

Structured Finite Model Theory

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Part I

FINITE MODEL THEORY?

Cornerstone Result of Model Theory

Theorem (Compactness Theorem)

Let T be a set of first-order sentences. The following are equivalent:

- *T has a model,*
- *every finite subset $T_0 \subseteq T$ has a model.*

When restricted to finite structures, it fails

Let $T = \{\varphi_1, \varphi_2, \dots\}$ where

$$\varphi_n = (\exists x_1) \cdots (\exists x_n) \left(\bigwedge_{i \neq j} x_i \neq x_j \right)$$

- every finite $T_0 \subseteq T$ has a finite model,
- T itself does not have a finite model.

A finite model theory?

Fact:

- The study of finite structures is important for computer science and discrete mathematics.

Unfortunately:

- Failure of the Compactness Theorem.
- No Completeness Theorem: the set of first-order sentences that are valid on finite structures is not r.e. (Trahtenbrot's Theorem).
- Most classical results fail as well, or are just meaningless.

Example 1: Łoś-Tarski Theorem

Definition

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Theorem (Łoś-Tarski Theorem)

Let φ be a first-order sentence. The following are equivalent:

- *φ is preserved under extensions,*
- *φ is equivalent to an existential sentence.*

Counterexample to Łoś-Tarski on finite structures

[Tait 1952, Gurevich 1984].

Let ψ be the sentence over $\sigma = \{R^{(2)}, S^{(2)}, T^{(1)}, \max, \min\}$ saying:

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- if S is total, then T is non-empty.

Counterexample to Łoś-Tarski on finite structures

ψ is the sentence:

- R is a linear order with endpoints max and min,
- S is a partial successor relation compatible with R ,
- if S is total, then T is non-empty.

Fact

ψ is preserved under substructures *on finite structures*.

$\neg\psi$ is preserved under extensions *on finite structures*.

Proof: Every proper $N \subset M$ of a **finite** $M \models \varphi$ has non-total S .

Counterexample to Łoś-Tarski on finite structures

Fact

$\neg\psi$ is not equivalent to an existential sentence *on finite structures*.

Proof: It has infinitely many minimal models: the finite linear orders with total successor and empty T .

Example 2: Order Invariance

Definition

$\varphi(<)$ is **order-invariant** if for every M and every two linear orders $<_1$ and $<_2$ on M we have

$$(M, <_1) \models \varphi \text{ iff } (M, <_2) \models \varphi$$

Notation: $M \models \varphi$ iff $(M, <) \models \varphi$ for some $<$.

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Theorem (consequence to Craig's Interpolation)

Order-invariant FO = FO

Counterexample to order invariance on finite structures

[Gurevich 1984]

Fact

*The finite Boolean algebras with an even number of atoms are not definable in FO **on finite structures**.*

Proof: An easy Ehrenfeucht-Fraïssé argument.

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Proof: Next slide.

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- for every atom $a \subset \bar{c}$, there exists an atom $a^- \subset c$ such that $a^- < a$ and there are no atoms in between.

Other failures

Some other 'celebrated' failures:

- Interpolation Theorem
- Lyndon's Positivity Theorem [Ajtai-Gurevich 1984]
- Homomorphism preservation? [Now solved! Rossman 2005]
- ...

Finite Model Theory since the 1970's

Descriptive Complexity and Expressive Power [1970's-90's]:

Fagin's Theorem, Immerman-Vardi Theorem,
monadic- $\Sigma_1^1 \neq$ monadic- Π_1^1 , ...

Asymptotic Probabilities [1970's-90's]:

0-1 laws, convergence laws, analysis of the random graph
 $G(n, n^{-\alpha})$, ...

Classical Results on Tame Classes [2000's-]:

Homomorphism preservation on excluded minors, Łoś-Tarski
Theorem on treewidth, order-invariance on trees, ...

Algorithmic Metatheorems [1990's-]:

Courcelle's Theorem, model-checking on bounded degree and
excluded minors, approximation algorithms, ...

Methods in Finite Model Theory

Each of the four areas has its own methods.
But there is one that permeates all four:

Locality of first-order logic.

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The **r -neighborhood** of a in M is

$$N_r^M(a) = \{b : d_G(a, b) \leq r\},$$

where $G = \mathcal{G}(M)$ and $d_G(a, b)$ denotes distance (length of the shortest path).

Locality

A first-order formula $\varphi(x)$ is called ***r*-local** if for every M and $a \in M$ we have

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A **basic local sentence** is one of the form:

$$(\exists x_1) \dots (\exists x_m) \left(\bigwedge_{i \neq j} d_G(x_i, x_j) > 2r \wedge \bigwedge_i \psi(x_i) \right)$$

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Theorem (Gaifman's Locality)

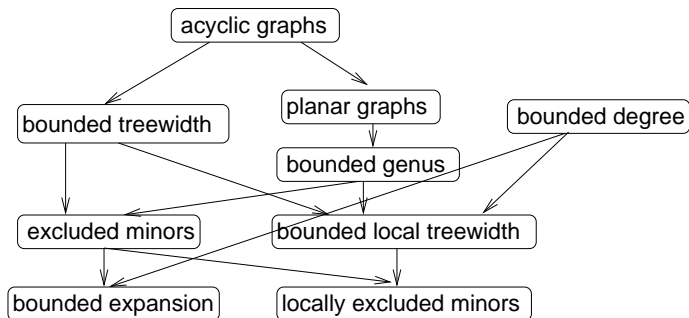
Every first-order sentence is equivalent to a Boolean combination of basic local sentences.

Part II

CLASSICAL RESULTS ON TAME CLASSES

Tame classes of structures

We study classes of finite structures whose Gaifman graphs belong to classes of interest in graph theory:



Treewidth

Definition

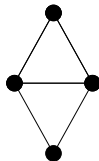
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- if G is a *k-tree*, then adding a vertex connected to all vertices of a K_k -subgraph of G is a *k-tree*.



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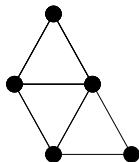
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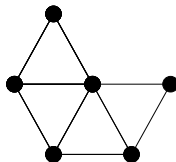
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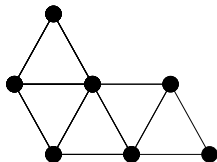
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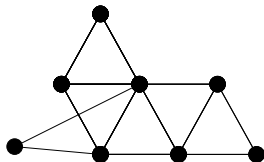
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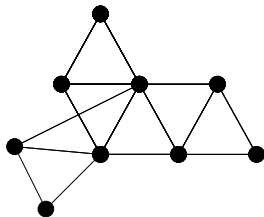
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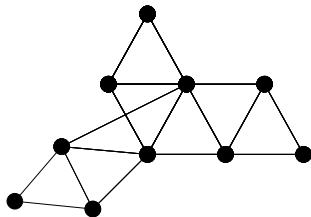
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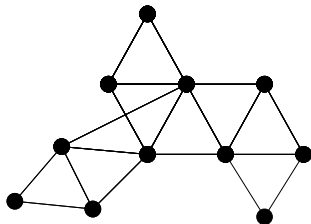
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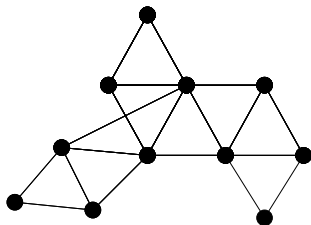
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Definition (Robertson and Seymour)

The **treewidth** of a graph G , denoted by $tw(G)$, is the smallest k such that G is the subgraph of a k -tree.

Notation for classes

\mathcal{T}_k : class of all finite structures M with $tw(\mathcal{G}(M)) \leq k$.

\mathcal{D}_k : class of all finite structures M with $\Delta(\mathcal{G}(M)) \leq k$.

\mathcal{P} : class of all finite structures M with planar $\mathcal{G}(M)$.

\mathcal{F}_k : class of all finite structures M with $K_k \not\prec \mathcal{G}(M)$.

Łoś-Tarski Theorem on bounded treewidth

Theorem (AA.-Dawar-Grohe 2005)

Let φ be a first-order sentence and k an integer. The following are equivalent:

- 1. φ is preserved under extensions on \mathcal{T}_k*
- 2. φ is equivalent to an existential sentence on \mathcal{T}_k .*

Proof Ingredients and Architecture

Suppose φ is preserved under extensions on \mathcal{T}_k .

We want to put a bound B on the size of the minimal models of φ as a function of $|\varphi|$.

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If we succeed, then

$$\varphi \equiv \bigvee_{\substack{M \models \varphi \\ |M| \leq B}} (\exists x_1) \cdots (\exists x_{|M|}) (\text{diagram}(M)).$$

Proof Ingredients and Architecture

Combinatorial part:

Lemma

For every d and m , every sufficiently large graph $G = (V, E)$ of treewidth at most k contains vertices $a_1, \dots, a_k \in V$ such that $G \setminus \{a_1, \dots, a_k\}$ contains m points b_1, \dots, b_m with

$$d_G(b_i, b_j) > d$$

for every $i \neq j$.

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Proof requires the Sunflower Lemma of Erdős and Rado.

Proof Ingredients and Architecture

Apply Gaifman's locality:

Apply Gaifman's locality and write φ as a Boolean combination

$$\bigvee_{i=1}^q \left(\bigwedge_{j \in J_i} \tau_j \wedge \bigwedge_{j \in K_i} \neg \tau_j \right)$$

where each τ_j is a basic local sentence.

Proof Ingredients and Architecture

Model construction part:

Huge simplifying assumption: Assume φ is just a basic local sentence or its negation:

$$(\exists x_1) \dots (\exists x_m) \left(\bigwedge_{i \neq j} d_G(x_i, x_j) > 2r \wedge \bigwedge_i \psi^{\leq r}(x_i) \right).$$

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Contradiction.

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General case requires building a **chain** of submodels.

Proof Ingredients and Architecture

We build a chain of proper submodels of M :

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where M_0 is the 'exceptional neighborhoods of M ' (which is small).

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By **closure under extensions** of φ , if M_t is not yet a model of φ , then it must be distinguished from $M + M_t$ by some

$$\left(\bigwedge_{j \in J_t} \tau_j \wedge \bigwedge_{j \in K_t} \neg \tau_j \right).$$

We build M_{t+1} out of the witnesses as follows.

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If the construction exhausts all disjuncts of φ , then

$$M_{last} + M \not\models \varphi$$

A contradiction.

Preservation under extensions on other classes

Same methods apply to other classes of structures:

Theorem (AA.-Dawar-Grohe 2005)

The preservation-under-extensions property holds for:

- *classes $\mathcal{K} \subseteq \mathcal{D}_k$ closed under \subseteq and $+$,*
- *classes $\mathcal{K} \subseteq \mathcal{T}_1$ closed under \subseteq and $+$,*
- *classes \mathcal{T}_k for every fixed k .*

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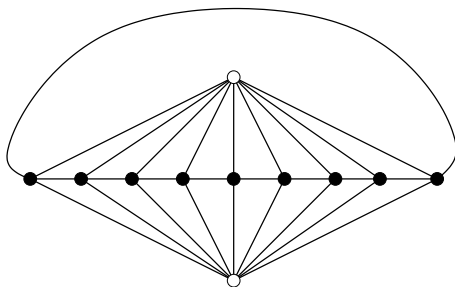
Question:

What about planar graphs?

Counterexample for planar graphs

ψ is the sentence:

there are at least two different white points such that either some point is not connected to both, or every black point has exactly two black neighbors.



Other preservation theorems

Homomorphisms vs existential-positive sentences.

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Note 1: Second includes bounded treewidth and planar graphs.

Note 2: For \mathcal{F}_k , the hard part is the combinatorial part.
Uses finite Ramsey theory.

Note 3: Also uses Gaifman's locality.

Order invariance on restricted classes

Recall: Order-invariant FO is more powerful than FO on finite structures.

Upper bound: Order-invariant FO $\subseteq \Sigma_1^1 \cap \Pi_1^1$.

Theorem (Benedikt-Segoufin 2006)

The following hold:

- *Order-invariant FO = FO on \mathcal{T}_1*
- *Order-invariant FO \subseteq MSO on \mathcal{T}_k*
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Open: Are inclusions proper in the last two cases?

Proof Ingredients

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Theorem (McNaughton-Papert)

Let $L \subseteq \mathcal{W}$ be a class of word structures (a language). The following are equivalent:

- L is first-order definable on \mathcal{W}
- there exists p such that for every $u, v, w \in \mathcal{W}$ we have

$$uv^p w \in L \iff uv^{p+1} w \in L$$

Proof Ingredients

First ingredient: An analogue of the McNaughton-Papert theorem for **trees** [Benedikt and Segoufin 2005]

Second ingredient: **Locality** theorem for Order-invariant FO:

Theorem (Grohe-Schwentick 2000)

Let \mathcal{K} be a class of finite structures and let $\varphi(x_1, \dots, x_k)$ be a first-order formula that is order-invariant on \mathcal{K} . There exists an integer r such that, for every $M \in \mathcal{K}$ and $\mathbf{a}, \mathbf{b} \in M^k$, if

$$N_r^M(\mathbf{a}) \cong N_r^M(\mathbf{b})$$

then for every linear order $<$ on M ,

$$(M, <) \models \varphi(\mathbf{a}) \leftrightarrow \varphi(\mathbf{b}).$$

Part III

ALGORITHMIC META-THEOREMS

Combinatorial Optimization Problems

MAX INDEPENDENT SET:

Given a graph $G = (V, E)$, find the largest independent set of G (largest set of pairwise non-adjacent points).

From the logic point of view, this problem asks for the largest set $X \subseteq V$ such that

$$(G, X) \models (\forall x)(\forall y)(X(x) \wedge X(y) \rightarrow \neg E(x, y))$$

General framework

MAX: For a fixed FO sentence $\varphi(X)$ that is **negative** in X .

Given a finite structure M , find the largest set $X \subseteq M$ such that $M \models \varphi(X)$.

MIN: For a fixed FO sentence $\varphi(X)$ that is **positive** in X .

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MIN: For a fixed FO sentence $\varphi(X)$ that is **positive** in X .

Given a finite structure M , find the smallest set $X \subseteq M$ such that $M \models \varphi(X)$.

Let $C \geq 1$. For a maximization problem, we say that an algorithm is a **C -approximation algorithm** if it returns a solution A such that

$$|A| \leq OPT \leq C \cdot |A|.$$

Hardness and Easiness to Approximate

The MAX INDEPENDENT SET problem is a hard optimization problem:

Theorem (consequence to the PCP Theorem 1990's)

For every constant $C \geq 1$, there is no polynomial-time C -approximation algorithm for MAX INDEPENDENT SET, unless $P = NP$.

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Question:

Is this is a general phenomenon?

Algorithm meta-theorem for optimization problems

Recall: \mathcal{F}_k is the class of structures M with $K_k \not\prec \mathcal{G}(M)$.

Theorem (Dawar-Grohe-Kreutzer-Schweikardt 2006)

For every FO-sentence $\varphi(X)$ that is positive (resp. negative) in X , every $k \geq 2$, and every $C > 1$, there exists a **polynomial-time C -approximation algorithm** for $\text{MAX } \varphi(X)$ (resp. $\text{MIN } \varphi(X)$) when the inputs are restricted to \mathcal{F}_k .

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Examples:

- MAX INDEPENDENT SET on graphs of bounded genus
- MIN VERTEX COVER on planar graphs
- MIN DOMINATING SET on bounded treewidth graphs
- ...

Proof Ingredients

Proof has two main parts:

- A new locality theorem for **monotone** formulas
- An adaptation of Baker's layer decomposition algorithmic technique

Monotone Locality Theorem

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Note: The proof of this locality result is **not** an modification of Gaifman's original theorem.

Surprisingly, the proof required the ideas that were developed for the Łoś-Tarski Theorem restricted to structures of bounded degree!

Other Algorithmic Meta-Theorems

The precursor of all algorithmic meta-theorems is:

Theorem (Courcelle 1980's)

*Every MSO-definable property is **decidable in linear time** when the inputs are restricted to \mathcal{T}_k .*

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The precursor of all algorithmic meta-theorems is:

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*Every MSO-definable property is **decidable in linear time** when the inputs are restricted to \mathcal{T}_k .*

Examples:

- 3-COLORABILITY
- BOOLEAN SATISFIABILITY
- ...

Proof does not use locality.

Two alternative proofs: (1) tree-automata, (2) Feferman-Vaught composition techniques.

Part IV

CONCLUDING REMARKS

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The class of **all** finite structures is not well-behaved. But **tame subclasses** are.

From the point of view of applications to computer science and discrete mathematics, this is precisely what one is expected to do.

- Structures as modelling databases (arbitrary shape?)
- Structures as modelling program traces (arbitrary shape?)
- Structures of interest for combinatorics (trees, topological embeddings, ...).

Concluding remarks

A few open problems:

- Lyndon's positivity theorem on tame classes?
- Order invariance on \mathcal{T}_k ? Further classes?
- Algorithmic meta-theorems for larger classes?
- Limits to algorithmic meta-theorems?
- More locality theorems? For structures with functions?
- Finite model theory of well-behaved finite algebras?