Structured Finite Model Theory

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Part I

FINITE MODEL THEORY?

Cornerstone Result of Model Theory

Theorem (Compactness Theorem)

Let T be a set of first-order sentences. The following are equivalent:

- *T has a model,*
- *every finite subset $T_0 \subseteq T$ has a model.*

When restricted to finite structures, it fails

Let $T = \{\varphi_1, \varphi_2, \ldots\}$ where

$$\varphi_n = (\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i \neq j} x_i \neq x_j \right)$$

- every finite $T_0 \subseteq T$ has a finite model,
- T itself does not have a finite model.

A finite model theory?

Fact:

- The study of finite structures is important for computer science and discrete mathematics.

Unfortunately:

- Failure of the Compactness Theorem.
- No Completeness Theorem: the set of first-order sentences that are valid on finite structures is not r.e. (Trahtenbrot's Theorem).
- Most classical results fail as well, or are just meaningless.

Example 1: Łoś-Tarski Theorem

Definition

A sentence φ is preserved under extensions if

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Theorem (Łoś-Tarski Theorem)

Let φ be a first-order sentence. The following are equivalent:

- *φ is preserved under extensions,*
- *φ is equivalent to an existential sentence.*

Counterexample to Łoś-Tarski on finite structures

[Tait 1952, Gurevich 1984].

Let ψ be the sentence over $\sigma = \{R^{(2)}, S^{(2)}, T^{(1)}, \max, \min\}$ saying:

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- R is a linear order with endpoints max and min,
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Fact

ψ is preserved under substructures on finite structures.
$\neg\psi$ is preserved under extensions on finite structures.

Proof: Every proper $N \subset M$ of a finite $M \models \varphi$ has non-total S.

Counterexample to Łoś-Tarski on finite structures

Fact

$\neg\psi$ is not equivalent to an existential sentence on finite structures.

Proof: It has infinitely many minimal models: the finite linear orders with total successor and empty T.

Example 2: Order Invariance

Definition

$\varphi(<)$ is order-invariant if for every M and every two linear orders $<_1$ and $<_2$ on M we have

$$(M, <_1) \models \varphi \quad \text{iff} \quad (M, <_2) \models \varphi$$

Notation: $M \models \varphi$ iff $(M, <) \models \varphi$ for some $<$.

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Theorem (consequence to Craig's Interpolation)

Order-invariant FO = FO

Counterexample to order invariance on finite structures

[Gurevich 1984]

Fact

The finite Boolean algebras with an even number of atoms are not definable in FO on finite structures.

Proof: An easy Enhrenfeucht-Fraïssé argument.

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Proof: Next slide.

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- for every atom $a \subset \overline{c}$, there exists an atom $a^- \subset c$ such that $a^- < a$ and there are no atoms in between.

Other failures

Some other 'celebrated' failures:

- Interpolation Theorem
- Lyndon's Positivity Theorem [Ajtai-Gurevich 1984]
- Homomorphism preservation? [Now solved! Rossman 2005]
- ...

Finite Model Theory since the 1970's

Descriptive Complexity and Expressive Power [1970's-90's]:
Fagin's Theorem, Immerman-Vardi Theorem,
monadic-$\Sigma^1_1 \neq$ monadic-Π^1_1, ...

Assymptotic Probabilities [1970's-90's]:
0-1 laws, convergence laws, analysis of the random graph $G(n, n^{-\alpha})$, ...

Classical Results on Tame Classes [2000's-]:
Homomorphism preservation on excluded minors, Łoś-Tarski Theorem on treewidth, order-invariance on trees, ...

Algorithmic Metatheorems [1990's-]:
Courcelle's Theorem, model-checking on bounded degree and excluded minors, approximation algorithms, ...

Methods in Finite Model Theory

Each of the four areas has its own methods.
But there is one that permeates all four:

Locality of first-order logic.

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- vertices: elements of M,
- edges: between any two elements that appear together in some tuple of M.

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The r-neighborhood of a in M is

$$N_r^M(a) = \{b : d_G(a, b) \leq r\},$$

where $G = \mathcal{G}(M)$ and $d_G(a, b)$ denotes distance (length of the shortest path).

Locality

A first-order formula $\varphi(x)$ is called *r-local* if for every M and $a \in M$ we have

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A basic local sentence is one of the form:

$$(\exists x_1) \ldots (\exists x_m) \left(\bigwedge_{i \neq j} d_G(x_i, x_j) > 2r \wedge \bigwedge_i \psi(x_i) \right)$$

where ψ is r-local (typically, by relativizing to $N_r(x_i)$).

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Theorem (Gaifman's Locality)

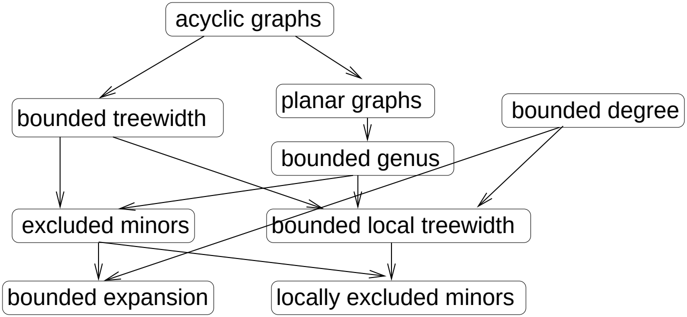
Every first-order sentence is equivalent to a Boolean combination of basic local sentences.

Part II

CLASSICAL RESULTS ON TAME CLASSES

Tame classes of structures

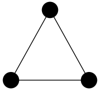
We study classes of finite structures whose Gaifman graphs belong to classes of interest in graph theory:



Treewidth

Definition

- K_{k+1} is a k-tree,
- if G is a k-tree, then adding a vertex connected to all vertices of a K_k-subgraph of G is a k-tree.



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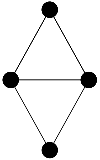

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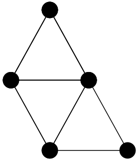

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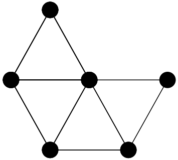

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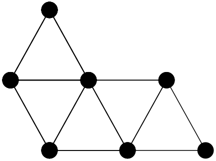

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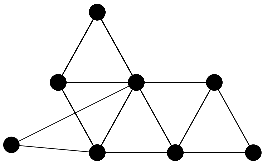

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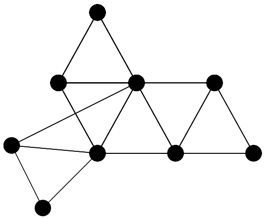

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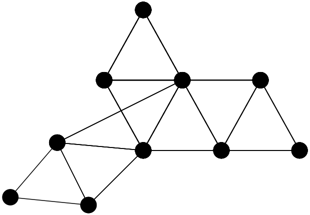

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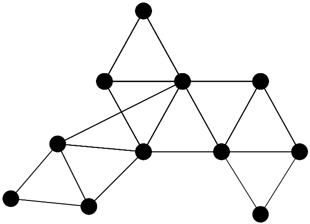

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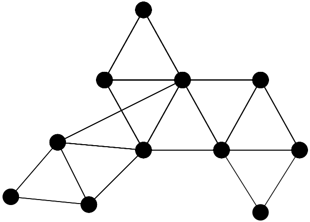

Definition (Robertson and Seymour)

The treewidth of a graph G, denoted by $tw(G)$, is the smallest k such that G is the subgraph of a k-tree.

Notation for classes

$\mathcal{T}_k$: class of all finite structures M with $tw(\mathcal{G}(M)) \leq k$.

$\mathcal{D}_k$: class of all finite structures M with $\Delta(\mathcal{G}(M)) \leq k$.

$\mathcal{P}$: class of all finite structures M with planar $\mathcal{G}(M)$.

$\mathcal{F}_k$: class of all finite structures M with $K_k \not\preceq \mathcal{G}(M)$.

Łoś-Tarski Theorem on bounded treewidth

Theorem (AA.-Dawar-Grohe 2005)

Let φ be a first-order sentence and k an integer. The following are equivalent:

1. *φ is preserved under extensions on $\mathcal{T}_k$*
2. *φ is equivalent to an existential sentence on $\mathcal{T}_k$.*

Proof Ingredients and Architecture

Suppose φ is preserved under extensions on $\mathcal{T}_k$.

We want to put a bound B on the size of the minimal models of φ as a function of $|\varphi|$.

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If we succeed, then

$$\varphi \equiv \bigvee_{\substack{M \models \varphi \\ |M| \leq B}} (\exists x_1) \cdots (\exists x_{|M|})(\text{diagram}(M)).$$

Proof Ingredients and Architecture

Combinatorial part:

Lemma

For every d and m, every sufficiently large graph $G = (V, E)$ of treewidth at most k contains vertices $a_1, \ldots, a_k \in V$ such that $G \setminus \{a_1, \ldots, a_k\}$ contains m points $b_1, \ldots, b_m$ with

$$d_G(b_i, b_j) > d$$

for every $i \neq j$.

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Proof requires the Sunflower Lemma of Erdös and Rado.

Proof Ingredients and Architecture

Apply Gaifman's locality:
Apply Gaifman's locality and write φ as a Boolean combination

$$\bigvee_{i=1}^{q} \left(\bigwedge_{j \in J_i} \tau_j \wedge \bigwedge_{j \in K_i} \neg \tau_j \right)$$

where each τ_j is a basic local sentence.

Proof Ingredients and Architecture

Model construction part:
Huge simplifying assumption: Assume φ is just a basic local sentence or its negation:

$$(\exists x_1)\dots(\exists x_m)\left(\bigwedge_{i\neq j} d_G(x_i,x_j) > 2r \wedge \bigwedge_i \psi^{\leq r}(x_i)\right).$$

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Contradiction.

General case requires building a chain of submodels.

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We build a chain of proper submodels of M:

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where M_0 is the 'exceptional neighborhoods of M' (which is small).

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By closure under extensions of φ, if M_t is not yet a model of φ, then it must be distinguished from $M + M_t$ by some

$$\left(\bigwedge_{j \in J_t} \tau_j \wedge \bigwedge_{j \in K_t} \neg \tau_j \right).$$

We build M_{t+1} out of the witnesses as follows.

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If the construction exhausts all disjuncts of φ, then

$$M_{last} + M \not\models \varphi$$

A contradiction.

Preservation under extensions on other classes

Same methods apply to other classes of structures:

Theorem (AA.-Dawar-Grohe 2005)

The preservation-under-extensions property holds for:

- *classes* $\mathcal{K} \subseteq \mathcal{D}_k$ *closed under* $\subseteq$ *and* $+$,
- *classes* $\mathcal{K} \subseteq \mathcal{T}_1$ *closed under* $\subseteq$ *and* $+$,
- *classes* $\mathcal{T}_k$ *for every fixed* k.

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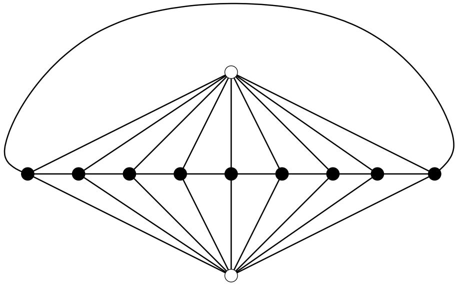
Question:

What about planar graphs?

Counterexample for planar graphs

ψ is the sentence:

> *there are at least two different white points such that either some point is not connected to both, or every black point has exactly two black neighbors.*



Other preservation theorems

Homomorphisms vs existential-positive sentences.

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Note 1: Second includes bounded treewidth and planar graphs.

Note 2: For $\mathcal{F}_k$, the hard part is the combinatorial part.
Uses finite Ramsey theory.

Note 3: Also uses Gaifman's locality.

Order invariance on restricted classes

Recall: Order-invariant FO is more powerful than FO on finite structures.

Upper bound: Order-invariant FO $\subseteq \Sigma_1^1 \cap \Pi_1^1$.

Theorem (Benedikt-Segoufin 2006)

The following hold:

- *Order-invariant FO $=$ FO on $\mathcal{T}_1$*
- *Order-invariant FO $\subseteq$ MSO on $\mathcal{T}_k$*
- *Order-invariant FO $\subseteq$ MSO on $\mathcal{D}_k$.*

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Open: Are inclusions proper in the last two cases?

Proof Ingredients

A word structure is a finite colored linear order. Let $\mathcal{W}$ be the class of word structures (over $\{0, 1\}$ say).

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Theorem (McNaughton-Papert)

Let $L \subseteq \mathcal{W}$ be a class of word structures (a language). The following are equivalent:

- *L is first-order definable on $\mathcal{W}$*
- *there exists p such that for every $u, v, w \in \mathcal{W}$ we have*

$$uv^p w \in L \iff uv^{p+1} w \in L$$

Proof Ingredients

First ingredient: An analogue of the McNaugthon-Papert theorem for trees [Benedikt and Segoufin 2005]

Second ingredient: Locality theorem for Order-invariant FO:

Theorem (Grohe-Schwentick 2000)

Let $\mathcal{K}$ be a class of finite structures and let $\varphi(x_1, \ldots, x_k)$ be a first-order formula that is order-invariant on $\mathcal{K}$. There exists an integer r such that, for every $M \in \mathcal{K}$ and $\mathbf{a}, \mathbf{b} \in M^k$, if

$$N_r^M(\mathbf{a}) \cong N_r^M(\mathbf{b})$$

then for every linear order $<$ on M,

$$(M, <) \models \varphi(\mathbf{a}) \leftrightarrow \varphi(\mathbf{b}).$$

Part III

ALGORITHMIC META-THEOREMS

Combinatorial Optimization Problems

MAX INDEPENDENT SET:

Given a graph $G = (V, E)$, find the largest independent set of G (largest set of pairwise non-adjacent points).

From the logic point of view, this problem asks for the largest set $X \subseteq V$ such that

$$(G, X) \models (\forall x)(\forall y)(X(x) \land X(y) \to \neg E(x, y))$$

General framework

MAX: For a fixed FO sentence $\varphi(X)$ that is negative in X.

Given a finite structure M, find the largest set $X \subseteq M$ such that $M \models \varphi(X)$.

MIN: For a fixed FO sentence $\varphi(X)$ that is positive in X.

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Let $C \geq 1$. For a maximization problem, we say that an algorithm is a C-approximation algorithm if it returns a solution A such that

$$|A| \leq OPT \leq C \cdot |A|.$$

Hardness and Easiness to Approximate

The MAX INDEPENDENT SET problem is a hard optimization problem:

Theorem (consequence to the PCP Theorem 1990's)

For every constant $C \geq 1$, there is no polynomial-time C-approximation algorithm for MAX INDEPENDENT SET, unless $P = NP$.

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Note: On planar graphs, MAX INDEPENDENT SET, MIN VERTEX COVER, ... have polynomial-time C-approximation algorithms for every $C > 1$.

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Question:

Is this is a general phenomenon?

Algorithm meta-theorem for optimization problems

Recall: $\mathcal{F}_k$ is the class of structures M with $K_k \not\preceq \mathcal{G}(M)$.

Theorem (Dawar-Grohe-Kreutzer-Schweikardt 2006)

For every FO-sentence $\varphi(X)$ that is positive (resp. negative) in X, every $k \geq 2$, and every $C > 1$, there exists a polynomial-time C-approximation algorithm for MAX $\varphi(X)$ (resp. MIN $\varphi(X)$) when the inputs are restricted to $\mathcal{F}_k$.

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Examples:

- MAX INDEPENDENT SET on graphs of bounded genus
- MIN VERTEX COVER on planar graphs
- MIN DOMINATING SET on bounded treewidth graphs
- ...

Proof Ingredients

Proof has two main parts:

- A new locality theorem for monotone formulas
- An adaptation of Baker's layer decomposition algorithmic technique

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Note: The proof of this locality result is not an modification of Gaifman's original theorem.

Surprisingly, the proof required the ideas that were developped for the Łoś-Tarski Theorem restricted to structures of bounded degree!

Other Algorithmic Meta-Theorems

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Theorem (Courcelle 1980's)

Every MSO-definable property is decidable in linear time when the inputs are restricted to $\mathcal{T}_k$.

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Examples:

- 3-COLORABILITY
- BOOLEAN SATISFIABILITY
- ...

Proof does not use locality.

Two alternative proofs: (1) tree-automata, (2) Feferman-Vaught composition techniques.

Part IV

CONCLUDING REMARKS

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The class of all finite structures is not well-behaved. But tame subclasses are.

From the point of view of applications to computer science and discrete mathematics, this is precisely what one is expected to do.

- Structures as modelling databases (arbitrary shape?)
- Structures as modelling program traces (arbitrary shape?)
- Structures of interest for combinatorics (trees, topological embeddings, ...).

Concluding remarks

A few open problems:

- Lyndon's positivity theorem on tame classes?
- Order invariance on $\mathcal{T}_k$? Further classes?
- Algorithmic meta-theorems for larger classes?
- Limits to algorithmic meta-theorems?
- More locality theorems? For structures with functions?
- Finite model theory of well-behaved finite algebras?