Difference sheaf cohomology

(joint work with Marcin Chałupnik)

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Linearly closed difference fields and difference Picard group

- K. and Pillay proved that models of ACFA and difference fields of the form $(\mathbb{F}_p^{\mathrm{alg}}, \mathrm{Fr})$ are linearly closed i.e. any finite-dimensional σ-module is isomorphic to a trivial one.
- The proof actually shows that for any difference field $(\mathbf{k}, \sigma)$, the isomorphism classes of one-dimensional σ-modules correspond to the cokernel of the "Artin-Schreier map" $x \mapsto \sigma(x)x^{-1}$. This cokernel is the same as the group of *coinvariants* of the action of σ on $\mathbb{G}_{\mathrm{m}}(\mathbf{k})$. Hence, we get

$$\mathrm{Pic}_\sigma(\mathbf{k}) \cong \mathbb{G}_{\mathrm{m}}(\mathbf{k})_\sigma.$$

- Since $\mathrm{Pic}(\mathbf{k}) = 0$, we have the following short exact sequence

$$0 \to \mathbb{G}_{\mathrm{m}}(\mathbf{k})_\sigma \to \mathrm{Pic}_\sigma(\mathbf{k}) \to \mathrm{Pic}(\mathbf{k})^\sigma \to 0.$$

Difference Picard group and difference cohomology

- Using the cohomological interpretation of the Picard group, this exact sequence takes the following form ($X = \mathrm{Spec}(\mathbf{k})$):

$$0 \to H^0\left(X, \mathcal{O}_X^*\right)_\sigma \to \mathrm{Pic}_\sigma(k) \to H^1\left(X, \mathcal{O}_X^*\right)^\sigma \to 0.$$

- The above exact sequence has the same shape as the short exact sequence for difference group cohomology (developed by Chałupnik and K. in *Algebra & Number Theory* 12-7 (2018)):

$$0 \to H^{n-1}(G, M)_\sigma \to H^n_\sigma(G, M) \to H^n(G, M)^\sigma \to 0.$$

- It is natural then to search for a difference sheaf cohomology which would explain these "cohomological difference Picard phenomena". This is what we did and what I will describe in this talk. For several reasons (to be explained soon), our sheaves often live on sites rather than just topological spaces.

Sites: definition

Definition

A site $\mathbf{C}$ is a pair $(\mathcal{C}, J)$ such that $\mathcal{C}$ is a category and J is a Grothendieck topology on $\mathcal{C}$ that is J is a set consisting of families $\{U_i \to U\}_i$ of morphisms in $\mathcal{C}$ (coverings) such that:

1. For $\{U_i \to U\}_i$ in J and a morphism $V \to U$ in $\mathcal{C}$ all fibre products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}_i$ is again in J.
2. Given $\{U_i \to U\}_i$ in J, and for each i, a family $\{V_{ij} \to U_i\}_j$ in J, the family $\{V_{ij} \to U\}_{ij}$, obtained by composition of morphisms, also belongs to J.
3. If $U \to V$ is an isomorphism in $\mathcal{C}$, then $\{U \to V\} \in J$.

Property (1) is about "intersections", property (2) is about "unions", and property (3) is about "the whole space".

Sites: examples

Let X be a scheme and E be a class of scheme morphisms. A family of E-morphisms $\{f_i : U_i \to U\}_i$ is an E-covering if we have

$$U = \bigcup_i f_i(U_i).$$

Example

- The Zariski site $\mathbf{Z}(X)$ consists of the category of Zariski open sets in X (with inclusions), and the set of Zariski coverings. This can be done for any topological space.
- The étale site $\mathbf{E}(X)$ consists of the category of étale schemes over X (with étale morphisms), and the set of étale coverings.
- The (big) flat site $\mathbf{F}(X)$ consists of the full subcategory of the category of schemes over X whose structure morphism is locally of finite-type, and the set of flat coverings.

Sites: (pre)sheaves and some properties

- A presheaf on a site $\mathbf{C} = (\mathcal{C}, J)$ is just a contravariant functor whose domain is $\mathcal{C}$. A sheaf is a presheaf satisfying the sheaf conditions with respect to J.
- The category of sheaves of Abelian groups on a site $\mathbf{C}$ is a Grothendieck category (AB5 category with a generator), e.g. it is Abelian with enough injectives. We denote it by $\mathrm{Sh}(\mathbf{C})$.
- Similarly for presheaves and the category $\mathrm{PSh}(\mathbf{C})$.
- Any morphism of sites ρ induces a pair of adjoint functors ρ_*, ρ^* on the categories of sheaves. All adjointness bijections will be denoted by $f \mapsto \tilde{f}$.
- We tacitly assume that the underlying categories of all the sites we consider are *small* (even if they do not look small). Otherwise, set-theoretic problems. (Grothendieck used the notion of *universe*, which resembles a monster model of ZFC.)

Sites: our motivation

- We will interpret the category of difference sheaves (to be defined soon) as a topos on a certain site.
- Even if we start from the Zariski site, the above site does not come from a topological space.
- We will consider difference principal homogenous spaces. But the very definition of PHS (and its cohomological interpretation) requires sites, that is, the flat site.
- To understand difference algebraic groups as difference sheaves, we need to use the flat site.

Sites: our set-up

We fix a scheme X, a self-morphism $\sigma : X \to X$, and a "site on X", that is a site $\mathbf{C}(X) = (\mathcal{C}, J)$ such that the following holds.

1. The category $\mathcal{C}$ is a subcategory of $\mathbf{Sch}_X$.
2. The category $\mathcal{C}$ is closed under any base extensions in $\mathbf{Sch}_X$.
3. The Grothendieck topology J on $\mathcal{C}$ is subcanonical, that is all representable presheaves on $\mathcal{C}$ are sheaves with respect to J.
4. The fiber product functor $\sigma^{-1} : \mathcal{C} \to \mathcal{C}$ (well-defined thanks to Item (2) above) induces a morphism of sites, which we denote by the same letter as the fixed self-morphism of X:

$$\sigma : \mathbf{C}(X) \to \mathbf{C}(X).$$

We will mainly consider the Zariski site $\mathbf{Z}(X)$, the étale site $\mathbf{E}(X)$, and the flat site $\mathbf{F}(X)$.

(left) Difference sheaves: definition

1. We call a sheaf $\mathcal{F}$ of Abelian groups on $\mathbf{C}(X)$ together with a sheaf morphism
$$\sigma_{\mathcal{F}} : \mathcal{F} \to \sigma_*(\mathcal{F}),$$
a *left difference sheaf of Abelian groups on* $\mathbf{C}(X)$.
2. A morphism between left difference sheaves $(\mathcal{F}, \sigma_{\mathcal{F}})$ and $(\mathcal{G}, \sigma_{\mathcal{G}})$ is a sheaf morphism $\alpha : \mathcal{F} \to \mathcal{G}$ such that
$$\sigma_{\mathcal{G}} \circ \alpha = \sigma_*(\alpha) \circ \sigma_{\mathcal{F}}.$$
3. The difference sheaves of Abelian groups on X form a category, which we denote $\mathrm{Sh}_\sigma(\mathbf{C}(X))$. Similarly for the difference presheaves and the category $\mathrm{PSh}_\sigma(\mathbf{C}(X))$.
4. We will mainly consider left difference sheaves during this talk, hence we will usually skip the adjective “left” here.

Difference sheaves: examples

A morphism $\sigma^*(\mathcal{F}) \to \mathcal{F}$ also gives $\mathcal{F}$ a difference sheaf structure.

Example

1. A: Abelian group, A_X: constant sheaf, $\sigma^*(A_X) = A_X$. Then $\mathrm{id} : A_X \to A_X$ makes A_X a difference sheaf.
2. R: a ring, $X = \mathrm{Spec}(R)$, $\sigma = \mathrm{Spec}(s : R \to R)$, (M, s_M): a difference (R, s)-module. Then the quasi-coherent sheaf $\widetilde{M}$ has a natural structure of a difference sheaf.
3. (X, σ) as in item (2) above for $R = \mathbf{k}$: a field, $\mathbf{C}(X) = \mathbf{E}(X)$. Then difference sheaves are (almost) the same as difference $\mathrm{Gal}(\mathbf{k})$-modules in the sense of our ANT-paper.
4. Let G be a group scheme over X and $\mathcal{R}(G)$ be the presheaf represented by G. Then $\mathcal{R}(G)$ is a sheaf, and we have a morphism $\sigma^*(\mathcal{R}(G)) \to \mathcal{R}({}^{\sigma}G)$. If G is "defined over constants", then ${}^{\sigma}G \cong G$ and $\mathcal{R}(G)$ is a difference sheaf.

Difference site and difference sheaves

We are in our set-up, so we have: X, $\mathbf{C}(X) = (\mathcal{C}, J)$ and σ. We define the difference site $\mathbf{C}_\sigma(X) = (\mathcal{C}', J')$ in the following way.

- $\mathrm{Ob}(\mathcal{C}') := \mathrm{Ob}(\mathcal{C})$.
- For $u : U \to X, v : V \to X$ in $\mathrm{Ob}(\mathcal{C}')$:

$$\mathrm{Hom}_{\mathcal{C}'}(u, v) := \{(f, n) \in \mathrm{Hom}_{\mathrm{Sch}}(U, V) \times \mathbb{N} \mid v \circ f = \sigma^n \circ u\},$$

$$(f, n) \circ (g, m) := (f \circ g, n + m).$$

- $J' := J$.

Theorem

The pair $\mathbf{C}_\sigma(X) = (\mathcal{C}', J')$ is a site and we have:

$$\mathrm{PSh}(\mathbf{C}_\sigma(X)) \simeq \mathrm{PSh}_\sigma(\mathbf{C}(X)), \quad \mathrm{Sh}(\mathbf{C}_\sigma(X)) \simeq \mathrm{Sh}_\sigma(\mathbf{C}(X)).$$

Hence $\mathrm{PSh}_\sigma(\mathbf{C}(X))$ and $\mathrm{Sh}_\sigma(\mathbf{C}(X))$ are Grothendieck categories.

Intuitions about difference site

- The category $\mathcal{C}$ can be considered as a subcategory of $\mathcal{C}'$ using $f \mapsto (f, 0)$. This is also a morphism of sites.
- The difference site is a graded category, which is somehow analogous to the ring of twisted polynomials (non-commutative nature of difference geometry).
- Any morphism in $\mathcal{C}'$ decomposes as:

$$(f, n) = \sigma_V^n \circ f_0,$$

and we have the composition formula:

$$(\sigma_W^m \circ g) \circ (\sigma_V^n \circ f) = \sigma_W^{n+m} \circ \sigma^n(g) \circ f,$$

which looks the same as the formula for multiplying twisted monomials:

$$X^m a \cdot X^n b = X^{n+m} \sigma^n(a) b.$$

Difference sheaf cohomology

For $(\mathcal{F}, \sigma_{\mathcal{F}}) \in \mathrm{Sh}_\sigma(\mathbf{C}(X))$, $\sigma_{\mathcal{F}}$ acts on $\Gamma(\mathcal{F}) = \mathcal{F}(X)$, since $\Gamma(\sigma_*(\mathcal{F})) = \Gamma(\mathcal{F})$ (we denote this action just by σ). We define:

- the difference global section functor:

$$\Gamma^\sigma : \mathrm{Sh}_\sigma(\mathbf{C}(X)) \longrightarrow \mathbf{Ab}, \quad \Gamma^\sigma(\mathcal{F}) := \mathcal{F}(X)^\sigma;$$

- the *n*-th difference sheaf cohomology:

$$H^n_\sigma(\mathbf{C}(X), \mathcal{F}) := \mathrm{Ext}^n_{\mathrm{Sh}_\sigma(\mathbf{C}(X))}(\mathbb{Z}_X, \mathcal{F}).$$

It is easy to see that there is a natural isomorphism:

$$\Gamma^\sigma \cong \mathrm{Hom}_{\mathrm{Sh}_\sigma(\mathbf{C}(X))}(\mathbb{Z}_X, -),$$

and *n*-th difference sheaf cohomology is *n*-th derived functor of Γ^σ.

Example

- Let $X = \mathrm{Spec}(\mathbf{k})$, $\sigma = \mathrm{Spec}(s : \mathbf{k} \to \mathbf{k})$, $\mathbf{C}(X) = \mathbf{E}(X)$. We know that in this case the difference sheaves correspond to difference $\mathrm{Gal}(\mathbf{k})$-modules. Formally, the classical equivalence of Abelian categories:

$$\Psi : \mathrm{Sh}(\mathbf{E}(X)) \simeq \mathrm{Gal}(\mathbf{k}) - \mathrm{Mod},$$

$$\Psi(\mathcal{F}) = \varinjlim_{K/\mathbf{k}} \mathcal{F}(\mathrm{Spec}(K))$$

  induces an equivalence of difference Abelian categories.
- For a difference sheaf $\mathcal{F}$, we get:

$$H^n_\sigma(\mathbf{C}(X), \mathcal{F}) \cong H^n_{\sigma-\mathrm{top}}(\mathrm{Gal}(\mathbf{k}), \Psi(\mathcal{F})),$$

  where $H^n_{\sigma-\mathrm{top}}$ is a topological version of the difference group cohomology from our ANT-paper.

Grothendieck spectral sequence and short exact sequence

- The difference global section functor is the composition of two functors, where the first functor takes injective objects to acyclic ones, hence we can use the Grothendieck spectral sequence to compute the derived functors, and we obtain the following short exact sequence:

$$0 \to H^{n-1}(\mathbf{C}(X), \mathcal{F})_\sigma \to H^n_\sigma(\mathbf{C}(X), \mathcal{F}) \to H^n(\mathbf{C}(X), \mathcal{F})^\sigma \to 0.$$

- The above short exact sequence has the same shape as the short exact sequence we have obtained in the context of difference group cohomology.

Difference Čech cohomology

For $(\mathcal{F}, \sigma_{\mathcal{F}}) \in \mathrm{PSh}_\sigma(\mathbf{C}(X))$ and a covering (of X) $\mathcal{U}$, we define:

- The presheaf of invariants

$$\mathcal{F}^\sigma(U) := \{s \in \mathcal{F}(U) : s|_{U \times_{X^\sigma} U} = \sigma_{\mathcal{F}}(s)|_{U \times_{X^\sigma} U}\}.$$

- The difference Čech functor

$$\check{H}^0_\sigma(\mathcal{U}, \mathcal{F}) := \check{H}^0(\mathcal{U}, \mathcal{F}) \cap \prod_{U \in \mathcal{U}} \mathcal{F}^\sigma(U)$$

  (coinciding with $\Gamma^\sigma(\mathcal{F})$, whenever $\mathcal{F}$ is a sheaf).

- The n-th difference Čech cohomology functor

$$\check{H}^n_\sigma(\mathcal{U}, -) := R^n\left(\check{H}^0_\sigma(\mathcal{U}, -)\right).$$

- Taking direct limit over coverings $\mathcal{U}$, we obtain $\check{H}^n_\sigma(\mathbf{C}(X), \mathcal{F})$, which can be computed using an explicit bicomplex, and which coincides with $H^n_\sigma(\mathbf{C}(X), \mathcal{F})$, whenever it is true classically.

Difference sheaf torsors

$(\mathcal{G}, \sigma_{\mathcal{G}})$ is a difference sheaf of (not necessarily Abelian!) groups.

- A sheaf of sets $\mathcal{P}$ on $\mathbf{C}(X)$ is a $\mathcal{G}$-sheaf if there is a sheaf morphism $\mu : \mathcal{G} \times \mathcal{P} \to \mathcal{P}$ such that μ induces an action of the group $\mathcal{G}(U)$ on the set $\mathcal{P}(U)$ for any $U \in \mathrm{Ob}(\mathcal{C})$.
- A $\mathcal{G}$-sheaf $\mathcal{P}$ is a trivial $\mathcal{G}$-torsor, if $\mathcal{P} \cong \mathcal{G}$ as $\mathcal{G}$-sheaves.
- A $\mathcal{G}$-sheaf $\mathcal{P}$ is a sheaf $\mathcal{G}$-torsor, if there is a covering $\mathcal{U}$ (of X) in $\mathbf{C}(X)$ such that for $U \in \mathcal{U}$, $\mathcal{P}|_U$ is a trivial $\mathcal{G}|_U$-torsor.
- A difference sheaf $\mathcal{G}$-torsor is a pair $(\mathcal{P}, \sigma_{\mathcal{P}})$, where

$$\sigma_{\mathcal{P}} : \mathcal{P} \longrightarrow (\widetilde{\sigma_{\mathcal{G}}})_* (\sigma^*(\mathcal{P}))$$

is an morphism of sheaf $\mathcal{G}$-torsors.

Difference Čech cohomology and difference sheaf torsors

Let $\mathcal{G}$ be difference sheaf of (not necessarily Abelian) groups.

Definition

We denote the pointed set of isomorphism classes of difference sheaf $\mathcal{G}$-torsors by $\mathrm{PHSh}_\sigma(\mathcal{G}/X)$.

There is a natural definition of the first (pointed set) difference Čech cohomology with coefficients in a sheaf of groups, which coincides with the previous definition in the commutative case.

Theorem

There is an isomorphism of pointed sets, which is an isomorphism of Abelian groups when $\mathcal{G}$ is commutative:

$$\mathrm{PHSh}_\sigma(\mathcal{G}/X) \cong \check{H}^1_\sigma(\mathbf{C}(X), \mathcal{G}).$$

Difference torsors over difference groups over constants

- Let G be a group scheme over X which is defined over constants (of σ), that is, there is a scheme morphism $t : X \to F$ such that $\sigma \circ t = t$ and

$$G \cong X \times_F G_F,$$

where G_F is a group scheme over F.
- Then there is $\tau : {}^{\sigma}G \cong G$, so G becomes an isotrivial difference group scheme over X.
- A pair (P, σ_P) is called a difference G-torsor if:
  - P is a scheme G-torsor over X ($\mathbf{C}(X) = \mathbf{F}(X)$);
  - $\sigma_P : P \to \tau_*({}^{\sigma}P)$ is a morphism of G-torsors over X.

  We denote the pointed set of isomorphism classes of difference G-torsor by $\mathrm{PHS}_\sigma(G/X)$.
- The last definition coincides (in this case of isotrivial difference groups schemes) with the one from "Torsors for Difference Algebraic Groups" by Bachmayr and Wibmer.

Difference torsors and difference cohomology

- Let G be a group scheme over X which is defined over constants. We make the standard assumptions implying that sheaf torsors are representable by scheme torsors. We have:

$$\mathrm{PHS}_{\sigma}(G/X) \cong \check{H}^1_{\sigma}(\mathbf{F}(X), \mathcal{R}(G)).$$

- If G is commutative, we get:

$$\mathrm{PHS}_{\sigma}(G/X) \cong H^1_{\sigma}(\mathbf{F}(X), \mathcal{R}(G)).$$

- If G is smooth, then we can replace $\mathbf{F}(X)$ with $\mathbf{E}(X)$ above.
- If $G = \mathrm{GL}_{n,X}$, then we can replace $\mathbf{F}(X)$ with $\mathbf{Z}(X)$ above.
- For G commutative, we have our short exact sequence giving:

$$0 \to G(X)_{\sigma} \to \mathrm{PHS}_{\sigma}(G/X) \to \mathrm{PHS}(G/X)^{\sigma} \to 0.$$

Difference sheaf and Galois cohomology I

- Let $(\mathbf{k}, s)$ be a difference field and G be a group scheme over $\mathbf{k}$, which is defined over $\mathrm{Fix}(s)$. Then we have:

$$H^1_\sigma(\mathbf{k}, G) \cong \check{H}^1_\sigma(\mathbf{F}(\mathrm{Spec}(\mathbf{k})), \mathcal{R}(G)),$$

  where $H^1_\sigma(\mathbf{k}, G)$ is the "difference Galois cohomology" (pointed set or group) in the sense of Bachmayr and Wibmer.
- In the case when G is commutative and smooth, we have the following short exact sequence:

$$0 \to G(\mathbf{k})_\sigma \to H^1_\sigma(\mathbf{k}, G) \to H^1(\mathbf{k}, G)^\sigma \to 0,$$

  where $H^1(\mathbf{k}, G)$ is the usual Galois cohomology.

Computing difference torsors: example

- Bachmayr and Wibmer considered the following difference group scheme (functor):

$$(R,\sigma_R)\mapsto\{g\in\mathbb{G}_{\mathrm{m}}(R)\mid\sigma_R(g)=g,\ g^2=1\}.$$

- This functor is represented by the isotrivial difference group scheme μ_2 (second roots of unity), so we obtain the following short exact sequence:

$$0\to\mu_2(\mathbf{k})_\sigma\to H^1_\sigma(\mathbf{k},\mu_2)\to H^1(\mathbf{k},\mu_2)^\sigma\to 0.$$

- Using the classical description of the Galois cohomology of μ_2, we get (assuming $\mathrm{char}(\mathbf{k})\neq 2$):

$$0\to\mathbb{Z}/2\mathbb{Z}\to H^1_\sigma(\mathbf{k},\mu_2)\to\left(\mathbf{k}^*/(\mathbf{k}^*)^2\right)^\sigma\to 0,$$

which coincides with the description of Bachmayr/Wibmer.

- We define a difference bundle as a difference sheaf (E, σ_E) such that E is a locally free sheaf of $\mathcal{O}_X$-modules and:

$$\widetilde{\sigma_E} : \sigma^{\diamond}(E) \cong E$$

($\sigma^{\diamond}(E)$: pull-back in the category of sheaves of $\mathcal{O}_X$-modules).

- We have:

$$\mathrm{Bun}^n_\sigma(X) \cong \mathrm{PHS}_\sigma(\mathsf{GL}_n / X) \cong \check{H}^1_\sigma(\mathbf{F}(X), \mathcal{R}(\mathsf{GL}_n)),$$

where $\mathbf{F}(X)$ can be replaced with $\mathbf{E}(X)$ or with $\mathbf{Z}(X)$.

- For $n = 1$, we get the difference Picard group and:

$$\mathrm{Pic}_\sigma(X) \cong H^1_\sigma(\mathbf{F}(X), \mathcal{R}(\mathbb{G}_\mathrm{m})),$$

$$0 \longrightarrow \mathbb{G}_\mathrm{m}(X)_\sigma \longrightarrow \mathrm{Pic}_\sigma(X) \longrightarrow \mathrm{Pic}(X)^\sigma \longrightarrow 0.$$

Affine case and Chevalley's ambiguous class formula

For a difference ring (R, s), we define in a natural way the (affine) difference Picard group $\mathrm{Pic}_s(R)$ of isomorphism classes of invertible difference (R, s)-modules, and we have: $\mathrm{Pic}_s(R) \cong \mathrm{Pic}_\sigma(\mathrm{Spec}(R))$.

Example

- For a difference field $(\mathbf{k}, s)$, we have:

$$\mathrm{Pic}_s(\mathbf{k}) \cong \mathbb{G}_{\mathrm{m}}(\mathbf{k})_s \cong H^1_\sigma(\mathbf{k}, \mathbb{G}_{\mathrm{m}}),$$

  which fits to the original motivation.

- Let L/K be a cyclic extension of number fields and $G := \mathrm{Gal}(L/K) = \langle s \rangle$. Then $(\mathcal{O}_L, s)$ is a difference ring, and we get the following exact sequence:

$$1 \longrightarrow \mathbb{G}_{\mathrm{m}}(\mathcal{O}_L)_G \longrightarrow \mathrm{Pic}_s(\mathcal{O}_L) \longrightarrow \mathrm{Cl}(L)^G \longrightarrow 1,$$

  which is related to *Chevalley's ambiguous class formula*.

Difference group schemes and right difference sheaves I

Let (X, σ) be a difference scheme and (G, σ_G) be a difference group scheme over (X, σ) that is:

- G is a group scheme over X;
- $\sigma_G : G \to {}^{\sigma}G$ is a morphism of X-group schemes.

There is one problem with the theory developed so far: (G, σ_G) does *not* yield a (left) difference sheaf, unless σ_G is an isomorphism. Therefore we introduce the following:

Definition

A right difference presheaf on $\mathbf{C}(X)$ is a pair $(\mathcal{F}, \sigma^{\mathcal{F}})$, where $\mathcal{F}$ is a sheaf on $\mathbf{C}(X)$ and we have a sheaf morphism:

$$\sigma^{\mathcal{F}} : \mathcal{F} \longrightarrow \sigma^*(\mathcal{F}).$$

However, since $\mathcal{R}({}^{\sigma}G) \ncong \sigma^*(\mathcal{R}(G))$, the sheaf $\mathcal{R}(G)$ does not become even a right difference sheaf!

Difference group schemes and right difference sheaves II

- There is a natural morphism:

$$\phi_G : \sigma^*(\mathcal{R}(G)) \longrightarrow \mathcal{R}({}^{\sigma}G).$$

- If $\{G \to X\} \in J$ (covering), then ϕ_G *is* an isomorphism.
- It is natural to assume that G is flat over X.
- Because of the existence of the identity section, if G is flat over X, then G is *faithfully flat* over X, so, in this case, $\{G \to X\}$ is a covering in $\mathbf{F}(X)$.
- Hence the following composition:

$$\mathcal{R}(G) \xrightarrow{\mathcal{R}(\sigma_G)} \mathcal{R}({}^{\sigma}G) \xrightarrow{\phi_G^{-1}} \sigma^*(\mathcal{R}(G))$$

gives $\mathcal{R}(G)$ the right difference sheaf structure on the flat site $\mathbf{F}(X)$. This is our main and motivating example.

Category of right difference sheaves

- It is not clear, whether for the category of right difference sheaves (denoted $\mathrm{Sh}^{\sigma}(\mathbf{C}(X))$) one can find a natural "right difference site" $\mathbf{C}^{\sigma}(X)$ such that:

$$\mathrm{Sh}^{\sigma}(\mathbf{C}(X)) \simeq \mathrm{Sh}(\mathbf{C}^{\sigma}(X))$$

  (as it was the case for the category of left difference sheaves).
- Still, it is easy to see that the category of right difference sheaves is Abelian.
- However, it is not immediate to check whether there are enough injective objects. We follow the strategy from Grothendieck's Tohoku paper, that is we find a family of generators for the category $\mathrm{Sh}^{\sigma}(\mathbf{C}(X))$, which, together with the existence of inductive limits and one more property, implies that the category $\mathrm{Sh}^{\sigma}(\mathbf{C}(X))$ has enough injectives.

Right difference sheaf cohomology

- Let $(\mathcal{F}, \sigma^{\mathcal{F}})$ be a right difference sheaf. Then we define:

  $$\Gamma^{\sigma}(\mathcal{F}) = \ker(\mathrm{res} - \sigma^{\mathcal{F}})$$

  for the evident maps:

  $$\mathrm{res}, \sigma^{\mathcal{F}} : \Gamma(\mathcal{F}) \longrightarrow \Gamma(\sigma^*(\mathcal{F})).$$

- Let $\mathbb{Z}_X$ denote the right difference constant sheaf on X. Then it is easy to see that there is a natural isomorphism:

  $$\Gamma^{\sigma} \cong \mathrm{Hom}_{\mathrm{Sh}^{\sigma}(\mathbf{C}(\mathrm{X}))}(\mathbb{Z}_{\mathrm{X}}, -).$$

  We define the n-th right difference sheaf cohomology:

  $$H^n_{\sigma}(X, \mathcal{F}) := \mathrm{Ext}^n_{\mathrm{Sh}^{\sigma}(\mathbf{C}(X))}(\mathbb{Z}_X, \mathcal{F}).$$

- There is a similar short exact sequence as before.

Right difference Čech and right difference torsors

- We define right difference Čech cohomology, which coincides with right difference derived cohomology as before.
- The first right difference Čech cohomology classifies the isomorphism classes of right difference sheaf torsors.
- For a flat difference group scheme (G, σ_G), the sheaf $\mathcal{R}(G)$ becomes a right difference sheaf on $\mathbf{F}(X)$ and difference torsors over (G, σ_G) (introduced by Bachmayr/Wibmer) coincide with right difference $\mathcal{R}(G)$-torsors. Hence, they are classified by the first right difference Čech cohomology.
- We denote the set of the isomorphism classes of difference torsors over (G, σ_G) by $\mathrm{PHS}_\sigma(G/X)$.

Theorem

Let (G, σ_G) be a commutative difference group scheme over a difference field $(\mathbf{k}, s)$. We have the following isomorphisms:

$$H^n_\sigma\left(\mathbf{F}(\mathrm{Spec}(\mathbf{k})), \mathcal{R}(G)\right) \cong H^n_{\sigma-\mathrm{top}}\left(\mathrm{Gal}(\mathbf{k}), G(\mathbf{k}^{\mathrm{sep}})\right),$$

$$H^1_\sigma\left(\mathbf{F}(\mathrm{Spec}(\mathbf{k})), \mathcal{R}(G)\right) \cong H^1_\sigma(\mathbf{k}, G) \cong \mathrm{PHS}_\sigma(G/\mathbf{k}),$$

and the following short exact sequence:

$$0 \longrightarrow G(\mathbf{k})_\sigma \to \mathrm{PHS}_\sigma(G/\mathbf{k}) \to H^1(\mathbf{k}, G)^\sigma \to 0.$$

One more example

- Bachmayr and Wibmer considered the following difference group scheme (functor):

$$(R,\sigma_R) \mapsto \left\{ r \in R \mid \sigma^n(r) = \lambda_0 r + \lambda_1 \sigma(r) + \ldots + \lambda_{n-1}\sigma^{n-1}(r) \right\}.$$

- This functor is represented by the difference group scheme:

$$(\mathbb{G}_{\mathrm{a}}^n, (a_1, \ldots, a_{n-1}, a_n) \mapsto (a_2, \ldots, a_n, \lambda_0 a_1 + \ldots + \lambda_{n-1} a_n)).$$

- We get the following short exact sequence:

$$0 \to \mathbb{G}_{\mathrm{a}}^n(\mathbf{k})_\sigma \to H^1_\sigma(\mathbf{k}, \mathbb{G}_{\mathrm{a}}^n) \to H^1(\mathbf{k}, \mathbb{G}_{\mathrm{a}}^n)^\sigma \to 0.$$

- By (additive) Hilbert 90, we get:

$$H^1_\sigma(\mathbf{k}, \mathbb{G}_{\mathrm{a}}^n) \cong \mathbb{G}_{\mathrm{a}}^n(\mathbf{k})_\sigma \cong \operatorname{coker}\left(x \mapsto \lambda_0 x + \ldots + \lambda_{n-1}\sigma^{n-1}(x)\right).$$