Automatic structures, Part 1

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Plan: Tutorial 1
Basics

- History and motivation
- Basic definitions and examples
- Decidability and definability theorems

Plan: Tutorial 2
Proof methods

- Proving non-automaticity
- Automatic well founded partially ordered sets
- Automatic Boolean algebras
- Automatic Finitely generated groups

Plan: Tutorial 3
Complexity

- Rabin automatic structures
- Scott ranks of automatic structures
- The isomorphism problem
- Heights of automatic well founded relations
- Cantor-Bendixson ranks of trees

- Characterize structures by their isomorphism types.
- Characterize elementary equivalent structures.
- For a sentence ϕ and a structure $\mathcal{A}$, decide if $\mathcal{A} \models \phi$.
- Want to know if the theory of a given structure is decidable.
- Want to understand definable relations in the structure.

Notations

A structure $(A; R_0, R_1, \ldots, R_m)$ is denoted by $\mathcal{A}$.

- We always assume that structures are relational.
- We work with infinite structures of finite language.

Notations:

- Let Σ be a finite alphabet.
- Let Σ^ω be all infinite words over Σ.
- α, β, $\gamma, \ldots$ denote variables for infinite words.

Definition

A **Büchi automaton** $\mathcal{M}$ is (S, ι, Δ, F), where S is a set of **states**, $\iota \in S$ is the **initial state**, $\Delta \subset S \times \Sigma \times S$ is the **transition table**, and $F \subset S$ is the set of **accepting states**.

Büchi automata

Definition

A **run** of $\mathcal{M}$ on $\alpha = \sigma_1\sigma_2\ldots$ is a sequence of states

$$q_1, q_2, q_3, \ldots, q_i, q_{i+1}, \ldots$$

such that $q_1 = \iota$ and $(q_i, \sigma_i, q_{i+1}) \in \Delta$ for all $i \in \omega$. The run is **accepting** if the set $\{s \mid \exists^\omega i(q_i = s)\}$ contains a state from F.

Definition

The **language** accepted by the automaton $\mathcal{M}$, denoted $L(\mathcal{M})$, is the set of all infinite words accepted by $\mathcal{M}$.

Examples of Büchi recognizable languages over $\{0, 1\}$:

1. $\{\alpha \mid \alpha$ has finitely many 1s $\}$.
2. $\{\alpha \mid \alpha$ has infinitely many 1s and infinitely many 0s $\}$.

Fact

1. **Decidability of the emptiness problem**: *There is an algorithm that, given a Büchi automaton $\mathcal{M}$, decides if there is an infinite word that the automaton accepts.*
2. *If $\mathcal{M}$ accepts a word then $\mathcal{M}$ accepts an ultimately periodic word.*

Theorem (Büchi, 1960)

The class of all Büchi recognizable languages is closed under the operations of union, intersection, and complementation.

- Consider the structure (ω, S). Consider the MSO logic defined to be the extension of the *FO* logic with (monadic) variables for subsets over ω.
- On (ω, S) the MSO logic can express many interesting relations such as $X \subseteq Y$, Line(X), $x \leq y$, Finite(X), and Add(X, Y, Z).

History and motivation: Büchi's Theorem

Theorem (Büchi, 1960)

A relation $R \subseteq P(\omega)^n$ is definable in the MSO logic if and only if R is Büchi recognizable.

We thus have an understanding of definable relations on (ω, S). As a corollary we have the following famous theorem:

Theorem (Büchi, 1960)

The monadic second order theory of (ω, S), denoted by $S1S$, is decidable.

- Let $\Sigma^\star$ be the set all finite words over Σ.
- **Word automata** $\mathcal{M}$ are the same as Büchi automata.
- A **run** $q_1, q_2, \ldots, q_n$ of a word automaton on $v \in \Sigma^\star$ is **successful** if q_n is accepting.
- The language recognized by $\mathcal{M}$ is the set of all words accepted by $\mathcal{M}$. Call the language a **regular** language.

Word Automata

Examples:

1. $\{w101 \mid w$ has no sub-word $101\}$.
2. $\{w \mid w$ is a reverse binary representation of integers $\geq 0\}$.
3. W is regular if and only if $W\diamondsuit^{\omega}$ is Büchi recognizable.

Theorem

1. *The emptiness problem for word automata is decidable.*
2. *Regular languages are closed under Boolean operations.*

Definition

A **structure** $\mathcal{A} = (A; R_0, R_1, \ldots, F_0, F_1, \ldots)$ is **computable** if the domain A, relations R_i and functions F_j are all computable.

- Van Der Waerden (1930).
- Frölich and Shepherdson, later M. Rabin (1950s).
- A. Malcev (1960s).
- Yu. Ershov (USSR) and A. Nerode (USA) (1970s).

- Given a structure, is it computable?
- Describe computable structures from a given class.
- Given two isomorphic computable structures, are they computably isomorphic?
- What is the complexity of the isomorphism problem?
- What is the complexity of the model checking problem?

- In computable model theory we assume the most general model of computation.
- Nerode suggested to study structures with resource-bounded machines (late 1970).
- Khoussainov and Nerode refined the idea and started a systematic development of the theory of computable structures when the underlying computation models are finite state machines (1994).

Use of automata for studying structures

- Büchi(1950s). Decidability of $S1S$.
- Rabin (1960s). Decidability of $S2S$.
- Büchi (1960s). Automata and Presburger arithmetic.
- A. Cobham (1969) and A. Semenov (1977).
- B. Hodgson (1983). Automata decidable theories.
- D. Epstein, W. Thurston (1990s). Automatic groups.
- B. Khoussainov, A. Nerode (1994). Automatic structures.
- E. Gradel, A. Blumensath (2000). LICS paper.
- A. Blumensath (1999). Diploma thesis [Aachen].
- S. Rubin (2004). PhD thesis [Auckland].
- V. Barany(2007). PhD thesis [Aachen].
- J. Liu (ongoing). PhD thesis [Auckland].
- M. Minnes (ongoing). PhD thesis [Cornell]

Definitions of Automatic Structure

Definition

A structure $\mathcal{A} = (A; R_0, R_1, \ldots, R_m)$ is **word automatic** over Σ if its domain A and all relations R_0, R_1, ..., R_m are regular over Σ.

Definition

A structure $\mathcal{A} = (A; R_0, R_1, \ldots, R_m)$ is **Büchi automatic** over Σ if its domain A and all relations R_0, R_1, ..., R_m are all Büchi automata recognizable (over Σ).

Which automaticity we use will be clear from the context.

Automata for relations

- The **convolution of a tuple** $(\alpha_1, \cdots, \alpha_n) \in (\Sigma^\omega)^n$ is the infinite word $c(\alpha_1, \cdots, \alpha_n)$ whose k'th symbol is $(\alpha_1(k), \ldots, \alpha_n(k))$.
- The **convolution of a relation** $R \subset (\Sigma^\omega)^n$ is the language formed as the set of convolutions of all the tuples in R.
- An n–ary relation $R \subset (\Sigma^\omega)^n$ is **Büchi recognisable** if its convolution $c(R)$ is a Büchi recognizable language.

Convolution of relations over $\Sigma^\star$ is defined similarly.

1. $(1^{\star}; \leq, S)$, where S is the successor function.
2. $(1^{\star}; mod(1), mod(2), \ldots, mod(n))$, where n is fixed.
3. $(\{0,1\}^{\star}; \vee, \wedge, \neg)$, where the operations are Boolean.
4. $(\{0,1\}^{\star} \cdot 1; +_2, \leq)$, where $+_2$ is the binary addition.

1. The word structure:
$$(\{0,1\}^\star; \preceq; Left, Right, EqL).$$
2. The configuration spaces of Turing machines T:
$$(Conf(T), E).$$

Examples of Büchi automatic structures

1. Every word automatic structure is Büchi automatic.
2. The Boolean algebra $(P(\omega), \cup, \cap, \neg, 0, 1)$.
3. The real numbers under the binary addition.

Calculus of automatic relations:

The class of all automatic relations (on $\Sigma^\star$ or Σ^ω) is closed under the following:

- The **union**, **intersection**, and **complementaion** operations on relations of the same arity.
- The **cylindrification** operation takes a relation R of arity k and outputs

  $$c(R) = \{(a_1, \ldots, a_k, a) \mid (a_1, \ldots, a_k) \in R \text{ and } a \in A\},$$

  where A is a regular or Büchi recognizable language.

Calculus of automatic relations:

- The $\exists$ operation. For a relation R of arity k, $\exists x_i R = \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k) \mid (a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_k) \in R$ for some $a \in A\}$
- The $\forall$ operation. For a relation R of arity k, $\forall x_i R = \{(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k) \mid$ for all $a \in A$ we have $(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_k) \in R\}$.

In both cases A is a regular or Büchi recognizable language.

Calculus of automatic relations:

- The **instantiation** operation. Given a relation R of arity k and a word c, the operation produces the relation $\{(x_1, \ldots, x_{k-1}) \mid (x_1, \ldots, x_{k-1}, c) \in R\}$.

  In case of relations on words c can be any word. In case of Büchi recognizable relations c is an ultimately periodic word.
- The **rearrangement** operations.

Calculus of automatic relations:

- The **linkage** operation. This operation links two relations on specified coordinates.

  For example, for $R(x, y, z)$ and $S(a, b, c, d)$, we link R and S on (y, z)-coordinates of R and (a, b)-coordinates of S. So, $(e_1, e_2, e_3, e_4, e_5)$ is in the new relation iff $(e_1, e_2, e_3) \in R$ and $(e_2, e_3, e_4, e_5) \in S$.

Decidability Theorem 1

Theorem (Hodgson, 1984; Khoussainov-Nerode, 1994; Blumensath-Gradel, 2000)

There is an algorithm that given an (word or Büchi) automatic structure $\mathcal{A}$ and a first order formula $\Phi(x_1, \ldots, x_n)$ produces an automaton recognizing exactly those tuples $(a_1, \ldots, a_n)$ in the structure that make the formula true.

Decidability Theorem 1: Corollaries

Corollary

If $\mathcal{A}$ is automatic and $\mathcal{B} = (B; R_1^{n_1}, \ldots, R_m^{n_m})$ is FO definable in $\mathcal{A}$ by formulas $D(\bar{x})$, $\Phi_1(\bar{x}_1, \ldots, \bar{x}_{n_1})$, ..., $\Phi_m(\bar{x}_1, \ldots, \bar{x}_{n_m})$ then $\mathcal{B}$ is also automatic.

Corollary

The FO theory of any automatic structure is decidable.

Decidability Theorem 1: Corollaries

Corollary

Let $\mathcal{A}$ be a word automatic structure, and let $\Phi(\bar{x})$ be a FO formula. There exists a linear time algorithm that given a tuple $\bar{a}$ from the structure checks if the tuple satisfies $\Phi(\bar{x})$.

Corollary

The FO theory of the Presburger arithmetic is decidable.

Decidability Theorem 2

Let $(FO + \exists^{\infty} + \exists^{n,m})$ be the first order logic with $\exists^{\infty}$ (there are ω many) and $\exists^{n,m}$ (there are m many mod n) quantifiers.

Theorem (Khoussainov, Rubin, Stephan; 2003)

There is an algorithm that given a word automatic structure $\mathcal{A}$ and a $(FO + \exists^{\infty} + \exists^{n,m})$-definition of any relation R, produces an automaton that recognizes the relation. In particular, the $(FO + \exists^{\infty} + \exists^{n,m})$-theory of $\mathcal{A}$ is decidable.

- A. Blumensath noted the case $\exists^{\infty}$ first.
- D. Kuske and M. Lohrey extend this theorem to Büchi automatic structures(2006).

Decidability Theorem 2: Corollaries

Corollary

If $(T, \leq)$ is an automatic finitely branching tree then it has a regular infinite path.

This is an automatic version of König's lemma. Note that a computable version of this lemma fails dramatically.

Corollary

Let L be an automatic partially ordered set. The set of all pairs (x, y) such that the interval $[x, y]$ has an even number of elements is regular.

Löwenheim-Skolem Theorem for Büchi structures

Theorem (Khoussainov, Nies, 2006)

Let $\mathcal{A}$ be a Büchi automatic structure. Consider

$$A' = \{\alpha \in A \mid \alpha \text{ is ultimately periodic word } \}.$$

The structure $\mathcal{A}'$ is a computable elementary substructure of $\mathcal{A}$.

This is also noted independently by V. Barany and S. Rubin.

Definability Theorems

From Büchi's theorem we immediately obtain the following:

Corollary

A structure is Büchi automatic iff it is definable in the monadic second order logic of the successor structure (ω, S).

Theorem (Definability theorem; Blumensath & Gradel, 2000)

A structure $\mathcal{A}$ *is word automatic iff it is first order definable in the word structure* $(\{0,1\}^{\star}; \preceq; Left, Right, EqL)$.

Proof of the definability theorem

One direction is clear. For the other, observe definability of:

- $|p| \leq |x|$.
- The digit of x at position $|p|$ is 0.
- The digits of x_1 and x_2 at position $|p|$ are distinct.

Let $\mathcal{M} = (S, \iota, \Delta, F)$ be a word automaton recognizing L. Assume $S = \{1, \ldots, n\}$ with 1 being the initial state. The formula $\Phi(v)$ states:

Definition of $\Phi(v)$:

1. There are elements $x_1, \ldots, x_n$ all of length $|v| + 1$.
2. For $|p| \leq |v| + 1$, exactly one of $x_1, \ldots, x_n$ has digit 1 at $|p|$.
3. If x_i has 1 at $|p|$, σ is the symbol of v at $|p|$, and $\Delta(i, \sigma) = j$ then x_j has 1 at $|p| + 1$.
4. The first position of x_1 is 1.
5. The last position of one of x_js, where $j \in F$, is 1. □

Definability Theorems

We interpret the structure $(\{0,1\}^\star; \preceq; \mathit{Left}, \mathit{Right}, \mathit{EqL})$ in (ω, S):

For $v \in \{0,1\}^\star$ set $\mathit{Rep}(v) = \{i \mid v(i) = 1\} \cup \{|v| + 1\}$.

1. $\mathit{Rep}(v)$ is a finite set.
2. For each finite $X \neq \emptyset$ there is a v such that $\mathit{Rep}(v) = X$.
3. Left, Right, $\preceq$, and EqL are all definable. Hence:

Corollary

A structure is word automatic if and only if the structure is definable in the weak monadic second order logic in (ω, S).

Definition

A structure $\mathcal{A}$ is **(word, Büchi) automata presentable** if it is isomorphic to a (word, Büchi) automatic structure $\mathcal{B}$.

We abuse notation and identify automatic and automata presentable structures. Here are more examples:

1. Finitely generated Abelian groups.
2. The additive group Q_p, where p is a prime number.
3. The Boolean algebra of finite and co-finite subsets of ω.
4. The linear order of rational numbers $(Q, \leq)$.

Properties of automatic structures

- If $\mathcal{A}$ and $\mathcal{B}$ are automatic then so is $\mathcal{A} \times \mathcal{B}$.
- If $\mathcal{A}$ and $\mathcal{B}$ are automatic then so is the disjoint union $\mathcal{A} + \mathcal{B}$.
- For countable structure $\mathcal{A}$, $\mathcal{A}$ is word automatic if and only if $\mathcal{A}$ is Büchi automatic.

Factorizations of automatic structures

Fact

If $\mathcal{A}$ is word automatic and E is a regular congruence relation, then $\mathcal{A}/E$ is word automatic.

Question: Is $\mathcal{A}/E$ Büchi automatic structures if $\mathcal{A}$ is Büchi automatic and E is Büchi recognizable?

Example: Consider $(P(\omega), \cup, \cap, \neg)$. The set $\{(X, Y) \mid X =^\star Y\}$ is Büchi recognizable.

Is $P(\omega)/=^\star$ Büchi automatic?

Factorizations of automatic structures

Theorem (Barany, Kaiser, Rubin, 2007)

If $\mathcal{A}$ is countable and Büchi automatic and E is a Büchi recognizable congruence in $\mathcal{A}$ then $\mathcal{A}/E$ is also Büchi automatic. □