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**Multiplicative quantifiers
in fuzzy and substructural logics**

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Substructural logics (of Ono 2003) = logics of residuated lattices

This talk focuses on the following subclass:

Deductive fuzzy logics = Ono's substructural logics with

(i) exchange (commutative conjunction)

(ii) prelinearity ... $\models (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$

They include the usual systems of t-norm fuzzy logics:

Łukasiewicz logic, Gödel–Dummett logic, Hájek's BL, ...

Some definitions and results can be extended to broader classes of substructural logics

For simplicity, in this talk we assume

weakening and full propositional language $(\&, \rightarrow, \wedge, \vee, 0, 1)$

Recall:

Substructural logics have *two* naturally defined conjunctions and disjunctions:

\wedge ... weak / lattice / “additive” conjunction

$$\varphi \otimes \psi \rightarrow \chi \equiv \varphi \rightarrow (\psi \rightarrow \chi)$$

\otimes ... strong / group / “multiplicative” conjunction

$$\varphi \wedge \psi \rightarrow \chi \equiv (\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)$$

$$\varphi \otimes \psi = \text{both } \varphi \text{ and } \psi$$

$$\varphi \wedge \psi = \text{any of } \varphi \text{ and } \psi$$

Denote $\underbrace{\varphi \otimes \dots \otimes \varphi}_n$ by φ^n

First-order substructural logics:

Easy to define \forall, \exists as the lattice infima and suprema \wedge, \vee

Rasiowa: *An Algebraic Approach to Non-Classical Logics*, 1974

$$(\forall x)\varphi(x) \rightarrow \varphi(t) \quad \text{if } t \text{ free for } x \text{ in } \varphi(x)$$

$$\varphi(t) \rightarrow (\exists x)\varphi(x) \quad \text{"}$$

$$(\forall x)(\chi \rightarrow \varphi(x)) \rightarrow (\chi \rightarrow (\forall x)\varphi(x)) \quad \text{if } x \text{ not free in } \varphi(x)$$

$$(\forall x)(\varphi(x) \rightarrow \chi) \rightarrow ((\exists x)\varphi(x) \rightarrow \chi) \quad \text{"}$$

$$\varphi / (\forall x)\varphi$$

Subtlety:

In incomplete lattices, the required \wedge, \vee need not be defined

Logics of complete lattices need not be axiomatizable (BL, Ł)

\Rightarrow use *Rasiowa's interpretations* = Hájek's *safe structures*

= those in which all necessary \wedge, \vee exist

\wedge, \vee are the *weak* quantifiers:

$$\vdash (\forall x)\varphi(x) \rightarrow \varphi(a) \wedge \varphi(b) \wedge \dots$$

$$\nVdash (\forall x)\varphi(x) \rightarrow \varphi(a) \otimes \varphi(b) \otimes \dots$$

$\forall = \text{ANY}$ (rather than ALL):

$(\forall x)\varphi(x)$ implies *any* single instance of $\varphi(x)$,
but not *all* of them *at once* (ie, with \otimes)

Question: How should strong quantifiers be defined?

- Long-standing problem in substructural logics
- Without strong quantifiers,
substructural quantification theory is incomplete
- First-order substructural logics with only weak quantifiers
are viewed as a cheat by many

Requirements of *strong quantifiers*

(to be well-defined, well-behaved, and well-motivated)

- To be *universal*, a quantifier Π should satisfy:

If $\models \varphi(x)$, then $\models (\Pi x)\varphi(x)$

- To be *multiplicative*, Π should satisfy:

$\models (\Pi x)\varphi(x) \rightarrow \bigotimes_{t \in M} \varphi(t)$ for any multiset M of terms

- To be semantically well-defined, the truth value of $(\Pi x)\varphi(x)$ in a model M should be determined by the truth values of $\varphi(a)$ for all individuals $a \in M$ (*truth-functionality*):

$$\|(\Pi x)\varphi(x)\|_{M,v} = F_{\Pi}(\{\langle a, \|\varphi(a)\|_{M,v} \rangle \mid a \in M\})$$

- It is natural to assume *monotony*:

If $\|\varphi(a)\|_{M,v} \leq \|\psi(a)\|_{M,v}$ for all $a \in M$
then $\|(\Pi x)\varphi(x)\|_{M,v} \leq \|(\Pi x)\psi(x)\|_{M,v}$

On single-element universes, truth-functional quantifiers
reduce to unary propositional connectives

⇒ Strong quantifiers generate unary connectives $*$ such that

$$\models \varphi^* \rightarrow \varphi^n \text{ for all } n$$

$$\text{if } \|\varphi\| \leq \|\psi\| \text{ then } \|\varphi^*\| \leq \|\psi^*\|$$

$$\text{if } \models \varphi \text{ then } \models \varphi^*$$

We call them *exponentials* here

(cf. Girard's exponentials; better terminology?)

For a strong quantifier Π , define:

$$\varphi^{*\Pi} \equiv_{\text{df}} (\Pi x)\varphi \quad \text{if } x \text{ is not free in } \varphi$$

Vice versa, if $*$ is an exponential, then

$$(\Pi_* x)\varphi(x) \equiv_{\text{df}} [(\forall x)\varphi(x)]^* \quad \text{is a strong quantifier}$$

not $(\forall x)\varphi^*(x)$

Examples:

- Girard's exponentials (! in linear logic):

Introduced proof-theoretically

Essentially, just $!\varphi \rightarrow \varphi$ and $!\varphi \rightarrow !\varphi \otimes !\varphi$ required

Truth value: any \otimes -idempotent below φ

not necessarily the weakest one

- Globalization

$\Box x = 1$ iff $x = 1$, otherwise $\Box x = 0$

Adding \Box to a fuzzy logic need not yield a fuzzy logic

- Baaz Δ operator

The strongest exponential preserving fuzziness

Coincides with globalization in linear algebras

Too strong unless $\text{Crisp}(\varphi^*)$ is required

(notice: conditions of Girard's ! satisfied by \Box, Δ)

- Montagna's storage operator

(Journal of Logic and Computation, 2004)

φ^* = the largest \otimes -idempotent below φ

(in algebras where it exists)

However, exponentials need not be idempotent

\Rightarrow still unnecessarily strong,

unless repeatable usage is required of φ^* , too

$$\varphi^* \otimes \varphi^* = \varphi^*, \quad (\varphi^*)^* = \varphi^*$$

Question:

optimal (ie, the weakest) exponential (or strong quantifier)...?

The condition of optimality of $*$ is expressed by

the infinitary rule $\{\psi \rightarrow \varphi^n \mid n \in \omega\} \vdash \psi \rightarrow \varphi^*$

This defines the **optimal (weakest) exponential** φ^ω

(as far as we know, not studied in fuzzy logic as yet)

The corresponding multiplicative quantifier:

$$(\Omega x)\varphi(x) \equiv_{\text{df}} ((\forall x)\varphi(x))^\omega$$

In semantics: $\varphi^\omega =_{\text{df}} \inf_{n \in \omega} \varphi^n$ (in “ ω -safe” algebras)

Not every algebra can be extended with $^\omega$

(cf Chang’s MV-algebra: co-infinitesimals have no inf),

but if it can, then $^\omega$ is its weakest exponential

Example:

$\varphi^\omega = \varphi^n$ in n -contractive logics (ie, such that $\models \varphi^n \rightarrow \varphi^{n+1}$)

In general, Montagna's \star differs from ω

Counter-example by Montagna (2004)

If they exist,

φ^\star is the nearest \otimes -idempotent below φ

φ^ω is the supremum of the first Archimedean class below φ

Recall: ω is introduced by an infinitary rule

Question: Can it be axiomatized (or approximated) finitarily?

Consider an operator $\bar{\omega}$ with the following axioms and rules:

$$\vdash \varphi^{\bar{\omega}} \rightarrow \varphi$$

$$\vdash ((\varphi \rightarrow \varphi^{\bar{\omega}}) \rightarrow \varphi^{\bar{\omega}}) \vee (\varphi^{\bar{\omega}} \rightarrow (\varphi^{\bar{\omega}})^2)$$

$$\psi \rightarrow \varphi, ((\varphi \rightarrow \psi) \rightarrow \psi) \vee (\psi \rightarrow \psi^2) \vdash \psi \rightarrow \varphi^{\bar{\omega}}$$

Then ω satisfies the rules for $\bar{\omega}$

In semantics, $\bar{\omega}$ coincides with ω if the latter is defined

However, ω need not be defined even if $\bar{\omega}$ is

(in Chang's MV-algebra: $\varphi^{\bar{\omega}} = \Delta\varphi$, while φ^{ω} is undefined)

Recall: In semantics, quantifiers are fuzzy sets of fuzzy sets

Why:

- quantifiers are operators on predicates
- semantic values of predicates are fuzzy sets
- ⇒ quantifiers take fuzzy sets to truth values
- ⇒ quantifiers are fuzzy sets of fuzzy sets

Recall: Sets of sets is the domain of higher-order logic

Notice: A system of Henkin-style higher-order fuzzy logic
(based on the weak quantifiers \forall, \exists only!)
has recently been developed

Behounek, Cintula: Fuzzy class theory. *Fuzzy Sets and Systems* 2004

⇒ Multiplicative quantifiers can conveniently be studied
in higher-order fuzzy logic

Propositional fuzzy logic:

any well-behaved expansion of MTL_Δ

First-order fuzzy logic (with weak quantifiers only)

add Rasiowa's axioms for \forall, \exists , crisp identity =

Henkin-style second-order fuzzy logic

= theory in 1st-order fuzzy logic:

- Sorts of objects (x, y, \dots) , fuzzy sets (X, Y, \dots) , tuples
- Axioms for tuples (crisp)
- Primitive membership predicate \in
- Comprehension axioms $(\exists Z)(\forall x) \Delta(x \in Z \leftrightarrow \varphi)$ for all φ
- Extensionality axiom $(\forall x) \Delta(x \in A \leftrightarrow x \in B) \rightarrow A = B$

Henkin-style higher-order fuzzy logic: iterate for all orders

Intended models = fuzzy subsets of all orders in a domain V

Fact: The definition of the weakest exponential ω can be internalized in higher-order fuzzy logic. The weakest multiplicative quantifier is thus **definable** in higher-order fuzzy logic.

Subtlety: Henkin-style \Rightarrow non-standard models
 \Rightarrow possibly non-standard semantics of the defined notions

Moral:

The lattice quantifiers \forall, \exists suffice for developing higher-order fuzzy logic, in which multiplicative quantifiers become definable

\Rightarrow Multiplicative quantifiers need not be present as primitives in first-order fuzzy logic: they can be bypassed by using lattice quantifiers, developing higher-order fuzzy logic by means of the latter, and defining the former within its framework

A similar approach should work for other substructural logics