On Ershov Semilattices of Degrees of Σ-Definability of Structures

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Theory KPU, admissible sets, $\mathbb{HF}(\mathfrak{M})$

Let $\sigma' = \sigma \cup \{U^1, \in^2, \varnothing\}$, where σ is a finite signature.

Definition

*The class of Δ_0-**formulas** of signature σ' is the least class of formulas containing all atomic formulas of signature σ' and closed under $\wedge, \vee, \neg$, $\exists x \in y$ and $\forall x \in y$.*

Definition

*The class of Σ-**formulas** of signature σ' is the least class of formulas containing all Δ_0-formulas of signature σ' and closed under $\wedge, \vee$, $\exists x \in y$, $\forall x \in y$ and $\exists x$.*

Definition

The **axioms of KPU** *(for signature σ') are the universal closures of the following formulas:*

Empty set: $\neg\exists x(x \in \varnothing) \wedge \neg U(\varnothing)$

Extensionality: $(\neg U(a) \wedge \neg U(b)) \rightarrow (\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$

Foundation: $\exists x \varphi(x) \rightarrow \exists x[\varphi(x) \wedge \forall y \in x \neg\varphi(y)]$ *for all formulas* $\varphi(x)$ *in which* y *does not occurs free*

Pair: $\exists a(x \in a \wedge y \in a)$

Union: $\exists b \forall y \in a \forall x \in y(x \in b)$

Δ_0-Separation: $\exists b \forall x(x \in b \leftrightarrow x \in a \wedge \varphi(x))$ *for all* Δ_0*-formulas* $\varphi(x)$ *in which b does not occurs free*

Δ_0-Collection: $\forall x \in a \exists y\, \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b\, \varphi(x, y)$ *for all* Δ_0*-formulas* $\varphi(x)$ *in which b does not occurs free.*

Admissible sets

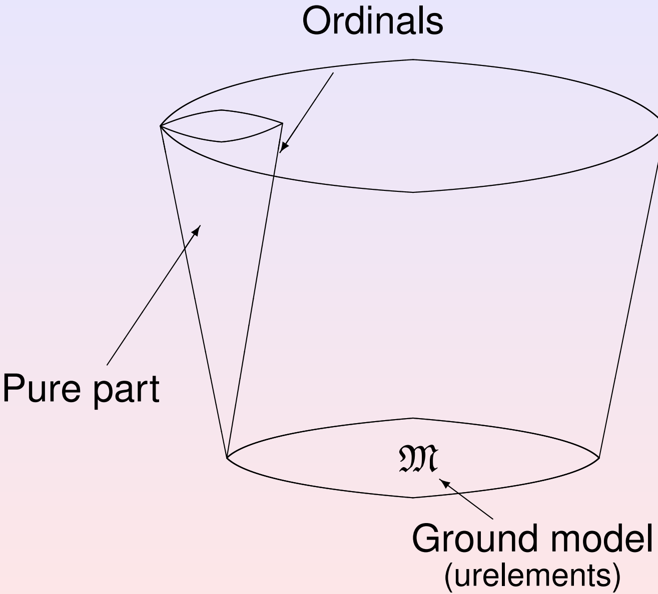
Let $\mathrm{Tran}(a)$ be the formula $\forall x \in a \forall y \in x (y \in a)$ and let

$$\mathrm{Ord}(a) \rightleftharpoons \mathrm{Tran}(a) \wedge \forall x \in a\, \mathrm{Tran}(x).$$

Definition

A structure $\mathbb{A}$ of signature σ' is called an **admissible set** *if*

1) $\mathbb{A} \models \mathrm{KPU}$
2) $\mathrm{Ord}\,\mathbb{A} = \{a \mid \mathbb{A} \models \mathrm{Ord}(a)\}$ *is wellfounded*


Ordinals
Pure part
$\mathfrak{M}$
Ground model
(urelements)

$\mathbb{HF}(\mathfrak{M})$

For a set M, consider the set $\mathrm{HF}(M)$ of hereditary finite sets over M defined as follows: $\mathrm{HF}(M) = \bigcup_{n\in\omega} \mathrm{HF}_n(M),$ where

$\mathrm{HF}_0(M) = \{\varnothing\} \cup M,$
$\mathrm{HF}_{n+1}(M) = \mathrm{HF}_n(M) \cup \{a \mid a \text{ is a finite subset of } \mathrm{HF}_n(M)\}.$

For a structure $\mathfrak{M} = \langle M, \sigma^{\mathfrak{M}}\rangle$ of signature σ, **hereditary finite superstructure**

$$\mathbb{HF}(\mathfrak{M}) = \langle \mathrm{HF}(M); \sigma^{\mathfrak{M}}, U, \in, \varnothing\rangle$$

is a structure of signature σ' (with $\mathbb{HF}(\mathfrak{M}) \models U(a) \iff a \in M$).

$\mathbb{HF}(\mathfrak{M})$ is the least admissible set over $\mathfrak{M}$.

Computability on admissible sets

Definition

For an admissible set $\mathbb{A} = \langle A, (\sigma')^{\mathbb{A}} \rangle$ of signature σ', a subset $R \subseteq A$ is called a **Σ-set** *in $\mathbb{A}$ if, for some Σ-formula $\Phi(x, \bar{y})$ of signature σ' and some $\bar{c} \in A^{<\omega}$,*

$$R = \{a \in A | \mathbb{A} \models \Phi(a, \bar{c})\}.$$

$R \subseteq A$ is called a **Δ-set** *in $\mathbb{A}$ if R and $A \setminus R$ are Σ-sets in $\mathbb{A}$.*

For a subset $R \subseteq \omega$ of natural numbers,

R is a Σ-set in $\mathbb{HF}(\varnothing) \iff R$ is computably enumerable,

R is a Δ-set in $\mathbb{HF}(\varnothing) \iff R$ is computable.

Σ-definability of structures in admissible sets

Let $\mathfrak{M}$ be a structure of relational computable signature $\langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots \rangle$ and let $\mathbb{A}$ be an admissible set.

Definition

$\mathfrak{M}$ is called Σ-definable in $\mathbb{A}$ if there exists a computable sequence of Σ-formulas $\varphi(x_0, y), \psi(x_0, x_1, y), \psi^*(x_0, x_1, y)$, $\varphi_0(x_0, \ldots, x_{n_0-1}, y), \varphi_0^*(x_0, \ldots, x_{n_0-1}, y), \ldots, \varphi_k(x_0, \ldots, x_{n_k-1}, y)$, $\varphi_k^*(x_0, \ldots, x_{n_k-1}, y), \ldots$ such that, for some parameter $a \in A$, $M_0 \leftrightharpoons \varphi^{\mathbb{A}}(x_0, a) \neq \varnothing$, $\eta \leftrightharpoons \psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$ is a congruence on the structure $\mathfrak{M}_0 \leftrightharpoons \langle M_0, P_0^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0}, \ldots \rangle$, where

$$P_k^{\mathfrak{M}_0} \leftrightharpoons \varphi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1}) \cap M_0^{n_k}, \ k \in \omega,$$
$$\psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \psi^{\mathbb{A}}(x_0, x_1, a),$$

$\varphi_k^{*\mathbb{A}}(x_0, \ldots, x_{n_k-1}, a) \cap M_0^{n_k} = M_0^{n_k} \setminus \varphi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1})$ for all $k \in \omega$, and the structure $\mathfrak{M}$ is isomorphic to the quotient structure $\mathfrak{M}_0 / \eta$.

For a countable structure $\mathfrak{M}$, the following are equivalent:

- $\mathfrak{M}$ is constructivizable (computable);
- $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\varnothing)$.

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_{\Sigma} \mathfrak{N}$ the fact that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$.

For arbitrary cardinal α, let $\mathcal{K}_\alpha$ be the class of all structures (of computable signatures) of cardinality $\leqslant \alpha$. We define on $\mathcal{K}_\alpha$ an equivalence relation $\equiv_\Sigma$ as follows: for $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$,

$$\mathfrak{M} \equiv_\Sigma \mathfrak{N} \text{ if } \mathfrak{M} \leqslant_\Sigma \mathfrak{N} \text{ and } \mathfrak{N} \leqslant_\Sigma \mathfrak{M}.$$

A structure

$$\mathcal{S}_\Sigma(\alpha) = \langle \mathcal{K}_\alpha / \equiv_\Sigma, \leqslant_\Sigma \rangle$$

is an upper semilattice with the least element, and, for any $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$,

$$[\mathfrak{M}]_\Sigma \vee [\mathfrak{N}]_\Sigma = [(\mathfrak{M}, \mathfrak{N})]_\Sigma,$$

where $(\mathfrak{M}, \mathfrak{N})$ denotes the model-theoretic pair of $\mathfrak{M}$ and $\mathfrak{N}$.

Theorem (Ershov 1985, 1994)

For the field $\mathbb{C}$ of complex numbers,

- $\mathbb{C} \not\leqslant_\Sigma \mathbb{S}$ *for any infinite set* $\mathbb{S}$*;*
- $\mathbb{C} \leqslant_\Sigma \mathbb{L}$ *for any dense linear order* $\mathbb{L}$ *of cardinality* 2^ω*.*

For the field $\mathbb{R}$ of real numbers,

- $\mathbb{R} \not\leqslant_\Sigma \mathbb{L}$ *for any linear order* $\mathbb{L}$*.*

Definition

We call a theory T c-simple (computably simple) if

1) *T is decidable;*
2) *T is ω-categorical and model complete;*
3) *the family of prime formulas is decidable.*

Theorem (S. 2002)

Let T be a c-simple theory and $\mathfrak{M}$ be any computable model of T.

i) *T has uncountable model Σ-definable in $\mathbb{HF}(\mathbb{L})$ for some $\mathbb{L} \models \mathrm{DLO}$ iff there exists an infinite computable set of order indiscernibles in $\mathfrak{M}$.*

ii) *T has uncountable model Σ-definable in $\mathbb{HF}(\mathbb{S})$ for some infinite set $\mathbb{S}$ iff there exists an infinite computable set of total indiscernibles in $\mathfrak{M}$.*

Corollary (S. 2002)

There exists a c-simple theory (of infinite signature) such that none of it's uncountable models is Σ-definable in $\mathbb{HF}(\mathbb{L})$ for any uncountable dense linear order $\mathbb{L}$.

Definition

The rank of inner constructivizability of an admissible set $\mathcal{A}$ is the ordinal

$$\mathrm{cr}(\mathcal{A}) = \inf\{\mathrm{rnk}(B) | \mathcal{A} \text{ is constructivizable inside } B\}.$$

The next theorem gives the precise estimates of the rank of inner constructivizability for hereditary finite superstructures.

Theorem (S. 2005)

Suppose $\mathfrak{M}$ is a structure of computable signature. Then
1) if $\mathfrak{M}$ is finite then $\mathrm{cr}(\mathbb{HF}(\mathfrak{M})) = \omega$,
2) if $\mathfrak{M}$ is infinite then $\mathrm{cr}(\mathbb{HF}(\mathfrak{M})) \leqslant 2$.

Theorem (S. 2005)

$\mathrm{cr}(\mathbb{HF}(\mathbb{R})) = 1$.

Morozov results (2007) on countable models Σ-definable (without parameters!) in $\mathbb{HF}(\mathbb{R})$:

1) any such model is hyperarithmetic;
2) for any hyperarithmetical degree d, there is a countable model $\mathfrak{M}$ Σ-definable in $\mathbb{HF}(\mathbb{R})$ such that any presentation of $\mathfrak{M}$ has degree bigger than d.

For arbitrary structure $\mathfrak{M}$ of a computable signature σ and an admissible set $\mathbb{A}$ with $M \subseteq U(\mathbb{A})$, we say that $\mathfrak{M}$ is **decidable in $\mathbb{A}$** if

$$\{ \langle \varphi, \bar{m} \rangle \mid \varphi \in F_\sigma,\ \bar{m} \in M^{<\omega},\ \mathfrak{M} \models \varphi(\bar{m}) \}$$

is Σ-subset of $\mathbb{A}$. In the same way the notions of ***n*-decidable** and **computable** (i.e. 0-decidable) structures in $\mathbb{A}$ could be defined.

If $\mathfrak{M}$ is 1-decidable in $\mathbb{HF}(\mathfrak{M})$ when $\mathbb{HF}(\mathfrak{M})$ has universal Σ-function and reduction property, but not necessarily uniformization property.

Let $\mathfrak{M}$ be a structure of signature σ, signature σ_* consists of all symbols from σ and function symbols $f_\varphi(x_1, \ldots, x_n)$ for all $\exists$-formulas $\varphi(x_0, x_1, \ldots, x_n) \in F_\sigma$. A structure $\mathfrak{M}_*$ of signature σ_* is called **existential Skolem expansion of** $\mathfrak{M}$ if $|\mathfrak{M}_*| = |\mathfrak{M}|$, $\mathfrak{M} \upharpoonright_\sigma = \mathfrak{M}_* \upharpoonright_\sigma$, and for any $\exists$-formula $\varphi(x_0, x_1, \ldots, x_n) \in F_\sigma$

$$\mathfrak{M}_* \models \forall x_1 \ldots \forall x_n (\exists x \varphi(x, x_1, \ldots, x_n) \rightarrow$$

$$\rightarrow \varphi(f_\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n)).$$

Theorem (S. 1996)

Let $\mathfrak{M}$ be 1-decidable in $\mathbb{HF}(\mathfrak{M})$. Then $\mathbb{HF}(\mathfrak{M})$ has the uniformization property iff some existential Skolem expansion of $\mathfrak{M}$ is computable in $\mathbb{HF}(\mathfrak{M})$.

Corollary (S. 1996, indep. Korovina 1996 for $\mathbb{HF}(\mathbb{R})$)

$\mathbb{HF}(\mathbb{R})$ *and* $\mathbb{HF}(\mathbb{Q}_p)$ *have the uniformization property and a universal* Σ*-function.*

Let $\mathfrak{M}$ be a structure of a computable signature and let $\mathbb{A}$ be an admissible set.

Definition

A presentation of $\mathfrak{M}$ in $\mathbb{A}$ is any structure $\mathcal{C}$ such that $\mathcal{C} \cong \mathfrak{M}$ and the domain of $\mathcal{C}$ is a subset of A.

We can treat (the atomic diagram of) a presentation $\mathcal{C}$ as a subset of A, using some Gödel numbering of the atomic formulas of the signature of $\mathfrak{M}$.

Definition

The problem of presentability of $\mathfrak{M}$ in $\mathbb{A}$ is the set $\mathrm{Pr}(\mathfrak{M}, \mathbb{A})$ consisting of the atomic diagrams of all possible presentations of $\mathfrak{M}$ in $\mathbb{A}$:

$$\mathrm{Pr}(\mathfrak{M}, \mathbb{A}) = \{\, \mathcal{C} \mid \mathcal{C} \textit{ is a presentation of } \mathfrak{M} \textit{ in } \mathbb{A} \,\}$$

Denote by $\underline{\mathfrak{M}}$ the set $\mathrm{Pr}(\mathfrak{M}, \mathbb{HF}(\varnothing))$ of all presentations of $\mathfrak{M}$ in the least admissible set.

A *mass problem* (Yu. T. Medvedev, 1955) is any set of total functions from ω to ω. A mass problem can be considered as a set of ”solutions” (in form of functions from ω to ω) of some ”informal problem”.

Examples of mass problems: suppose $A, B \subseteq \omega$

1) the *problem of solvability* of a set A is the mass problem $\mathcal{S}_A = \{\chi_A\}$, where χ_A is the characteristic function of A
2) the *problem of enumerability* of a set A is the mass problem

$$\mathcal{E}_A = \{f : \omega \to \omega \mid \mathrm{rng}(f) = A\}$$

3) the *problem of separability* of sets A, B is the mass problem

$$\mathcal{P}_{A,B} = \{f : \omega \to 2 \mid f^{-1}(0) = A,\ f^{-1}(1) = B\}$$

Theorem

Let $\mathfrak{M}$ be a countable structure, and $A \subseteq \omega$, $A \neq \varnothing$. The following are equivalent:

1) $\mathcal{E}_A \leqslant_w \underline{\mathfrak{M}}$
2) $\mathcal{E}_A \leqslant \underline{(\mathfrak{M}, \bar{m})}$ *for some* $\bar{m} \in M^{<\omega}$
3) A *is* Σ*-definable in* $\mathbb{HF}(\mathfrak{M})$

Theorem

Let $\mathfrak{M}$ be a countable structure, and $A \subseteq \omega$. The following are equivalent:

1) $\mathcal{S}_A \leqslant_w \underline{\mathfrak{M}}$
2) $\mathcal{S}_A \leqslant \underline{(\mathfrak{M}, \bar{m})}$ *for some* $\bar{m} \in M^{<\omega}$
3) A *is* Δ*-definable in* $\mathbb{HF}(\mathfrak{M})$

Definition

Let $\mathfrak{M}$ be a countable structure. $\mathfrak{M}$ is said to have a degree (e-degree) if there exists a least degree in the class of T-degrees (e-degrees) of all possible presentations of $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$.

Theorem

For a countable $\mathfrak{M}$, $\mathfrak{M}$ has a degree (e-degree) iff, for some $\mathcal{C} \in \underline{\mathfrak{M}}$, $\mathcal{C}$ is Δ-definable (Σ-definable) in $\mathbb{HF}(\mathfrak{M})$.

Let $\mathcal{D}$ denotes the semilattice of Turing degrees of unsolvability. A mapping $i : \mathcal{D} \to \mathcal{S}_\Sigma$ is defined as follows: for any T-degree $\mathbf{a}$, let

$$i(\mathbf{a}) = [\mathfrak{M}_{\mathbf{a}}]_\Sigma, \text{ where } \mathfrak{M}_{\mathbf{a}} \text{ has degree } \mathbf{a}.$$

Let $\mathcal{D}_e$ denotes the semilattice of enumeration degrees. A mapping $j : \mathcal{D}_e \to \mathcal{S}_\Sigma$ is defined as follows: for any e-degree $\mathbf{a}$, let

$$j(\mathbf{a}) = [\mathfrak{M}_{\mathbf{a}}]_\Sigma, \text{ where } \mathfrak{M}_{\mathbf{a}} \text{ has e-degree } \mathbf{a}.$$

Proposition

Mappigs $i : \mathcal{D} \to \mathcal{S}_\Sigma$ and $j : \mathcal{D}_e \to \mathcal{S}_\Sigma$ are semilattice embeddings. So, $\mathcal{D} \hookrightarrow \mathcal{D}_e \hookrightarrow \mathcal{S}_\Sigma$.

Σ-operators

A mapping $F : P(A)^n \to P(A)$ ($n \in \omega$) is called a Σ-*operator* if there is a Σ-formula $\Phi(x_0, \ldots, x_{n-1}, y)$ of the signature $\sigma_{\mathbb{A}}$ such that for all $S_0, \ldots, S_{n-1} \in P(A)$

$$F(S_0, \ldots, S_{n-1}) = \{ a \mid \exists a_0, \ldots, a_{n-1} \in A$$

$$(\bigwedge_{i<n} a_i \subseteq S_i \ \wedge \ \mathbb{A} \models \Phi(a_0, \ldots, a_{n-1}, a))\}.$$

Suppose $B, C \subseteq A$. *B is $e\Sigma$-reducible to C* ($B \leqslant_{e\Sigma} C$) if there exists a unary Σ-operator F such that $C \in \delta_c(F)$ and $B = F(C)$.

B is $T\Sigma$-reducible to C ($B \leqslant_{T\Sigma} C$) if there exist binary Σ-operators F_0 and F_1 such that $\langle C, A \setminus C \rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ for which $B = F_0(C, A \setminus C)$ and $A \setminus B = F_1(C, A \setminus C)$.

An operator $F : P(A) \to P(A)$ is *strongly continuous* in $S \in P(A)$, if

$$\text{for any } a \subseteq F(S),\ a \in A,\ \text{ there exists } a' \subseteq S,\ a' \in A,\ \text{s.t. } a \subseteq F(a').$$

For operator $F : P(A)^n \to P(A)$, $\delta_{\mathrm{c}}(F)$ is the set of elements of $P(A)^n$ in which F is strongly continuous.

A set $S \in P(A)^n$ is called a Σ_**-set* if $S \in \delta_c(F)$ for any Σ-operator $F : P(A)^n \to P(A)$.

It is easy to show that in $\mathbb{HF}(\mathfrak{M})$ any subset is a Σ_*-set.

Uniform reducibilities

Suppose $\mathcal{X}, \mathcal{Y} \subseteq P(A)$. $\mathcal{X}$ *is Medvedev reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant \mathcal{Y}$) if there exist binary Σ-operators F_0 and F_1 such that, for all $Y \in \mathcal{Y}$, $\langle Y, A \setminus Y \rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$.

$\mathcal{X}$ *is Dyment reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant_e \mathcal{Y}$) if there exists a unary Σ-operator F such that, for all $Y \in \mathcal{Y}$, $Y \in \delta_c(F)$ and $F(\mathcal{Y}) \subseteq \mathcal{X}$.

$\mathcal{X}$ *is Muchnik reducible to* $\mathcal{Y}$ ($\mathcal{X} \leqslant_w \mathcal{Y}$) if, for any $Y \in \mathcal{Y}$, there exist binary Σ-operators F_0 and F_1 such that $\langle Y, A \setminus Y \rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$.

For countable structure $\mathfrak{M}$ consider classes

$$\mathcal{K}_\Sigma(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \text{ is } \Sigma\text{-definable in } \mathbb{HF}(\mathfrak{M})\}$$
$$\mathcal{K}_e(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_e \underline{(\mathfrak{M}, \bar{m})} \text{ for some } \bar{m} \in M^{<\omega}\}$$
$$\mathcal{K}(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant \underline{(\mathfrak{M}, \bar{m})} \text{ for some } \bar{m} \in M^{<\omega}\}$$
$$\mathcal{K}_{ew}(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_{ew} \underline{\mathfrak{M}}\}$$
$$\mathcal{K}_w(\mathfrak{M}) = \{\mathfrak{N} \mid \underline{\mathfrak{N}} \leqslant_w \underline{\mathfrak{M}}\}$$

For any structure $\mathfrak{M}$,

$$\mathcal{K}_\Sigma(\mathfrak{M}) \subseteq \mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M}),$$

as well as $\mathcal{K}_e(\mathfrak{M}) \subseteq \mathcal{K}_{ew}(\mathfrak{M}) \subseteq \mathcal{K}_w(\mathfrak{M})$. In general, all these inclusions are proper.

For any $* \in \{e, \ , w, ew\}$, define the relation $\leqslant_*$ on $\mathcal{K}_\omega$ in the following way: $\mathfrak{M} \leqslant_* \mathfrak{N}$ if and only if $\mathcal{K}_*(\mathfrak{M}) \subseteq \mathcal{K}_*(\mathfrak{N})$, and let $\mathcal{S}_* = \langle \mathcal{K}_\omega / \equiv_*, \leqslant_* \rangle$ be a structure of degrees of presentability corresponding to this reducibility relation.

Theorem

Each of $\mathcal{S}_*$, $* \in \{e, \ , w, ew\}$, is an upper semilattice with 0, and there are following embeddings ($\hookrightarrow$) and homomorphisms ($\rightarrow$)

$$\mathcal{D} \hookrightarrow \mathcal{D}_e \hookrightarrow \mathcal{S}_\Sigma \rightarrow \mathcal{S}_e \rightarrow \mathcal{S} \hookrightarrow \mathcal{M}.$$

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{M}'$ of the same signature and any $n \in \omega$, we denote by $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{M}'$ the fact that $\mathbb{HF}(\mathfrak{M}) \preccurlyeq_n \mathbb{HF}(\mathfrak{M}')$. It is easy to verify that, for $n < 2$, $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{M}'$ if and only if $\mathfrak{M} \preccurlyeq_n \mathfrak{M}'$. For $n = 2$, $\mathfrak{M} \preccurlyeq_2^{\mathrm{HF}} \mathfrak{M}'$ if and only if $\mathfrak{M} \leqslant \mathfrak{M}'$ and for any computable sequence $\{\varphi_{mn}(\bar{x}_m, \bar{y}_n, \bar{z}) | m, n \in \omega\}$ of quantifier-free formulas of signature $\sigma_{\mathfrak{M}}$ and any $\bar{m} \in M^{<\omega}$,

$$\mathfrak{M}' \models \bigvee_{m\in\omega} \exists \bar{x}_m \bigwedge_{n\in\omega} \forall \bar{y}_n \varphi_{mn}(\bar{x}_m, \bar{y}_n, \bar{z})$$

implies that

$$\mathfrak{M} \models \bigvee_{m\in\omega} \exists \bar{x}_m \bigwedge_{n\in\omega} \forall \bar{y}_n \varphi_{mn}(\bar{x}_m, \bar{y}_n, \bar{z}).$$

Definition

A structure $\mathfrak{M}$ is called locally constructivizable of level n ($1 < n \leqslant \omega$), if, for any tuple $\bar{m} \in M^{<\omega}$, there exist a constructivizable structure $\mathfrak{N}$ and a tuple $\bar{n} \in N^{<\omega}$ such that $(\mathfrak{M}, \bar{m}) \equiv_n^{\mathrm{HF}} (\mathfrak{N}, \bar{n})$. Structure $\mathfrak{M}$ is called uniformly locally constructivizable of level n ($1 < n \leqslant \omega$) if there exists a constructivizable structure $\mathfrak{N}$ such that $\mathfrak{M} \preccurlyeq_n^{\mathrm{HF}} \mathfrak{N}$.

Example: $(\omega_1^{CK}, \leqslant) \preccurlyeq^{\mathrm{HF}} (\omega_1^{CK}(1+\eta), \leqslant)$.

Proposition

If $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ and $\mathfrak{N}$ is (uniformly) locally constructivizable of level n ($1 < n \leqslant \omega$) then $\mathfrak{M}$ is also (uniformly) locally constructivizable of level n.

Proposition

Let a structure $\mathfrak{N}$ *be such that* $\mathfrak{N}$ *is locally constructivizable of level* 1*, and let* $\mathfrak{M} \leqslant_{\Sigma} \mathfrak{N}$*. Then there exists a partial constructivizable structure* $\mathfrak{M}'$ *such that* $\mathfrak{M} \preccurlyeq_{\exists} \mathfrak{M}'$*.*

For structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_{\exists} \mathfrak{N}$ the fact that for any $\bar{m} \in M^{<\omega}$ there is $\bar{n} \in N^{<\omega}$ such that $\mathrm{Th}_{\exists}(\mathfrak{M}, \bar{m}) \leqslant_e \mathrm{Th}_{\exists}(\mathfrak{N}, \bar{n})$.

Proposition

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{N}$, $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$ implies that $\mathfrak{N} \leqslant_{\exists} \mathfrak{M}$. In particular, if $\mathfrak{M}$ is locally constructivizable, then any $\mathfrak{N} \in \mathcal{K}_w(\mathfrak{M})$ is also locally constructivizable.

Theorem

If a structure $\mathfrak{M}$ *is locally constructivizable of level* $n > 1$ *and not constructivizable, then there is a structure* $\mathfrak{M}_0 \in \mathcal{K}(\mathfrak{M})$ *which is locally constructivizable of level* 1 *sharply. In particular,* $\mathcal{K}_\Sigma(\mathfrak{M}) \subsetneq \mathcal{K}(\mathfrak{M})$.

Theorem

There exist a structure $\mathfrak{M}$ and a relation $P \subseteq M$ such that $\underline{(\mathfrak{M}, P)} \equiv \underline{\mathfrak{M}}$, but $(\mathfrak{M}, P)$ is not Σ-definable in $\mathbb{HF}(\mathfrak{M})$.

Theorem (Ash, Knight, Manasse, Slaman; Chisholm)

Let $\mathfrak{M}$ be a countable structure and let $P \subseteq M^n$. Then the following are equivalent:

1) *P is Σ-definable in $\mathbb{HF}(\mathfrak{M})$;*
2) *for any $\mathcal{C} \in \underline{(\mathfrak{M}, P)}$, $\quad R^{\mathcal{C}}$ is $\mathcal{C} \restriction \sigma_{\mathfrak{M}}$-c.e.*

Theorem

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2) *for any $\mathcal{C} \in \underline{(\mathfrak{M}, P)}$, $\quad R^{\mathcal{C}}$ is $\mathcal{C} \restriction \sigma_{\mathfrak{M}}$-c.e.*

Theorem

For any countable structures $\mathfrak{M}$ *and* $\mathfrak{N}$ *and any* $R \subseteq \mathbb{HF}(\mathfrak{N})$, *the following are equivalent:*

1) *for any presentation* $\mathcal{C}$ *of* $\mathfrak{M}$ *in* $\mathbb{HF}(\mathfrak{N})$, $R \leqslant_{e\Sigma} \mathcal{C}$;
2) R *is* Σ*-definable in* $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.

Definition

Let $\mathfrak{M}$ *and* $\mathfrak{N}$ *be a countable structures.* $\mathfrak{M}$ *is said to have a degree (e-degree) over* $\mathfrak{N}$ *if there exists a least degree in the class of* $T\Sigma$*-degrees (*$e\Sigma$*-degrees) of all possible presentations of* $\mathfrak{M}$ *in* $\mathbb{HF}(\mathfrak{N})$.

Theorem

Let $\mathfrak{M}$ and $\mathfrak{N}$ be a countable structure. The following are equivalent:

1) *$\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}$;*
2) *some presentation $\mathcal{C} \subseteq HF(N)$ of $\mathfrak{M}$ is Δ-definable (Σ-definable) in $\mathbb{HF}(\mathfrak{M}, \mathfrak{N})$.*

Corollary

For a countable $\mathfrak{M}$, $\mathfrak{M}$ has a degree (e-degree) iff, for some $\mathcal{C} \in \underline{\mathfrak{M}}$, $\mathcal{C}$ is Δ-definable (Σ-definable) in $\mathbb{HF}(\mathfrak{M})$.

Proposition

If $\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}$ and $\mathfrak{N}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N}')$ then $\mathfrak{M}$ has a degree (e-degree) over $\mathfrak{N}'$.

Proposition

For any countable structure $\mathfrak{A}$ there exists a structure $\mathfrak{M}$ which has a degree but is not Σ-definable in $\mathbb{HF}(\mathfrak{A})$.

Theorem

If $\mathfrak{M}$ has a degree then

$$\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}(\mathfrak{M}) = \mathcal{K}_w(\mathfrak{M}).$$

Theorem

If $\mathfrak{M}$ has an e-degree then

$$\mathcal{K}_\Sigma(\mathfrak{M}) = \mathcal{K}_e(\mathfrak{M}) = \mathcal{K}_{ew}(\mathfrak{M}).$$