturated e.c. model.
- The proof is quite mysterious. The action of our group on primitive 3rd roots of unity in a saturated model yields a contradiction.
- Hrushovski has also an argument for (as I learnt this morning): if $\mathbb{Z} \times \mathbb{Z}$ embeds into G, then G-TCF does not exist.

Two conjectures

Conjecture 1

Suppose that G is finitely generated. Then the theory G-TCF exists iff G is virtually free.

- There is a surprisingly long list of equivalent conditions for a group to be virtually free. It would be interesting to have one more coming from model theory.
- Main challenge for a proof: infinite Burnside groups (finitely generated and of bounded exponent).

Conjecture 2

Let G be an arbitrary group (e.g. $G = (\mathbb{Q}, +)$). Then the theory G-TCF exists iff G is locally virtually free.

Remark

$\mathbb{Q}$-TCF exists (Medvedev).