Completion of numberings

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Basic notions

Any surjective mapping $\alpha : N \mapsto \mathcal{A}$ is called a *numbering* of $\mathcal{A}$.

Numbering α is *reducible* to numbering β ($\alpha \leqslant \beta$) if for some computable function f and for all x $\alpha(x) = \beta(f(x))$.
If f can be chosen among computable permutations then α and β are called *computably isomorphic*.

Numberings α and β are *equivalent* ($\alpha \equiv \beta$) if $\alpha \leqslant \beta$ and $\beta \leqslant \alpha$.

Complete and precomplete numberings.

Fact. Not every partial computable function f is extendable to a (total) computable function g.

But what about extension for every f modulo some equivalence relation?

Complete and precomplete numberings.

Definition (A.I. Mal'tsev, 1960)

Numbering α of $\mathcal{A}$ is called *complete w.r.t. special object* $a \in \mathcal{A}$ if for every partial computable function $f(x)$ there exists total computable function $g(x)$ s.t.

$$\alpha g(x) = \begin{cases} \alpha f(x) & \text{if} \quad f(x)\downarrow, \\ a & \text{otherwise.} \end{cases}$$

Numbering α of $\mathcal{A}$ is called *precomplete* if for every partial computable function $f(x)$ there exists total computable function $g(x)$ s.t. for all $x \in \mathrm{dom}(f)$

$$\alpha g(x) = \alpha f(x).$$

Example

Standard numberings W and φ are complete w.r.t. $\emptyset$.

Example

Let A, B be c.e. sets, $A \subset B$, $\alpha : N \mapsto \{A, B\}$, and let $\alpha^{-1}(B)$ be creative set. Then α is complete w.r.t. A.

The most important theorems on precomplete numberings

Theorem (Yu.L. Ershov)

Let $\alpha : N \mapsto \mathcal{A}$ be any numbering. Then the following statements are equivalent

(1) α *is precomplete;*

(2) *there exists a computable function h such that for every e, $\varphi_{h(e)}$ is total and for all x,*

$$\varphi_e(x)\downarrow \Rightarrow \alpha(\varphi_{h(e)}(x)) = \alpha(\varphi_e(x));$$

(3) *(The Uniform Fixed Point Theorem) there exists computable function g such that for every e,*

$$\varphi_e(g(e))\downarrow \Rightarrow \alpha(g(e)) = \alpha(\varphi_e(g(e))).$$

The most important theorems on precomplete numberings

Theorem (A.I. Mal'tsev)

If two numberings are equivalent and one of them is precomplete then the second is also precomplete and they indeed are computably isomorphic.

Theorem (Yu.L. Ershov)

Degree of any precomplete numbering is not splittable.

Corollary (A. Lachlan)

$\mathbf{m}$-*degree of creative set is not splittable.*

Completion of numberings

Definition

Let α be a numbering of $\mathcal{A}$, and $a \in \mathcal{A}$. Let $U(x)$ be unary universal partial computable function, for instance, $U(< e, x >) = \varphi_e(x)$. Define

$$\alpha_a = \begin{cases} \alpha U(x) & \text{if} \quad U(x) \downarrow, \\ a & \text{otherwise.} \end{cases}$$

Numbering α_a is called *completion* of α w.r.t. a.

Fact. For every numbering α, numbering α_a is complete w.r.t. a.

Why was completion almost forgotten for a long time?

Because of the class of classical computable numberings is not closed under completion.

Generalized computable numberings

Definition

Numbering α of a family $\mathcal{A} \subseteq \Sigma^0_{n+1}$ is called Σ^0_{n+1}-*computable* if $x \in \alpha y$ is Σ^0_{n+1}-relation.

$\mathrm{Com}^0_{n+1}(\mathcal{A})$ stands for the set of Σ^0_{n+1}-computable numberings of $\mathcal{A}$.

Proposition. If $\mathcal{A} \subseteq \Sigma^0_{n+2}$ and $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ then $\alpha_A \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ for every set $A \in \mathcal{A}$.

Corollary

The mapping $\alpha \to \alpha_A$ induces an operator on $\mathcal{R}^0_{n+2}(\mathcal{A})$.

The same holds for the families of c.e. sets we choose A to be the least set under inclusion.

Properties of completion

Theorem (BGS, 2003)

Let α be any numbering of $\mathcal{A}$ and let a, b be any two elements of $\mathcal{A}$. Then

1. $\alpha \leqslant \alpha_a$;
2. $\alpha_a \equiv_{\mathbf{0}'} \alpha$;
3. $\alpha \equiv \alpha_a$ *iff* α *is complete w.r.t.* A;
4. *if* $\alpha \leqslant \beta$ *then* $\alpha_a \leqslant \beta_a$.
5. $\inf(\deg(\alpha_a), \deg(\alpha_b)) = \deg(\alpha)$.

Consequences

Numbering $\alpha \in \mathrm{Com}^0_{n+1}(\mathcal{A})$ is called *principal* if each numbering from $\mathrm{Com}^0_{n+1}(\mathcal{A})$ is reducible to α.

- Principal numbering of $\mathcal{A} \subseteq \Sigma^0_{n+2}$, if any, is complete w.r.t. every element of $\mathcal{A}$.
- For every $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ and every $A \in \mathcal{A}$, $\deg(\alpha_A)$ in Rogers semilattice $\mathcal{R}^0_{n+2}(\mathcal{A})$ is non-splittable. In particular, the greatest element of $\mathcal{R}^0_{n+2}(\mathcal{A})$, if any, is never splittable.
- Index set of the special object A relative to α_A is productive set.

Principal numberings of finite families

Theorem (BGS,2003)

Every finite family $\mathcal{A} \subseteq \Sigma^0_{n+1}$ has $\mathbf{0^{(n)}}$-principal numbering.

Theorem (BGS,2003)

Finite family $\mathcal{A} \subseteq \Sigma^0_{n+2}$ has principal numbering iff $\mathcal{A}$ contains the least element under inclusion.

Completion of minimal numberings

Friedberg and positive numberings are incomplete.

Σ^0_{n+2}-computable minimal numberings which are built by method of Badaev-Goncharov are also incomplete.

Theorem (Badaev and Sorbi, 2007)

No minimal numbering of any non-trivial set can be complete.

Intervals and segments

Theorem (BGS, 2007)

For every Friedberg numbering α, the interval $(deg(\alpha), deg(\alpha_a))$ consists of the degrees of incomplete numberings (w.r.t. any element of $\mathcal{A}$).

Theorem (BGS, 2007)

For some numberings α of some families $\mathcal{A}$, the segment $[deg(\alpha), deg(\alpha_a)]$ is isomorphic to the upper semilattice of c.e. **m***-degrees.*

Open questions

Question 1. Is it true that, for every numbering $\alpha \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ of non-trivial family $\mathcal{A}$, there exists a numbering $\beta \in \mathrm{Com}^0_{n+2}(\mathcal{A})$ s.t. $\alpha \leqslant \beta$ and β is complete w.r.t. every element from $\mathcal{A}$?

Question 2. Is it true that $((\alpha_a)_b)_a \equiv (\alpha_a)_b)$?

Question 3. In which cases finite families of Σ^{-1}_{n+2}-sets have principal numberings?

Conjecture. If $\alpha_a \equiv \beta_a$ for incomplete numberings α and β then the segments $[\alpha, \alpha_a]$ and $[\beta, \beta_a]$ are isomorphic upper semilattices.

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