Computable Analysis and Effective Descriptive Set Theory

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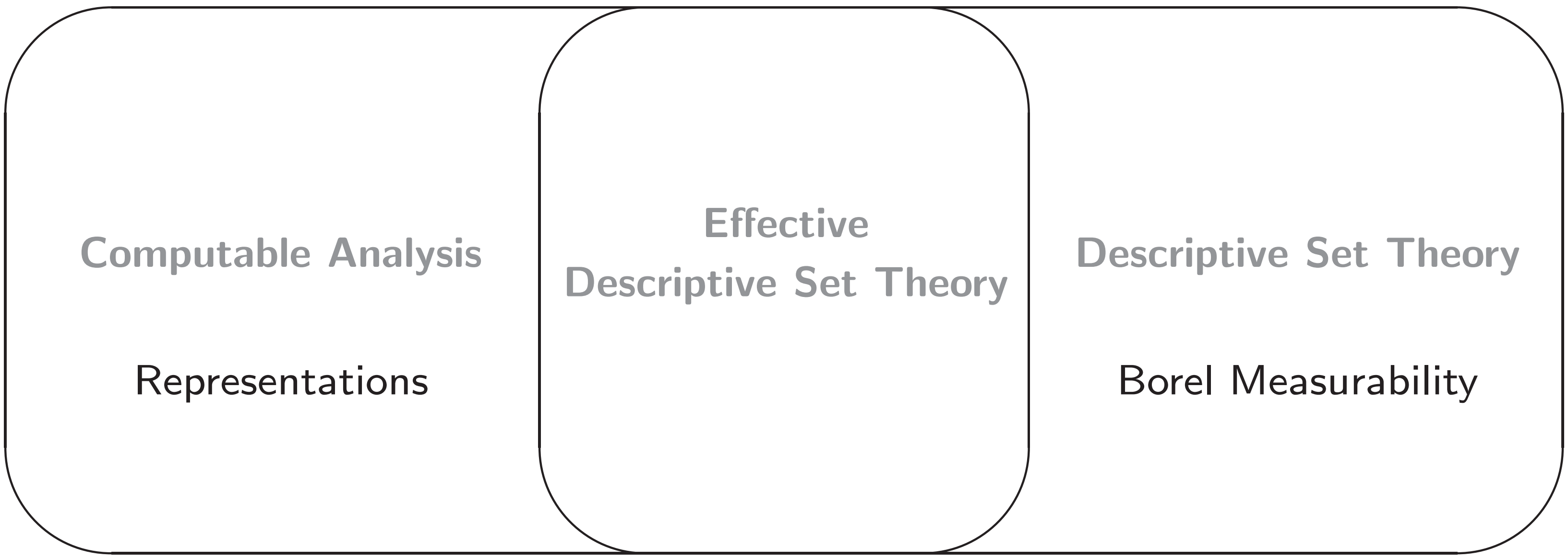
Survey

1. Basic Concepts
   - Computable Analysis
   - Computable Borel Measurability
   - The Representation Theorem
2. Classification of Topological Operations
   - Representations of Closed Subsets
   - Topological Operations
3. Classification of Theorems from Functional Analysis
   - Uniformity versus Non-Uniformity
   - Open Mapping and Closed Graph Theorem
   - Banach's Inverse Mapping Theorem
   - Hahn-Banach Theorem

Computable Analysis and Effective Descriptive Set Theory



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- This theory has been further extended by Pour-El and Richards, Hauck, Nerode, Kreitz, Weihrauch and many others.
- The representation based approach to computable analysis allows to describe computations in a large class of topological space that suffice for most applications in analysis.

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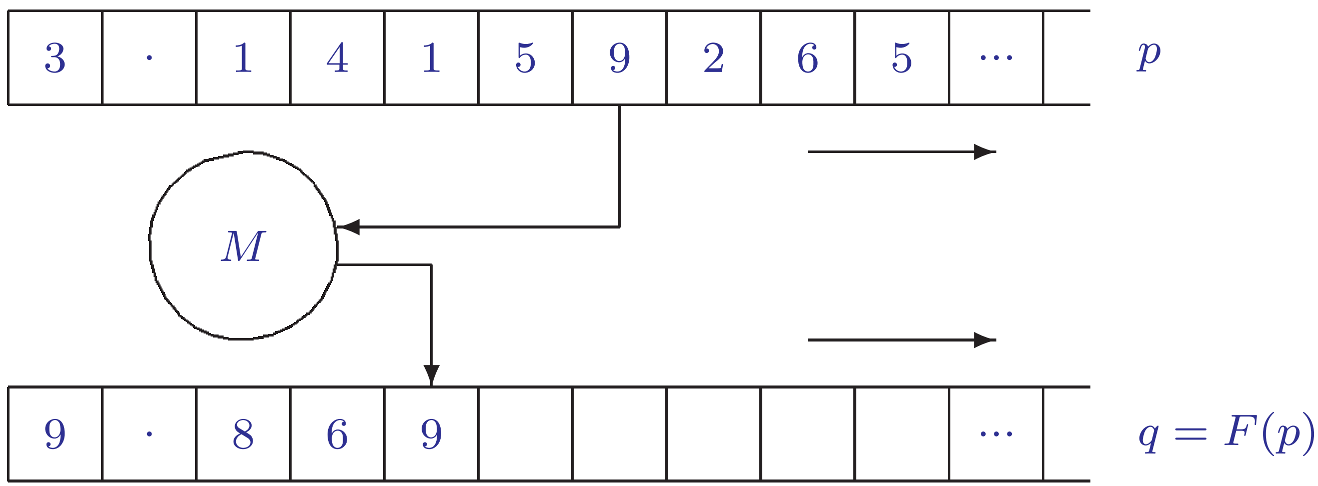
- Natural characterizations of the degree of difficulty of theorems in analysis.
- Uniform model to express computability, continuity and measurability and to provide counterexamples.
- Axiomatic choices do not matter.

Turing Machines

Definition 1 A function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is called *computable*, if there exists a Turing machine with one-way output tape which transfers each input $p \in \mathrm{dom}(F)$ into the corresponding output $F(p)$.



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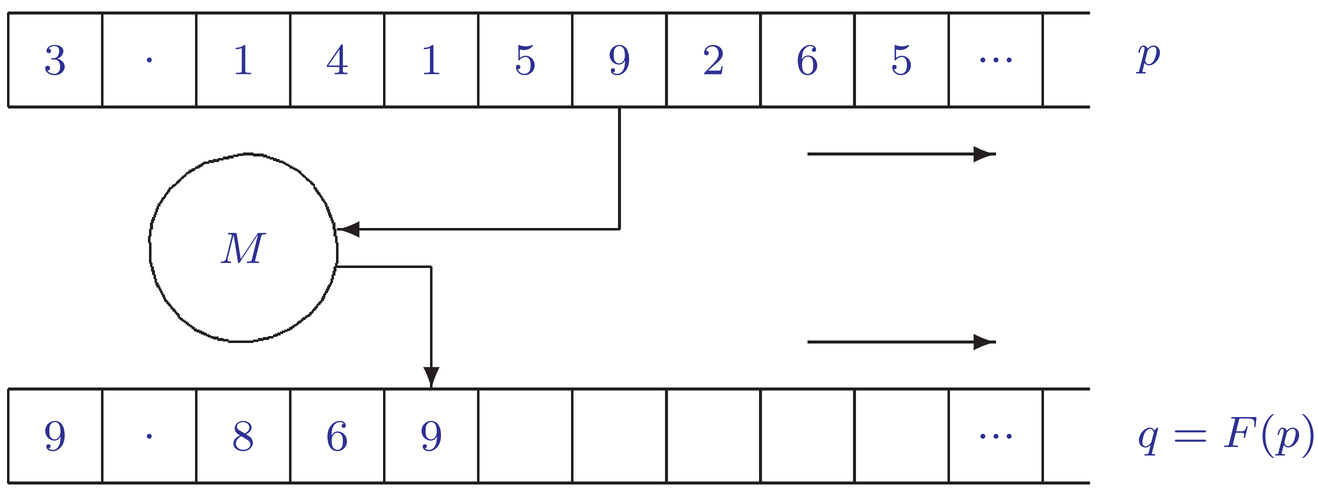

Proposition 2 *Any computable function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is continuous with respect to the Baire topology on $\mathbb{N}^{\mathbb{N}}$.*

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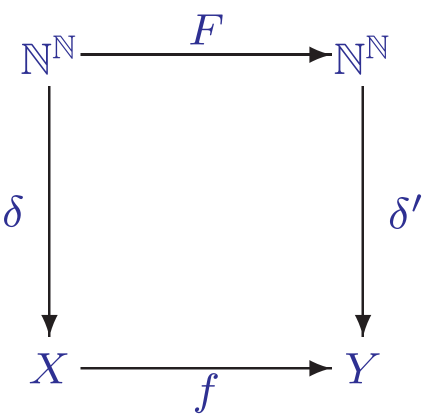

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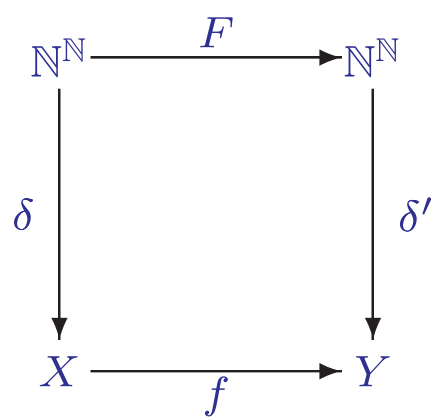

Definition 5 If δ, δ' are representations of X, Y, respectively, then there is a canonical representation $[\delta \to \delta']$ of the set of (δ, δ')–continuous functions $f : X \to Y$.

Admissible Representations

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- The representation $[\delta \to \delta']$ just includes sufficiently much information on operators T in order to evaluate them effectively.
- A computable description of an operator T with respect to $[\delta \to \delta']$ corresponds to a "program" of T.
- The underlying topology induced on $\mathcal{C}(X, Y)$ is typically the compact-open topology.

The Category of Admissibly Represented Spaces

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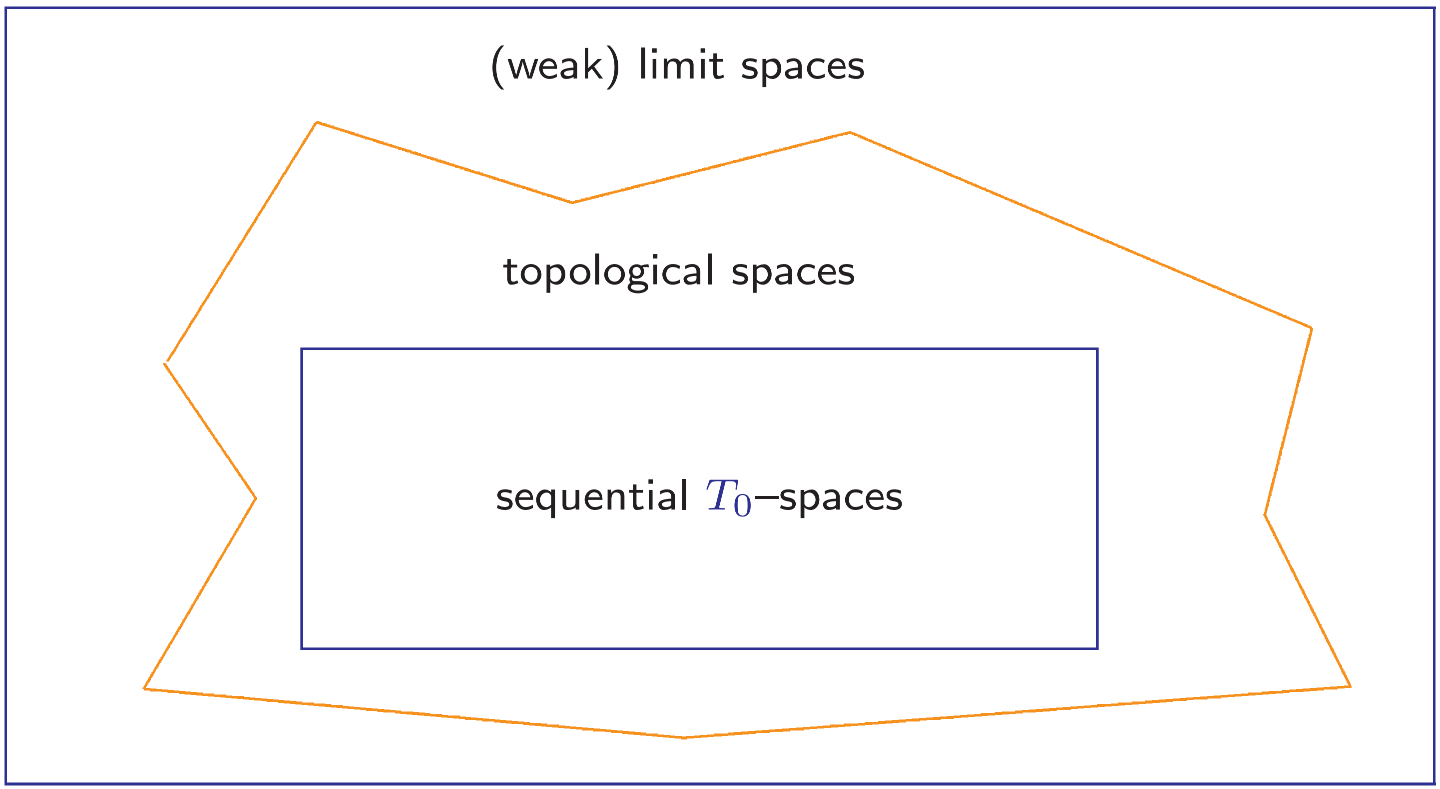

Computable Metric Spaces

Definition 9 A tuple (X, d, α) is called a *computable metric space*, if

1. $d : X \times X \to \mathbb{R}$ is a metric on X,
2. $\alpha : \mathbb{N} \to X$ is a sequence which is dense in X,
3. $d \circ (\alpha \times \alpha) : \mathbb{N}^2 \to \mathbb{R}$ is a computable (double) sequence in $\mathbb{R}$.

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Definition 10 Let (X, d, α) be a computable metric space. The *Cauchy representation* $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ of X is defined by

$$\delta_X(p) := \lim_{i \to \infty} \alpha p(i)$$

for all p such that $(\alpha p(i))_{i \in \mathbb{N}}$ converges and $d(\alpha p(i), \alpha p(j)) < 2^{-i}$ for all $j > i$ (and undefined for all other input sequences).

Examples of Computable Metric Spaces

Example 11 *The following are computable metric spaces:*

1. $(\mathbb{R}^n, d_{\mathbb{R}^n}, \alpha_{\mathbb{R}^n})$ *with the Euclidean metric*

$$d_{\mathbb{R}^n}(x, y) := \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

and a standard numbering $\alpha_{\mathbb{R}^n}$ *of* $\mathbb{Q}^n$.

2. $(\mathcal{K}(\mathbb{R}^n), d_{\mathcal{K}}, \alpha_{\mathcal{K}})$ *with the set* $\mathcal{K}(\mathbb{R}^n)$ *of non-empty compact subsets of* $\mathbb{R}^n$ *and the Hausdorff metric*

$$d_{\mathcal{K}}(A, B) := \max\left\{\sup_{a\in A} \inf_{b\in B} d_{\mathbb{R}^n}(a, b), \sup_{b\in B} \inf_{a\in A} d_{\mathbb{R}^n}(a, b)\right\}$$

and a standard numbering $\alpha_{\mathcal{K}}$ *of the non-empty finite subsets of* $\mathbb{Q}^n$.

3. $(\mathcal{C}(\mathbb{R}^n), d_{\mathcal{C}}, \alpha_{\mathcal{C}})$ *with the set* $\mathcal{C}(\mathbb{R}^n)$ *of continuous functions* $f : \mathbb{R}^n \to \mathbb{R}$,

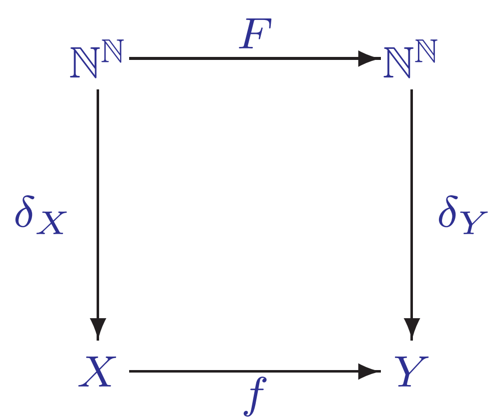
$$d_{\mathcal{C}}(f, g) := \sum_{i=0}^{\infty} 2^{-i-1} \frac{\sup_{x\in[-i,i]^n} |f(x)-g(x)|}{1+\sup_{x\in[-i,i]^n} |f(x)-g(x)|}$$

and a standard numbering $\alpha_{\mathcal{C}}$ *of* $\mathbb{Q}[x_1, ..., x_n]$.

Kreitz-Weihrauch Representation Theorem

Theorem 12 *Let X, Y be computable metric spaces and let $f :\subseteq X \to Y$ be a function. Then the following are equivalent:*

1. *f is continuous,*
2. *f admits a continuous realizer $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$.*



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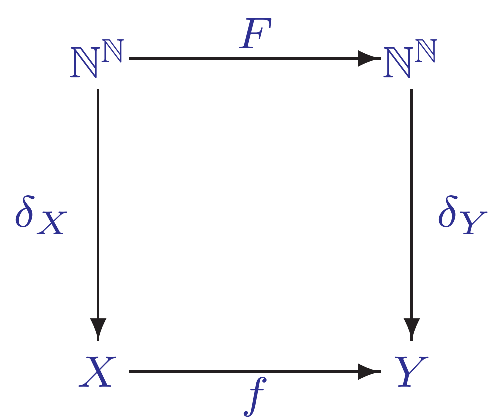
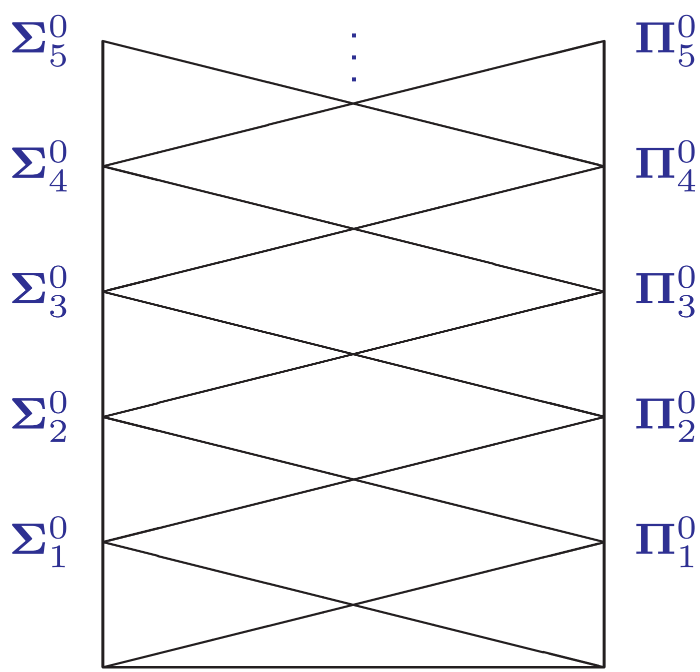


Question: Can this theorem be generalized to Borel measurable functions?

Borel Hierarchy

- $\mathbf{\Sigma}^0_1(X)$ is the set of open subsets of X,
- $\mathbf{\Pi}^0_1(X)$ is the set of closed subsets of X,
- $\mathbf{\Sigma}^0_2(X)$ is the set of F_σ subsets of X,
- $\mathbf{\Pi}^0_2(X)$ is the set of G_δ subsets of X, etc.
- $\mathbf{\Delta}^0_k(X) := \mathbf{\Sigma}^0_k(X) \cap \mathbf{\Pi}^0_k(X)$.



Representations of Borel Classes

Definition 13 Let (X, d, α) be a separable metric space. We define representations $\delta_{\mathbf{\Sigma}^0_k(X)}$ of $\mathbf{\Sigma}^0_k(X)$, $\delta_{\mathbf{\Pi}^0_k(X)}$ of $\mathbf{\Pi}^0_k(X)$ and $\delta_{\mathbf{\Delta}^0_k(X)}$ of $\mathbf{\Delta}^0_k(X)$ for $k \geq 1$ as follows:

- $\delta_{\mathbf{\Sigma}^0_1(X)}(p) := \bigcup_{\langle i,j\rangle \in \mathrm{range}(p)} B(\alpha(i), \bar{j})$,
- $\delta_{\mathbf{\Pi}^0_k(X)}(p) := X \setminus \delta_{\mathbf{\Sigma}^0_k(X)}(p)$,
- $\delta_{\mathbf{\Sigma}^0_{k+1}(X)}\langle p_0, p_1, ...\rangle := \bigcup_{i=0}^{\infty} \delta_{\mathbf{\Pi}^0_k(X)}(p_i)$,
- $\delta_{\mathbf{\Delta}^0_k(X)}\langle p, q\rangle = \delta_{\mathbf{\Sigma}^0_k(X)}(p) :\iff \delta_{\mathbf{\Sigma}^0_k(X)}(p) = \delta_{\mathbf{\Pi}^0_k(X)}(q)$,

for all $p, p_i, q \in \mathbb{N}^{\mathbb{N}}$.

Effective Closure Properties of Borel Classes

Proposition 14 *Let X, Y be computable metric spaces. The following operations are computable for any $k \geq 1$:*

1. $\mathbf{\Sigma}^0_k \hookrightarrow \mathbf{\Sigma}^0_{k+1}$, $\mathbf{\Sigma}^0_k \hookrightarrow \mathbf{\Pi}^0_{k+1}$, $\mathbf{\Pi}^0_k \hookrightarrow \mathbf{\Sigma}^0_{k+1}$, $\mathbf{\Pi}^0_k \hookrightarrow \mathbf{\Pi}^0_{k+1}$, $A \mapsto A$ *(injection)*
2. $\mathbf{\Sigma}^0_k \to \mathbf{\Pi}^0_k$, $\mathbf{\Pi}^0_k \to \mathbf{\Sigma}^0_k$, $A \mapsto A^{\mathrm{c}} := X \setminus A$ *(complement)*
3. $\mathbf{\Sigma}^0_k \times \mathbf{\Sigma}^0_k \to \mathbf{\Sigma}^0_k$, $\mathbf{\Pi}^0_k \times \mathbf{\Pi}^0_k \to \mathbf{\Pi}^0_k$, $(A, B) \mapsto A \cup B$ *(union)*
4. $\mathbf{\Sigma}^0_k \times \mathbf{\Sigma}^0_k \to \mathbf{\Sigma}^0_k$, $\mathbf{\Pi}^0_k \times \mathbf{\Pi}^0_k \to \mathbf{\Pi}^0_k$, $(A, B) \mapsto A \cap B$ *(intersection)*
5. $(\mathbf{\Sigma}^0_k)^{\mathbb{N}} \to \mathbf{\Sigma}^0_k$, $(A_n)_{n\in\mathbb{N}} \mapsto \bigcup_{n=0}^{\infty} A_n$ *(countable union)*
6. $(\mathbf{\Pi}^0_k)^{\mathbb{N}} \to \mathbf{\Pi}^0_k$, $(A_n)_{n\in\mathbb{N}} \mapsto \bigcap_{n=0}^{\infty} A_n$ *(countable intersection)*
7. $\mathbf{\Sigma}^0_k(X) \times \mathbf{\Sigma}^0_k(Y) \to \mathbf{\Sigma}^0_k(X \times Y)$, $(A, B) \mapsto A \times B$ *(product)*
8. $(\mathbf{\Pi}^0_k(X))^{\mathbb{N}} \to \mathbf{\Pi}^0_k(X^{\mathbb{N}})$, $(A_n)_{n\in\mathbb{N}} \mapsto \times_{n=0}^{\infty} A_n$ *(countable product)*
9. $\mathbf{\Sigma}^0_k(X \times \mathbb{N}) \to \mathbf{\Sigma}^0_k(X)$, $A \mapsto \{x \in X : (\exists n)(x, n) \in A\}$ *(countable projection)*
10. $\mathbf{\Sigma}^0_k(X \times Y) \times Y \to \mathbf{\Sigma}^0_k(X)$, $(A, y) \mapsto A_y := \{x \in X : (x, y) \in A\}$ *(section)*

Borel Measurable Operations

Definition 15 Let X, Y be separable metric spaces. An operation $f : X \to Y$ is called

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- *effectively* $\mathbf{\Sigma}^0_k$*–measurable* or $\mathbf{\Sigma}^0_k$*–computable*, if the map

$$\mathbf{\Sigma}^0_k(f^{-1}) : \mathbf{\Sigma}^0_1(Y) \to \mathbf{\Sigma}^0_k(X), U \mapsto f^{-1}(U)$$

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Definition 16 Let X, Y be separable metric spaces. We define representations $\delta_{\mathbf{\Sigma}^0_k(X \to Y)}$ of $\mathbf{\Sigma}^0_k(X \to Y)$ by

$$\delta_{\mathbf{\Sigma}^0_k(X \to Y)}(p) = f :\iff [\delta_{\mathbf{\Sigma}^0_1(Y)} \to \delta_{\mathbf{\Sigma}^0_k(X)}](p) = \mathbf{\Sigma}^0_k(f^{-1})$$

for all $p \in \mathbb{N}^{\mathbb{N}}$, $f : X \to Y$ and $k \geq 1$. Let $\delta_{\mathbf{\Sigma}^0_k(X \to Y)}$ denote the restriction to $\mathbf{\Sigma}^0_k(X \to Y)$.

Effective Closure Properties of Borel Measurable Operations

Proposition 17 *Let W, X, Y and Z be computable metric spaces. The following operations are computable for all $n, k \geq 1$:*

1. $\mathbf{\Sigma}^0_n(Y \to Z) \times \mathbf{\Sigma}^0_k(X \to Y) \to \mathbf{\Sigma}^0_{n+k-1}(X \to Z), (g, f) \mapsto g \circ f$ *(composition)*
2. $\mathbf{\Sigma}^0_k(X \to Y) \times \mathbf{\Sigma}^0_k(X \to Z) \to \mathbf{\Sigma}^0_k(X \to Y \times Z), (f, g) \mapsto (x \mapsto f(x) \times g(x))$ *(juxtaposition)*
3. $\mathbf{\Sigma}^0_k(X \to Y) \times \mathbf{\Sigma}^0_k(W \to Z) \to \mathbf{\Sigma}^0_k(X \times W \to Y \times Z), (f, g) \mapsto f \times g$ *(product)*
4. $\mathbf{\Sigma}^0_k(X \to Y^{\mathbb{N}}) \to \mathbf{\Sigma}^0_k(X \times \mathbb{N} \to Y), f \mapsto f_*$ *(evaluation)*
5. $\mathbf{\Sigma}^0_k(X \times \mathbb{N} \to Y) \to \mathbf{\Sigma}^0_k(X \to Y^{\mathbb{N}}), f \mapsto [f]$ *(transposition)*
6. $\mathbf{\Sigma}^0_k(X \to Y) \to \mathbf{\Sigma}^0_k(X^{\mathbb{N}} \to Y^{\mathbb{N}}), f \mapsto f^{\mathbb{N}}$ *(exponentiation)*
7. $\mathbf{\Sigma}^0_k(X \times \mathbb{N} \to Y) \to \mathbf{\Sigma}^0_k(X \to Y)^{\mathbb{N}}, f \mapsto (n \mapsto (x \mapsto f(x, n)))$ *(sequencing)*
8. $\mathbf{\Sigma}^0_k(X \to Y)^{\mathbb{N}} \to \mathbf{\Sigma}^0_k(X \times \mathbb{N} \to Y), (f_n)_{n \in \mathbb{N}} \mapsto ((x, n) \mapsto f_n(x))$ *(de-sequencing)*

Representation Theorem

Theorem 18 *Let X, Y be computable metric spaces, $k \geq 1$ and let $f : X \to Y$ be a total function. Then the following are equivalent:*

1. *f is (effectively) $\mathbf{\Sigma}^0_k$–measurable,*
2. *f admits an (effectively) $\mathbf{\Sigma}^0_k$–measurable realizer $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$.*

Proof.

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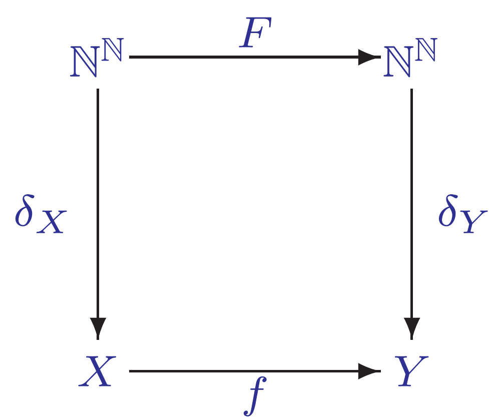
Proof.



The proof is based on effective versions of the

- Kuratowski-Ryll-Nardzewski Selection Theorem,
- Bhattacharya-Srivastava Selection Theorem. □

Weihrauch Reducibility of Functions

Definition 19 Let X, Y, U, V be computable metric spaces and consider functions $f :\subseteq X \to Y$ and $g :\subseteq U \to V$. We say that

- f is *reducible* to g, for short $f \leqslant_{\mathrm{t}} g$, if there are continuous functions $A :\subseteq X \times V \to Y$ and $B :\subseteq X \to U$ such that

  $$f(x) = A(x, g \circ B(x))$$

  for all $x \in \mathrm{dom}(f)$,

- f is *computably reducible* to g, for short $f \leqslant_{\mathrm{c}} g$, if there are computable A, B as above.

- The corresponding equivalences are denoted by $\cong_{\mathrm{t}}$ and $\cong_{\mathrm{c}}$.

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- The corresponding equivalences are denoted by $\cong_{\mathrm{t}}$ and $\cong_{\mathrm{c}}$.

Proposition 20 *The following holds for all $k \geq 1$:*

1. $f \leqslant_{\mathrm{t}} g$ *and* g *is* $\mathbf{\Sigma}^0_k$*–measurable* $\Longrightarrow$ f *is* $\mathbf{\Sigma}^0_k$*–measurable,*
2. $f \leqslant_{\mathrm{c}} g$ *and* g *is* Σ^0_k*–computable* $\Longrightarrow$ f *is* Σ^0_k*–computable.*

Completeness Theorem for Baire Space

Definition 21 For any $k \in \mathbb{N}$ we define $C_k : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by

$$C_k(p)(n) := \begin{cases} 0 & \text{if } (\exists n_k)(\forall n_{k-1})\dots p\langle n, n_1, \dots, n_k\rangle \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

for all $p \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$.

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for all $p \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$.

Theorem 22 *Let $k \in \mathbb{N}$. For any function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ we obtain:*

1. $F \leqslant_{\mathrm{t}} C_k \iff F$ *is $\mathbf{\Sigma}^0_{k+1}$–measurable,*
2. $F \leqslant_{\mathrm{c}} C_k \iff F$ *is Σ^0_{k+1}–computable.*

Proof. Employ the Tarski-Kuratowski Normal Form in the appropriate way. □

Realizer Reducibility

Definition 23 Let X, Y, U, V be computable metric spaces and consider functions $f : X \to Y$ and $g : U \to V$. We define

$$f \preceq_{\mathrm{t}} g :\iff f\delta_X \leqslant_{\mathrm{t}} g\,\delta_U$$

and we say that f is *realizer reducible* to g, if this holds. Analogously, we define $f \preceq_{\mathrm{c}} g$ with $\leqslant_{\mathrm{c}}$ instead of $\leqslant_{\mathrm{t}}$. The corresponding equivalences $\approx_{\mathrm{t}}$ and $\approx_{\mathrm{c}}$ are defined straightforwardly.

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Theorem 24 *Let X, Y be computable metric spaces and let $k \in \mathbb{N}$. For any function $f : X \to Y$ we obtain:*

1. $f \preceq_{\mathrm{t}} C_k \iff f$ *is $\mathbf{\Sigma}^0_{k+1}$–measurable,*
2. $f \preceq_{\mathrm{c}} C_k \iff f$ *is $\mathbf{\Sigma}^0_{k+1}$–computable.*

Characterization of Realizer Reducibility

Definition 25 Let X, Y, U, V be computable metric spaces, let $\mathcal{F}$ be a set of functions $F : X \to Y$ and let $\mathcal{G}$ be a set of functions $G : U \to V$. We define

$$\begin{aligned} \mathcal{F} \leqslant_{\mathrm{t}} \mathcal{G} \quad :\iff \quad & (\exists A, B \text{ computable})(\forall G \in \mathcal{G})(\exists F \in \mathcal{F}) \\ & (\forall x \in \mathrm{dom}(F))\, F(x) = A(x, GB(x)), \end{aligned}$$

where $A :\subseteq X \times V \to Y$ and $B :\subseteq X \to U$. Analogously, one can define $\leqslant_{\mathrm{c}}$ with computable A, B.

Characterization of Realizer Reducibility

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where $A :\subseteq X \times V \to Y$ and $B :\subseteq X \to U$. Analogously, one can define $\leqslant_{\mathrm{c}}$ with computable A, B.

Proposition 26 *Let X, Y, U, V be computable metric spaces and let $f : X \to Y$ and $g : U \to V$ be functions. Then*

$$f \preceq_{\mathrm{c}} g \iff \{F : F \vdash f\} \leqslant_{\mathrm{c}} \{G : G \vdash g\}.$$

An analogous statement holds with respect to $\preceq_{\mathrm{t}}$ and $\leqslant_{\mathrm{t}}$.

Completeness of the Limit

Proposition 27 *Let X be a computable metric space and consider* $c := \{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \text{ converges}\}$ *as computable metric subspace of $X^{\mathbb{N}}$. The ordinary limit map*

$$\lim : c \to X, (x_n)_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} x_n$$

is $\mathbf{\Sigma}^0_2$–computable and it is even $\mathbf{\Sigma}^0_2$–complete, if there is a computable embedding $\iota : \{0,1\}^{\mathbb{N}} \hookrightarrow X$.

Completeness of the Limit

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is $\mathbf{\Sigma}^0_2$–computable and it is even $\mathbf{\Sigma}^0_2$–complete, if there is a computable embedding $\iota : \{0,1\}^{\mathbb{N}} \hookrightarrow X$.

Proof. On the one hand, $\mathbf{\Sigma}^0_2$–computability follows from

$$\lim{}^{-1}(B(x,r)) = \left(\bigcup_{n=0}^{\infty} X^n \times \overline{B}(x, r-2^{-n})^{\mathbb{N}}\right) \cap c \in \mathbf{\Sigma}^0_2(c)$$

and on the other hand, $\mathbf{\Sigma}^0_2$–completeness follows from

$$C_1 \leqslant_{\mathrm{c}} \lim_{\{0,1\}^{\mathbb{N}}} \leqslant_{\mathrm{c}} \lim_X .$$

□

Lower Bounds for Unbounded Closed Linear Operators

Theorem 28 *Let X, Y be computable Banach spaces and let $f :\subseteq X \to Y$ be a closed linear and unbounded operator. Let $(e_n)_{n\in\mathbb{N}}$ be a computable sequence in $\mathrm{dom}(f)$ whose linear span is dense in X and let $f(e_n)_{n\in\mathbb{N}}$ be computable in Y. Then $C_1 \leqslant_{\mathrm{c}} f$.*

Lower Bounds for Unbounded Closed Linear Operators

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Corollary 29 (First Main Theorem of Pour-El and Richards) *Under the same assumptions as above f maps some computable input $x \in X$ to a non-computable output $f(x)$.*

Arithmetic Complexity of Points and the Invariance Theorem

Definition 30 Let X be a computable metric space and let $x \in X$. Then x is called Δ^0_n*–computable*, if there is a Δ^0_n–computable $p \in \mathbb{N}^{\mathbb{N}}$ such that $x = \delta_X(p)$.

Arithmetic Complexity of Points and the Invariance Theorem

Definition 30 Let X be a computable metric space and let $x \in X$. Then x is called Δ^0_n*–computable*, if there is a Δ^0_n–computable $p \in \mathbb{N}^{\mathbb{N}}$ such that $x = \delta_X(p)$.

Theorem 31 *Let X, Y be computable metric spaces.*

- *If $f : X \to Y$ is $\mathbf{\Sigma}^0_k$–computable, then it maps Δ^0_n–computable inputs $x \in X$ to Δ^0_{n+k-1}–computable outputs $f(x) \in Y$.*

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- *If $f : X \to Y$ is $\mathbf{\Sigma}^0_k$–computable, then it maps Δ^0_n–computable inputs $x \in X$ to Δ^0_{n+k-1}–computable outputs $f(x) \in Y$.*
- *If f is even $\mathbf{\Sigma}^0_k$–complete and $k \geq 2$, then there is some Δ^0_n–computable input $x \in X$ for any $n \geq 1$ which is mapped to some Δ^0_{n+k-1}–computable output $f(x) \in Y$ which is not Δ^0_{n+k-2}–computable.*

Arithmetic Complexity of Points and the Invariance Theorem

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- *If f is even $\mathbf{\Sigma}^0_k$–complete and $k \geq 2$, then there is some Δ^0_n–computable input $x \in X$ for any $n \geq 1$ which is mapped to some Δ^0_{n+k-1}–computable output $f(x) \in Y$ which is not Δ^0_{n+k-2}–computable.*

Corollary 32 *An $\mathbf{\Sigma}^0_2$–computable map f maps computable inputs $x \in X$ to outputs $f(x)$ that are computable in the halting problem $\emptyset'$. If f is even $\mathbf{\Sigma}^0_2$–complete, then there is some computable x which is mapped to a non-computable $f(x)$.*

Completeness of Differentiation

Proposition 33 (von Stein) *Let $\mathcal{C}^{(k)}[0,1]$ be the computable metric subspace of $\mathcal{C}[0,1]$ which contains the k–times continuously differentiable functions $f:[0,1]\to\mathbb{R}$. The operator of differentiation*

$$d^k:\mathcal{C}^{(k)}[0,1]\to\mathcal{C}[0,1], f\mapsto f^{(k)}$$

is $\mathbf{\Sigma}^0_{k+1}$–complete.

Completeness of Differentiation

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is $\mathbf{\Sigma}^0_{k+1}$*–complete.*

Corollary 34 *The operator of differentiation* $d:\mathcal{C}^{(1)}[0,1]\to\mathcal{C}[0,1]$ *is* $\mathbf{\Sigma}^0_2$*–complete.*

Completeness of Differentiation

Proposition 33 (von Stein) *Let $\mathcal{C}^{(k)}[0,1]$ be the computable metric subspace of $\mathcal{C}[0,1]$ which contains the k–times continuously differentiable functions $f:[0,1]\to\mathbb{R}$. The operator of differentiation*

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is $\mathbf{\Sigma}^0_{k+1}$–complete.

Corollary 34 *The operator of differentiation $d:\mathcal{C}^{(1)}[0,1]\to\mathcal{C}[0,1]$ is $\mathbf{\Sigma}^0_2$–complete.*

Corollary 35 (Ho) *The derivative $f':[0,1]\to\mathbb{R}$ of any computable and continuously differentiable function $f:[0,1]\to\mathbb{R}$ is computable in the halting problem $\emptyset'$.*

Completeness of Differentiation

Proposition 33 (von Stein) *Let $\mathcal{C}^{(k)}[0,1]$ be the computable metric subspace of $\mathcal{C}[0,1]$ which contains the k–times continuously differentiable functions $f:[0,1]\to\mathbb{R}$. The operator of differentiation*

$$d^k:\mathcal{C}^{(k)}[0,1]\to\mathcal{C}[0,1],f\mapsto f^{(k)}$$

is $\mathbf{\Sigma}^0_{k+1}$–complete.

Corollary 34 *The operator of differentiation $d:\mathcal{C}^{(1)}[0,1]\to\mathcal{C}[0,1]$ is $\mathbf{\Sigma}^0_2$–complete.*

Corollary 35 (Ho) *The derivative $f':[0,1]\to\mathbb{R}$ of any computable and continuously differentiable function $f:[0,1]\to\mathbb{R}$ is computable in the halting problem $\emptyset'$.*

Corollary 36 (Myhill) *There exists a computable and continuously differentiable function $f:[0,1]\to\mathbb{R}$ whose derivative $f':[0,1]\to\mathbb{R}$ is not computable.*

Survey

1. Basic Concepts
   - Computable Analysis
   - Computable Borel Measurability
   - The Representation Theorem
2. Classification of Topological Operations
   - Representations of Closed Subsets
   - Topological Operations
3. Classification of Theorems from Functional Analysis
   - Uniformity versus Non-Uniformity
   - Open Mapping and Closed Graph Theorem
   - Banach's Inverse Mapping Theorem
   - Hahn-Banach Theorem

Some Topological Operations

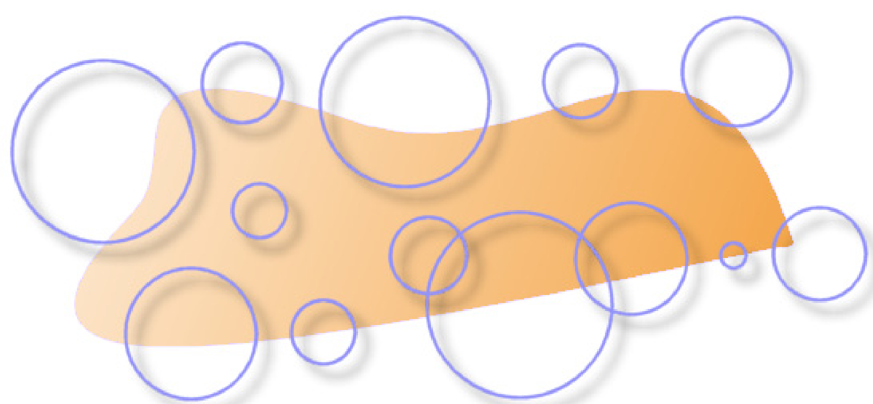
1. Union: $\cup : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathcal{A}(X), (A, B) \mapsto A \cup B$,
2. Intersection: $\cap : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathcal{A}(X), (A, B) \mapsto A \cap B$,
3. Complement: $c : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto \overline{A^{\mathrm{c}}}$,
4. Interior: $i : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto \overline{A^\circ}$,
5. Difference: $D : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathcal{A}(X), (A, B) \mapsto \overline{A \setminus B}$,
6. Symmetric Difference:
   $\Delta : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathcal{A}(X), (A, B) \mapsto \overline{A \Delta B}$,
7. Boundary: $\partial : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto \partial A$,
8. Derivative: $d : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto A'$.

All results in the second part of the talk are based on joint work with Guido Gherardi, University of Siena, Italy.

R.e. and Recursive Closed Subsets

Definition 37 Let (X, d, α) be a computable metric space and let $A \subseteq X$ a closed subset. Then

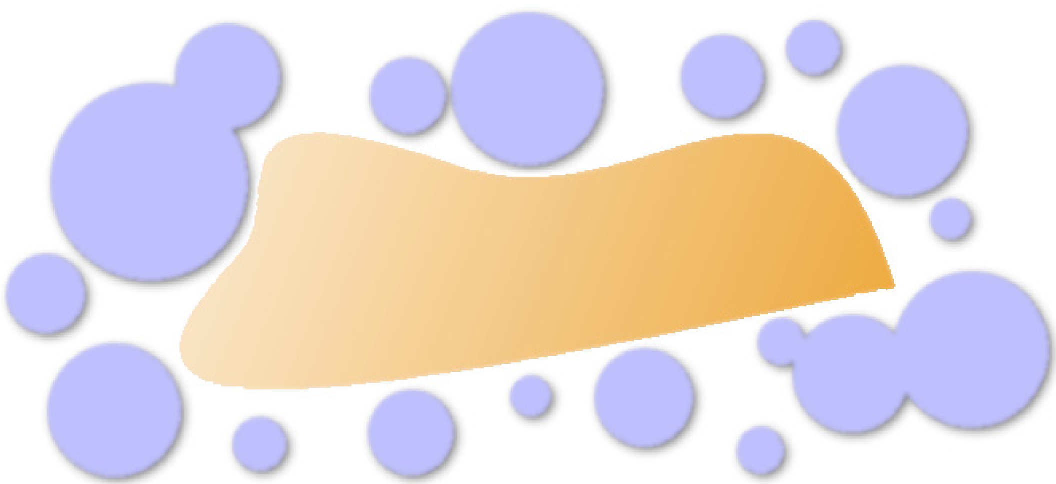
- A is called *r.e. closed*, if $\{(n,r) \in \mathbb{N} \times \mathbb{Q} : A \cap B(\alpha(n), r) \neq \emptyset\}$ is r.e.
- A is called *co-r.e. closed*, if there exists an r.e. set $I \subseteq \mathbb{N} \times \mathbb{Q}$ such that $X \setminus A = \bigcup_{(n,r)\in I} B(\alpha(n), r)$.
- A is called *recursive closed*, if A is r.e. and co-r.e. closed.



R.e. and Recursive Closed Subsets

Definition 38 Let (X, d, α) be a computable metric space and let $A \subseteq X$ a closed subset. Then

- A is called *r.e. closed*, if $\{(n, r) \in \mathbb{N} \times \mathbb{Q} : A \cap B(\alpha(n), r) \neq \emptyset\}$ is r.e.
- A is called *co-r.e. closed*, if there exists an r.e. set $I \subseteq \mathbb{N} \times \mathbb{Q}$ such that $X \setminus A = \bigcup_{(n,r) \in I} B(\alpha(n), r)$.
- A is called *recursive closed*, if A is r.e. and co-r.e. closed.



Definition 39 Let (X, d, α) be a computable metric space. We define representations of $\mathcal{A}(X) := \{A \subseteq X : A \text{ closed and non-empty}\}$:

1. $\psi_+(p) = A :\iff p$ is a "list" of all $\langle n, k\rangle$ with $A \cap B(\alpha(n), \overline{k}) \neq \emptyset$,
2. $\psi_-(p) = A :\iff p$ is a "list" of $\langle n_i, k_i\rangle$ with $X \setminus A = \bigcup_{i=0}^{\infty} B(\alpha(n_i), \overline{k_i})$,
3. $\psi\langle p, q\rangle = A :\iff \psi_+(p) = A$ and $\psi_-(q) = A$,

for all $p, q \in \mathbb{N}^{\mathbb{N}}$ and $A \in \mathcal{A}(X)$.

Some Hyperspace Representations

Definition 39 Let (X, d, α) be a computable metric space. We define representations of $\mathcal{A}(X) := \{A \subseteq X : A \text{ closed and non-empty}\}$:

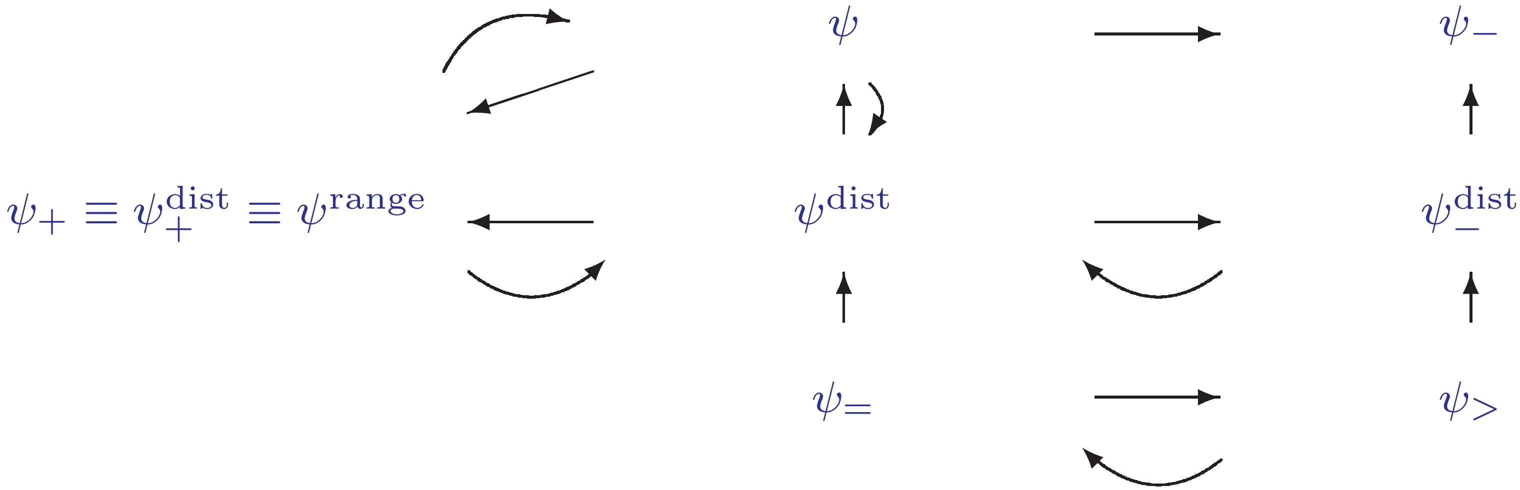
1. $\psi_+(p) = A :\iff p$ is a "list" of all $\langle n, k \rangle$ with $A \cap B(\alpha(n), \overline{k}) \neq \emptyset$,
2. $\psi_-(p) = A :\iff p$ is a "list" of $\langle n_i, k_i \rangle$ with $X \setminus A = \bigcup_{i=0}^{\infty} B(\alpha(n_i), \overline{k_i})$,
3. $\psi\langle p, q \rangle = A :\iff \psi_+(p) = A$ and $\psi_-(q) = A$,

for all $p, q \in \mathbb{N}^{\mathbb{N}}$ and $A \in \mathcal{A}(X)$.

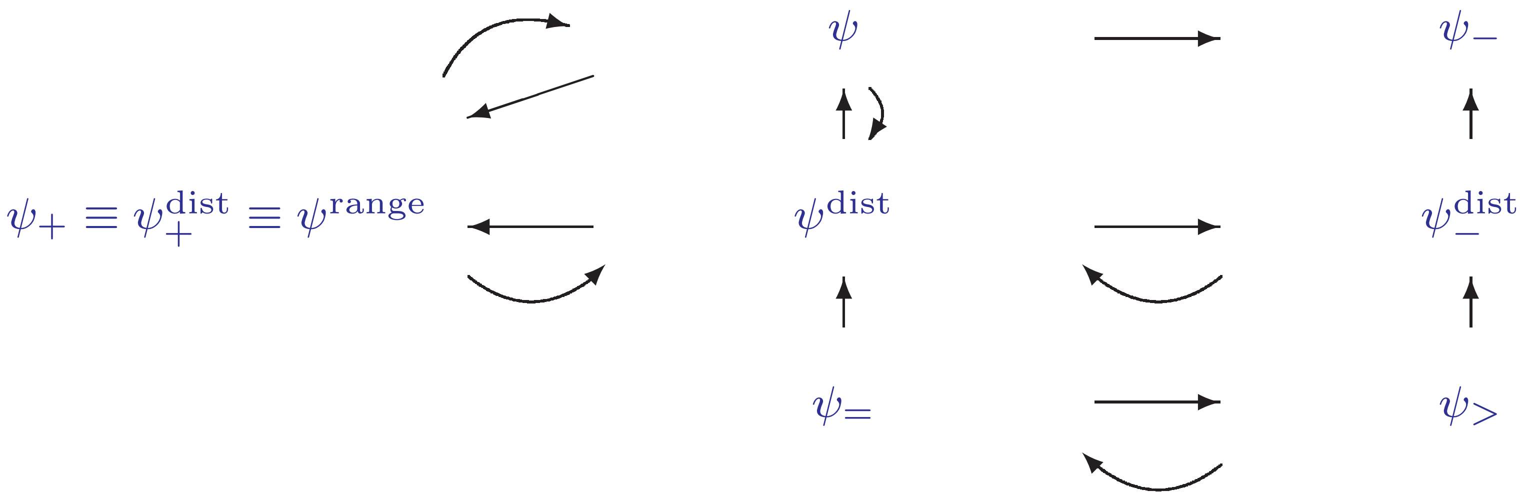
Remark 40

- *The representation ψ_+ of $\mathcal{A}(\mathbb{R}^n)$ is admissible with respect to the lower Fell topology (with subbase elements $\{A : A \cap U \neq \emptyset\}$ for any open U). The computable points are exactly the r.e. closed subsets.*
- *The representation ψ_- of $\mathcal{A}(\mathbb{R}^n)$ is admissible with respect to the upper Fell topology (with subbase elements $\{A : A \cap K = \emptyset\}$ for any compact K). The computable points are exactly the co-r.e. closed subsets.*
- *The representation ψ of $\mathcal{A}(\mathbb{R}^n)$ is admissible with respect to the Fell topology. The computable points are exactly the recursive closed subsets.*

Borel Lattice of Closed Set Representations for Polish Spaces

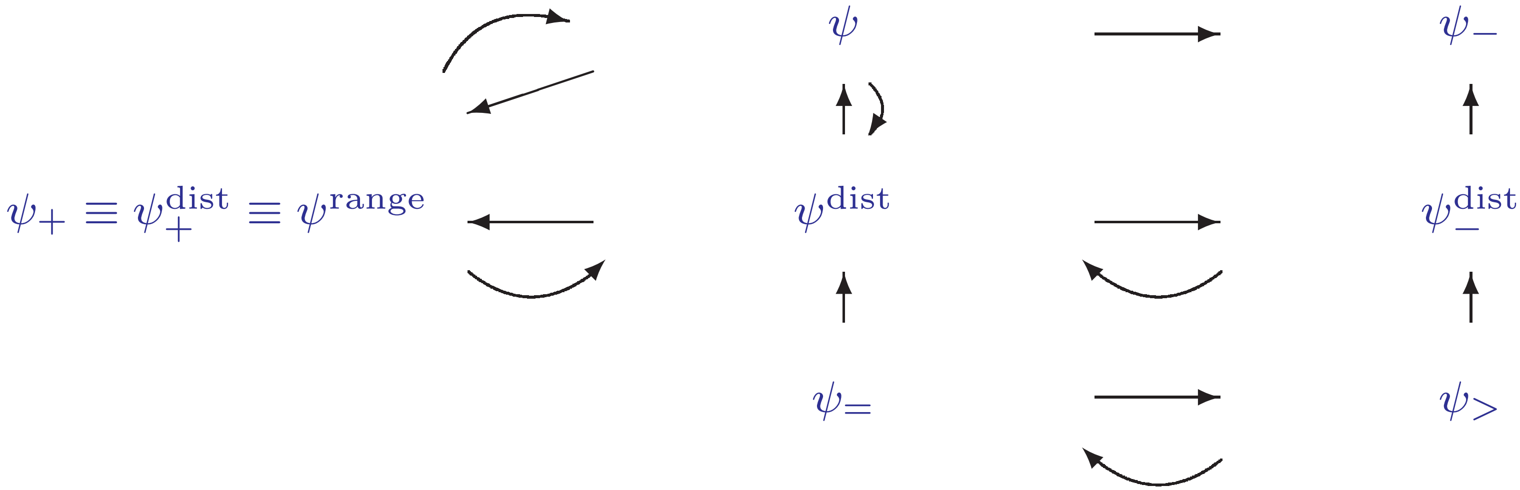


Borel Lattice of Closed Set Representations for Polish Spaces



- Straight arrows stand for computable reductions.
- Curved arrows stand for $\mathbf{\Sigma}_2^0$–computable reductions.

Borel Lattice of Closed Set Representations for Polish Spaces



- Straight arrows stand for computable reductions.
- Curved arrows stand for $\mathbf{\Sigma}_2^0$–computable reductions.

- The Borel structure induced by the final topologies of all representations except ψ_- is the Effros Borel structure.
- If X is locally compact, then this also holds true for ψ_-.

Intersection

Theorem 41 *Let X be a computable metric space. Then intersection $\cap : \mathcal{A}(X) \times \mathcal{A}(X) \to \mathcal{A}(X), (A, B) \mapsto A \cap B$ is*

1. *computable with respect to (ψ_-, ψ_-, ψ_-),*
2. *$\mathbf{\Sigma}^0_2$–computable with respect to (ψ_+, ψ_+, ψ_-),*
3. *$\mathbf{\Sigma}^0_2$–computable w.r.t. (ψ_-, ψ_-, ψ), if X is effectively locally compact,*
4. *$\mathbf{\Sigma}^0_3$–computable w.r.t. (ψ_+, ψ_+, ψ), if X is effectively locally compact,*
5. *$\mathbf{\Sigma}^0_3$–hard with respect to (ψ_+, ψ_+, ψ_+), if X is complete and perfect,*
6. *$\mathbf{\Sigma}^0_2$–hard with respect to (ψ, ψ, ψ_+), if X is complete and perfect,*
7. *not Borel measurable w.r.t. (ψ, ψ, ψ_+), if X is complete but not K_σ.*

Closure of the Complement

Theorem 42 *Let (X,d) be a computable metric space. Then the closure of the complement $c:\mathcal{A}(X)\to\mathcal{A}(X), A\mapsto\overline{A^c}$ is*

1. *computable with respect to (ψ_-,ψ_+),*
2. *$\mathbf{\Sigma}^0_2$–computable with respect to (ψ_+,ψ_+) and (ψ_-,ψ),*
3. *$\mathbf{\Sigma}^0_2$–complete with respect to (ψ_+,ψ_+), if X is complete and perfect,*
4. *$\mathbf{\Sigma}^0_2$–complete with respect to (ψ,ψ_-), if X is complete, perfect and proper.*

Closure of the Complement

Theorem 42 *Let* (X, d) *be a computable metric space. Then the closure of the complement* $c : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto \overline{A^c}$ *is*

1. *computable with respect to* (ψ_-, ψ_+),
2. $\mathbf{\Sigma}^0_2$*–computable with respect to* (ψ_+, ψ_+) *and* (ψ_-, ψ),
3. $\mathbf{\Sigma}^0_2$*–complete with respect to* (ψ_+, ψ_+)*, if* X *is complete and perfect,*
4. $\mathbf{\Sigma}^0_2$*–complete with respect to* (ψ, ψ_-)*, if* X *is complete, perfect and proper.*

Corollary 43 *Let* X *be a computable, perfect and proper Polish space. Then there exists a recursive closed* $A \subseteq X$ *such that* $\overline{A^c}$ *is not co-r.e. closed, but* $\overline{A^c}$ *is always co-r.e. closed in the halting problem* $\emptyset'$*. There exists a r.e. closed* $A \subseteq X$ *such that* $\overline{A^c}$ *is not r.e. closed, but* $\overline{A^c}$ *is always r.e. closed in the halting problem* $\emptyset'$*.*

Closure of the Interior

Theorem 44 *Let X be a computable metric space. Then the closure of the interior $i : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto \overline{A^\circ}$ is*

1. $\mathbf{\Sigma}^0_2$*–computable with respect to* (ψ_-, ψ_+),
2. $\mathbf{\Sigma}^0_3$*–computable with respect to* (ψ_+, ψ_+) *and* (ψ_-, ψ),
3. $\mathbf{\Sigma}^0_3$*–complete with respect to* (ψ_+, ψ_+)*, if* X *is complete and perfect,*
4. $\mathbf{\Sigma}^0_3$*–complete with respect to* (ψ, ψ_-)*, if* X *is complete, perfect and proper,*
5. $\mathbf{\Sigma}^0_2$*–complete with respect to* (ψ, ψ_+)*, if* X *is complete, perfect and proper.*

Closure of the Interior

Theorem 44 *Let X be a computable metric space. Then the closure of the interior $i : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto \overline{A^\circ}$ is*

1. *$\mathbf{\Sigma}^0_2$–computable with respect to (ψ_-, ψ_+),*
2. *$\mathbf{\Sigma}^0_3$–computable with respect to (ψ_+, ψ_+) and (ψ_-, ψ),*
3. *$\mathbf{\Sigma}^0_3$–complete with respect to (ψ_+, ψ_+), if X is complete and perfect,*
4. *$\mathbf{\Sigma}^0_3$–complete with respect to (ψ, ψ_-), if X is complete, perfect and proper,*
5. *$\mathbf{\Sigma}^0_2$–complete with respect to (ψ, ψ_+), if X is complete, perfect and proper.*

Corollary 45 *Let X be a computable, perfect and proper Polish space. Then there exists a recursive closed $A \subseteq X$ such that $\overline{A^\circ}$ is not r.e. closed, but $\overline{A^\circ}$ is always r.e. closed in the halting problem $\emptyset'$. There exists a recursive closed $A \subseteq X$ such that $\overline{A^\circ}$ is not even co-r.e. closed in the halting problem $\emptyset'$, but $\overline{A^\circ}$ is always co-r.e. closed in $\emptyset''$.*

Boundary

Theorem 46 *Let X be a computable metric space. Then the boundary $\partial : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto \partial A$ is*

1. *computable with respect to (ψ, ψ_+), if X is effectively locally connected,*
2. *$\mathbf{\Sigma}_2^0$–computable with respect to (ψ_+, ψ_+) and (ψ, ψ), if X is effectively locally connected,*
3. *$\mathbf{\Sigma}_2^0$–computable with respect to (ψ_-, ψ_-),*
4. *$\mathbf{\Sigma}_3^0$–computable w.r.t. (ψ_-, ψ), if X is effectively locally compact,*
5. *$\mathbf{\Sigma}_2^0$–computable with respect to (ψ_-, ψ), if X is effectively locally connected and effectively locally compact,*
6. *$\mathbf{\Sigma}_2^0$–complete w.r.t. (ψ, ψ_-), if X is complete, perfect and proper,*
7. *$\mathbf{\Sigma}_3^0$–complete with respect to (ψ, ψ_+), if $X = \{0,1\}^{\mathbb{N}}$,*
8. *not Borel measurable with respect to (ψ, ψ_+), if $X = \mathbb{N}^{\mathbb{N}}$.*

Boundary

Theorem 46 *Let X be a computable metric space. Then the boundary $\partial : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto \partial A$ is*

1. *computable with respect to (ψ, ψ_+), if X is effectively locally connected,*
2. *$\mathbf{\Sigma}^0_2$–computable with respect to (ψ_+, ψ_+) and (ψ, ψ), if X is effectively locally connected,*
3. *$\mathbf{\Sigma}^0_2$–computable with respect to (ψ_-, ψ_-),*
4. *$\mathbf{\Sigma}^0_3$–computable w.r.t. (ψ_-, ψ), if X is effectively locally compact,*
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8. *not Borel measurable with respect to (ψ, ψ_+), if $X = \mathbb{N}^{\mathbb{N}}$.*

Corollary 47 *Let X be a computable, perfect and proper Polish space. Then there exists a recursive closed $A \subseteq X$ such that ∂A is not co-r.e. closed, but ∂A is always co-r.e. closed in the halting problem $\emptyset'$.*

Derivative

Theorem 48 *Let X be a computable metric space. Then the derivative $d : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto A'$ is*

1. *$\mathbf{\Sigma}^0_2$–computable with respect to (ψ_+, ψ_-),*
2. *$\mathbf{\Sigma}^0_3$–computable with respect to (ψ_+, ψ) and (ψ_-, ψ_-), if X is effectively locally compact,*
3. *$\mathbf{\Sigma}^0_2$–complete with respect to (ψ, ψ_-), if X is complete and perfect,*
4. *$\mathbf{\Sigma}^0_3$–hard with respect to (ψ_-, ψ_-), if X is complete and perfect,*
5. *$\mathbf{\Sigma}^0_3$–hard with respect to (ψ, ψ_+), if X is complete and perfect,*
6. *not Borel measurable with respect to (ψ, ψ_+), if X is complete but not K_σ.*

Derivative

Theorem 48 *Let X be a computable metric space. Then the derivative $d : \mathcal{A}(X) \to \mathcal{A}(X), A \mapsto A'$ is*

1. $\mathbf{\Sigma}^0_2$*–computable with respect to* (ψ_+, ψ_-),
2. $\mathbf{\Sigma}^0_3$*–computable with respect to* (ψ_+, ψ) *and* (ψ_-, ψ_-), *if* X *is effectively locally compact,*
3. $\mathbf{\Sigma}^0_2$*–complete with respect to* (ψ, ψ_-), *if* X *is complete and perfect,*
4. $\mathbf{\Sigma}^0_3$*–hard with respect to* (ψ_-, ψ_-), *if* X *is complete and perfect,*
5. $\mathbf{\Sigma}^0_3$*–hard with respect to* (ψ, ψ_+), *if* X *is complete and perfect,*
6. *not Borel measurable with respect to* (ψ, ψ_+), *if* X *is complete but not* K_σ.

Corollary 49 *Let X be a computable and perfect Polish space. Then there exists a recursive closed $A \subseteq X$ such that A' is not r.e. closed in the halting problem $\emptyset'$, but any such A' is co-r.e. closed in the halting problem $\emptyset'$.*

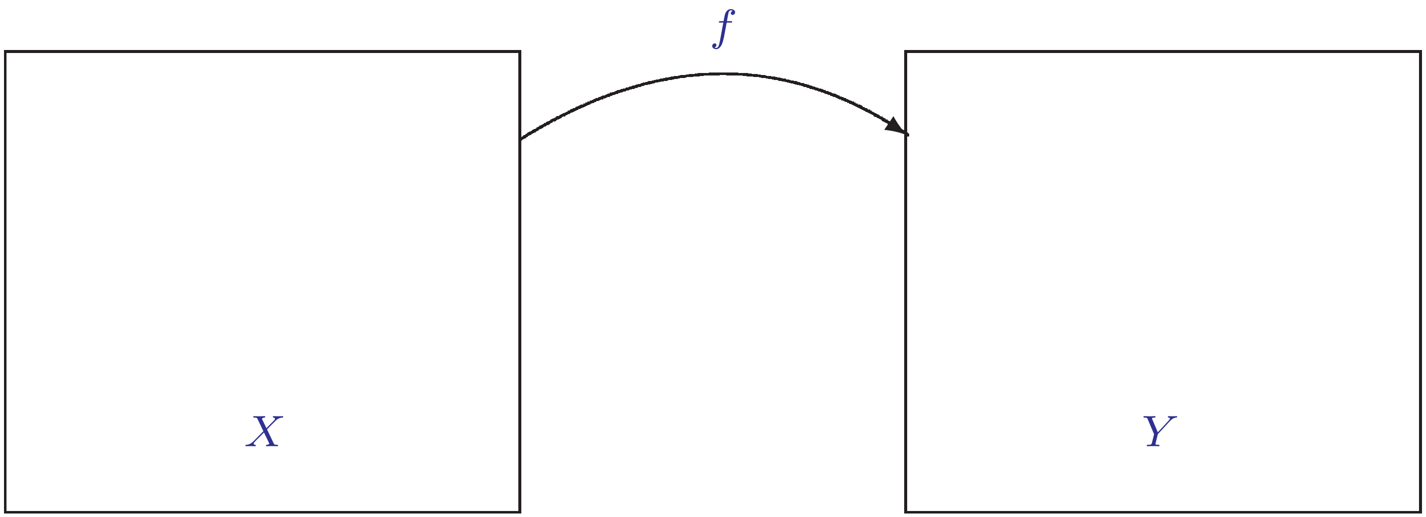
Survey on Results

	$\mathbb{N}$	$\{0,1\}^{\mathbb{N}}$	$\mathbb{N}^{\mathbb{N}}$	$[0,1]$	$[0,1]^{\mathbb{N}}$	$\mathbb{R}^n$	$\mathbb{R}^{\mathbb{N}}$	ℓ_2	$\mathcal{C}[0,1]$
$A \cup B$	1	1	1	1	1	1	1	1	1
$A \cap B$	1	2	∞	2	2	2	∞	∞	∞
$\overline{A^c}$	1	2	2	2	2	2	2	2	2
$\overline{A^\circ}$	1	3	3	3	3	3	3	3	3
$\overline{A \setminus B}$	1	2	2	2	2	2	2	2	2
$\overline{A \Delta B}$	1	2	2	2	2	2	2	2	2
∂A	1	3	∞	2	2	2	2	2	2
A'	1	3	∞	3	3	3	∞	∞	∞

Degrees of computability with respect to ψ

Survey

Uniform and Non-Uniform Computability



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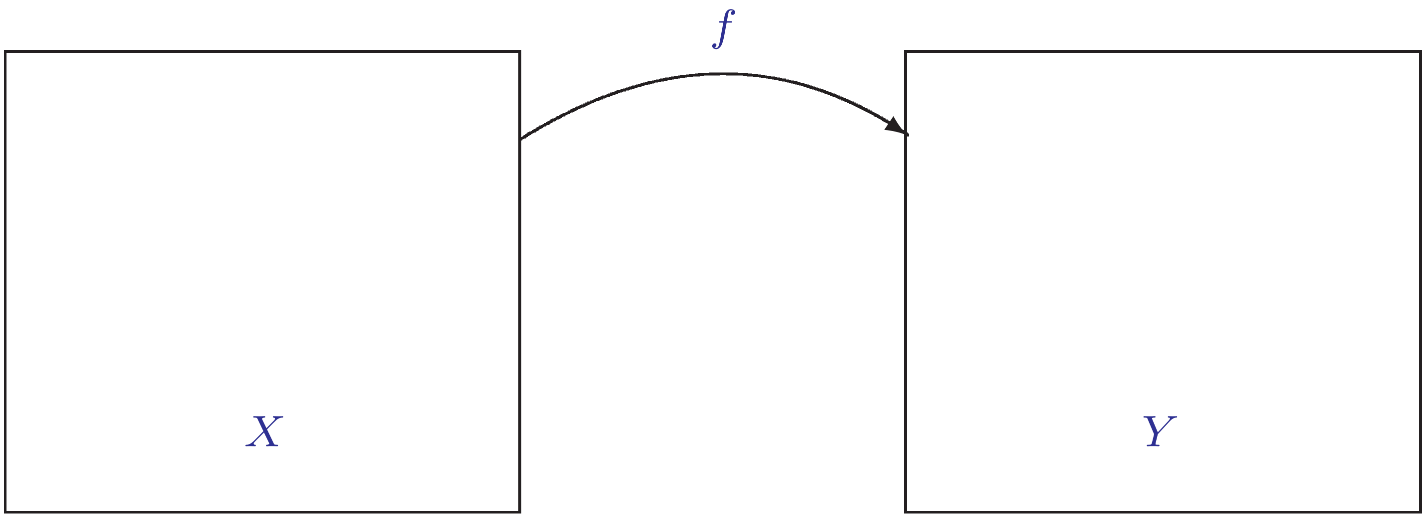
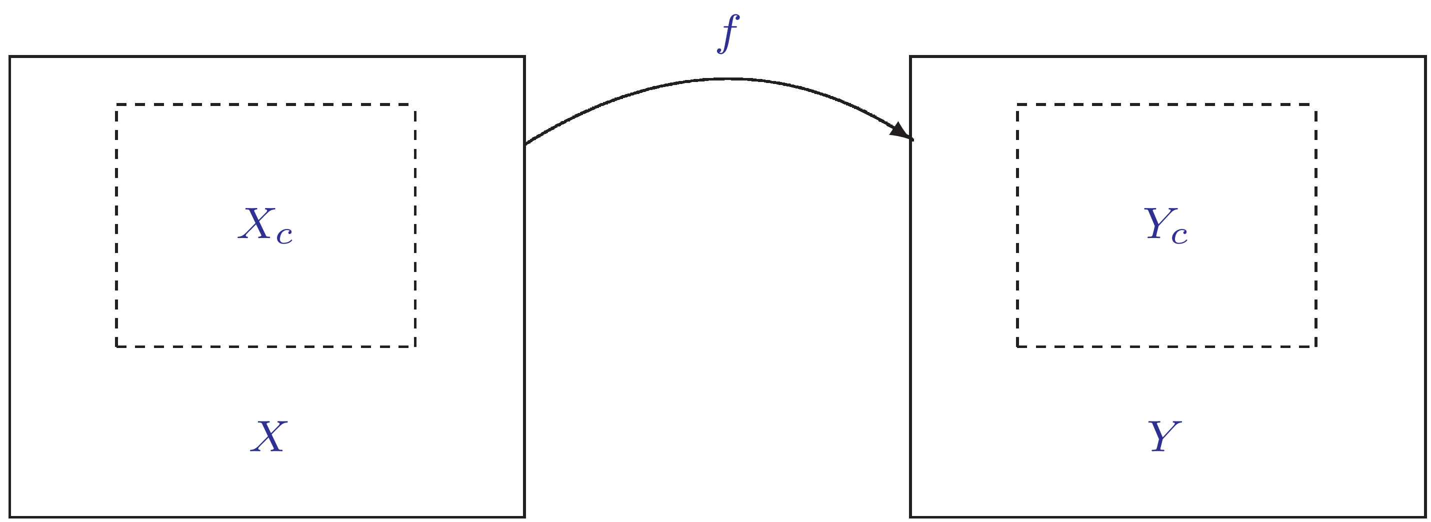


- *Uniform Computability:* The function $f : X \to Y$ is computable.

Uniform and Non-Uniform Computability



- *Uniform Computability:* The function $f : X \to Y$ is computable.
- *Non-Uniform Computability:* The function f maps computable elements to computable elements (i.e. $f(X_c) \subseteq f(Y_c)$).

Banach's Inverse Mapping Theorem

Definition 50 A Banach space or a normed space X together with a dense sequence is called *computable* if the induced metric space is a computable metric space.

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Theorem 51 *Let* X, Y *be Banach spaces and let* $T : X \to Y$ *be a linear operator. If* T *is bijective and bounded, then* $T^{-1} : Y \to X$ *is bounded.*

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Question: Given X and Y are computable Banach spaces, which of the following properties hold true under the assumptions of the theorem:

1. *Non-uniform inversion problem:*
   T computable $\Longrightarrow T^{-1}$ computable?

2. *Uniform inversion problem:*
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An Initial Value Problem

Theorem 53 *Let* $f_0, ..., f_n : [0, 1] \to \mathbb{R}$ *be computable functions with* $f_n \neq 0$. *The solution operator* $L : \mathcal{C}[0, 1] \times \mathbb{R}^n \to \mathcal{C}^{(n)}[0, 1]$ *which maps each tuple* $(y, a_0, ..., a_{n-1}) \in \mathcal{C}[0, 1] \times \mathbb{R}^n$ *to the unique function* $x = L(y, a_0, ..., a_{n-1})$ *with*

$$\sum_{i=0}^{n} f_i(t)x^{(i)}(t) = y(t) \text{ with } x^{(j)}(0) = a_j \text{ for } j = 0, ..., n-1,$$

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is computable.

Proof. The following operator is linear and computable:

$$L^{-1} : \mathcal{C}^{(n)}[0,1] \to \mathcal{C}[0,1] \times \mathbb{R}^n, x \mapsto \left(\sum_{i=0}^{n} f_i x^{(i)}, x^{(0)}(0), ..., x^{(n-1)}(0) \right)$$

Computability follows since the i–th differentiation operator is computable. By the computable Inverse Mapping Theorem it follows that L is computable too. □

Non-Constructive Existence Proofs of Algorithms

- The inverse $T^{-1} : Y \to X$ of any bijective and computable linear operator $T : X \to Y$ is computable.

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- This method is highly non-constructive: the existence of algorithms is ensured without any hint how they could look like.
- In the finite dimensional case the method is even constructive: an algorithm of T^{-1} can be effectively determined from an algorithm of T.

Operator Spaces in Computable Functional Analysis

- It is known that the map $\mathrm{Inv}:\mathcal{B}(X,Y)\to\mathcal{B}(Y,X), T\mapsto T^{-1}$ is continuous with respect to the operator norm $||T|| := \sup_{||x||=1} ||Tx||$ (Banach's Uniform Inversion Theorem)

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- However, $||\ || :\subseteq \mathcal{C}(X,Y) \to \mathbb{R}, T \mapsto ||T||$ is lower semi-computable.

Uniformity of Banach's Inverse Mapping Theorem

Theorem 54 *Let X, Y be computable normed spaces. The map*

$$\iota :\subseteq \mathcal{C}(X,Y) \times \mathbb{R} \to \mathcal{C}(Y,X), (T,s) \mapsto T^{-1},$$

defined for all (T,s) such that $T : X \to Y$ is a linear bounded and bijective operator such that $||T^{-1}|| \leq s$, is computable.

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Corollary 55 *Let X, Y be computable normed spaces. The map*

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Proof. The map $\mathrm{id} : \mathbb{R}_< \to \mathbb{R}_>$ is $\mathbf{\Sigma}^0_2$–computable and

$$||\mathrm{Inv}|| :\subseteq \mathcal{C}(X,Y) \to \mathbb{R}_<, T \mapsto ||T^{-1}|| = \sup_{||Tx|| \leq 1} ||x||$$

is computable. Altogether, this implies that Inv is $\mathbf{\Sigma}^0_2$–computable. □

Computable Linear Operators

Theorem 56 *Let X, Y be computable normed spaces, let $T : X \to Y$ be a linear operator and let $(e_n)_{n\in\mathbb{N}}$ be a computable sequence in X whose linear span is dense in X. Then the following are equivalent:*

1. *$T : X \to Y$ is computable,*
2. *$(T(e_n))_{n\in\mathbb{N}}$ is computable and T is bounded,*
3. *T maps computable sequences to computable sequences and is bounded,*
4. *$\mathrm{graph}(T)$ is a recursive closed subset of $X \times Y$ and T is bounded,*
5. *$\mathrm{graph}(T)$ is an r.e. closed subset of $X \times Y$ and T is bounded.*

In case that X and Y are even Banach spaces, one can omit boundedness in the last two cases.

The Uniform Closed Graph Theorem

Theorem 57 *Let X, Y be computable normed spaces. Then*

$$\mathrm{graph} : \mathcal{C}(X, Y) \to \mathcal{A}(X \times Y), f \mapsto \mathrm{graph}(f)$$

is computable. The partial inverse graph^{-1}, defined for linear bounded operators, is $\mathbf{\Sigma}^0_2$–computable.

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Proof. The following maps have the following computability properties:

- $\gamma :\subseteq \mathcal{A}(X \times Y) \times \mathbb{R} \to \mathcal{C}(X,Y), (\mathrm{graph}(T), s) \mapsto T$ is computable, (and defined for all graphs of linear bounded T such that $||T|| \leq s$),
- $N :\subseteq \mathcal{A}(X \times Y) \to \mathbb{R}_<, \mathrm{graph}(T) \mapsto ||T|| = \sup\limits_{||x|| \leq 1} ||Tx||$ is computable (and defined for all graphs of linear bounded T),
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□

The Open Mapping Theorem

Theorem 58 *Let* X, Y *be Banach spaces. If* $T : X \to Y$ *is a linear bounded and surjective operator, then* T *is open, i.e.* $T(U) \subseteq Y$ *is open for any open* $U \subseteq X$.

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Question: Given X and Y are computable Banach spaces, which of the following properties hold true under the assumptions of the theorem:

1. $U \subseteq X$ r.e. open $\Longrightarrow T(U) \subseteq Y$ r.e. open?
2. $\mathcal{O}(T) : \mathcal{O}(X) \to \mathcal{O}(Y), U \mapsto T(U)$ is computable?
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Note the different levels of uniformity: the Open Mapping Theorem is uniformly computable in U but only non-uniformly computable in T.

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- $T \mapsto \mathcal{O}(T)$ is $\mathbf{\Sigma}^0_2$–computable.

The Hahn-Banach Theorem

Theorem 59 (Hahn-Banach Theorem) *Let* X *be a normed space and* $Y \subseteq X$ *a linear subspace. Any linear bounded functional* $f : Y \to \mathbb{R}$ *admits a linear bounded extension* $g : X \to \mathbb{R}$ *with* $||g|| = ||f||$.

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   f computable $\Longrightarrow \exists$ a computable extension g?
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Nerode and Metakides also proved that the non-uniform version is computable in the finite dimensional case.

The Finite-Dimensional Case

Theorem 60 (Metakides and Nerode) *Let X be a finite-dimensional computable Banach space with some closed linear subspace $Y \subseteq X$. For any computable linear functional $f : Y \to \mathbb{R}$ with computable norm $||f||$ there exists a computable linear extension $g : X \to \mathbb{R}$ with $||g|| = ||f||$.*

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Lemma 61 *Let $(X, ||\ ||)$ be a normed space, $Y \subseteq X$ a linear subspace, $x \in X$ and let Z be the linear subspace generated by $Y \cup \{x\}$. Let $f : Y \to \mathbb{R}$ be a linear functional with $||f|| = 1$. A functional $g : Z \to \mathbb{R}$ with $g|_Y = f|_Y$ is a linear extension of f with $||g|| = 1$, if and only if*

$$\sup_{u \in Y}(f(u) - ||x - u||) \leq g(x) \leq \inf_{v \in Y}(f(v) + ||x - v||).$$

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Computable Hilbert Spaces

Definition 62 A computable Hilbert space is a computable Banach space which is a Hilbert space (i.e. whose norm is induced by a scalar product).

Theorem 63 (Hahn-Banach Theorem) *Let X be a Hilbert space and $Y \subseteq X$ a linear subspace. Any linear bounded functional $f : Y \to \mathbb{R}$ admits a uniquely determined linear bounded extension $g : X \to \mathbb{R}$ with $||g|| = ||f||$.*

Question: Given X and Y are computable Hilbert spaces, which of the following properties hold true:

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Survey on Results

	non-uniform		uniform	
dimension	finite	infinite	finite	infinite

Banach spaces

Open Mapping Theorem	computable		computable	Σ^0_2–computable
Banach's Inverse Mapping Theorem	computable		computable	Σ^0_2–computable
Closed Graph Theorem	computable		computable	Σ^0_2–computable
Hahn-Banach Theorem	computable	Σ^0_2–computable	Σ^0_2–computable	

Hilbert spaces

Hahn-Banach Theorem	computable	computable

The realizers of these theorems are not Σ^0_2–complete in general.

Survey on Different Types of Effective Mathematics

Effective Mathematics	Uniformity	Degrees of Effectivity
constructive analysis	fully uniform	principles of omniscience
reverse analysis over RCA_0	non-uniform	comprehension axioms
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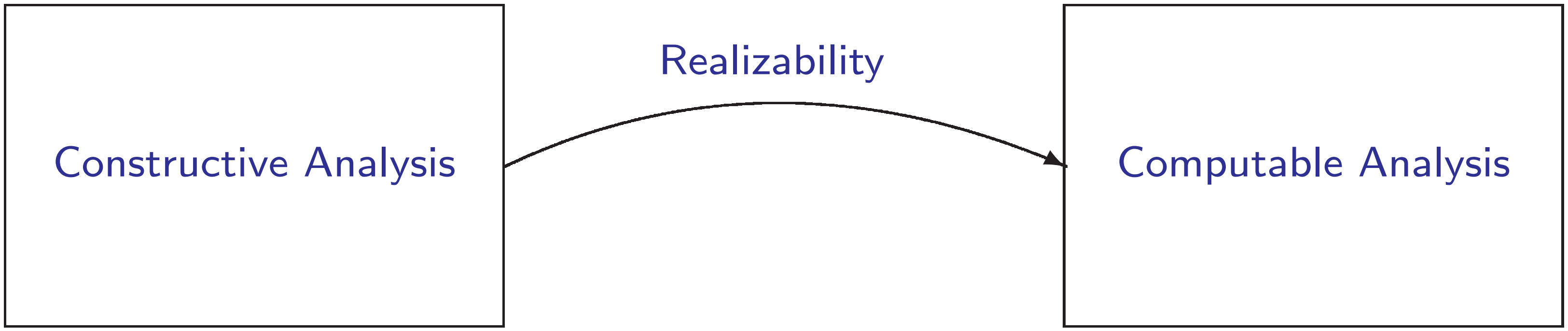
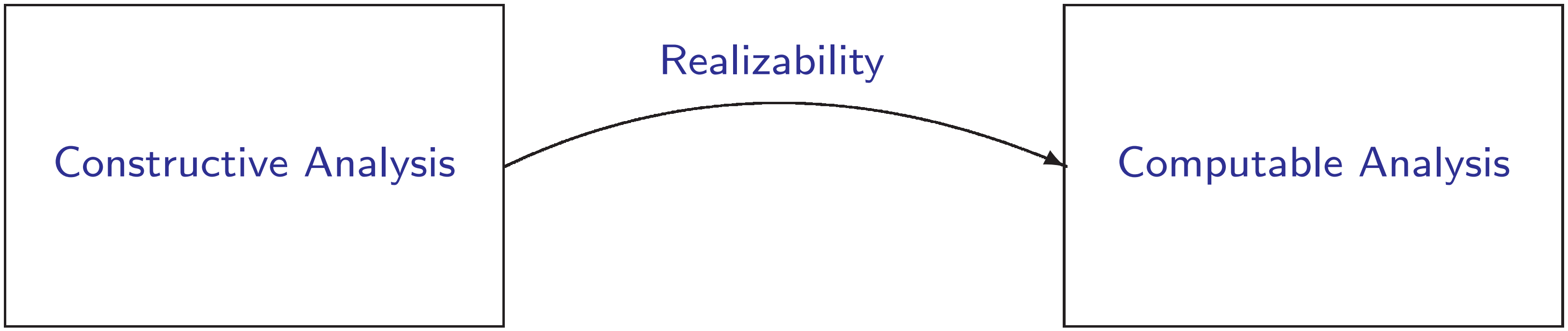
- Uniform reverse analysis (Kohlenbach) allows to express higher degrees of uniformity.
- Reverse analysis with intuitionistic logic (Ishihara) is automatically fully uniform.
- Constructive analysis allows to retranslate non-uniform results into (more complicated) double negation statements that might be provable intuitionistically.

Constructive and Computable Mathematics



Constructive and Computable Mathematics



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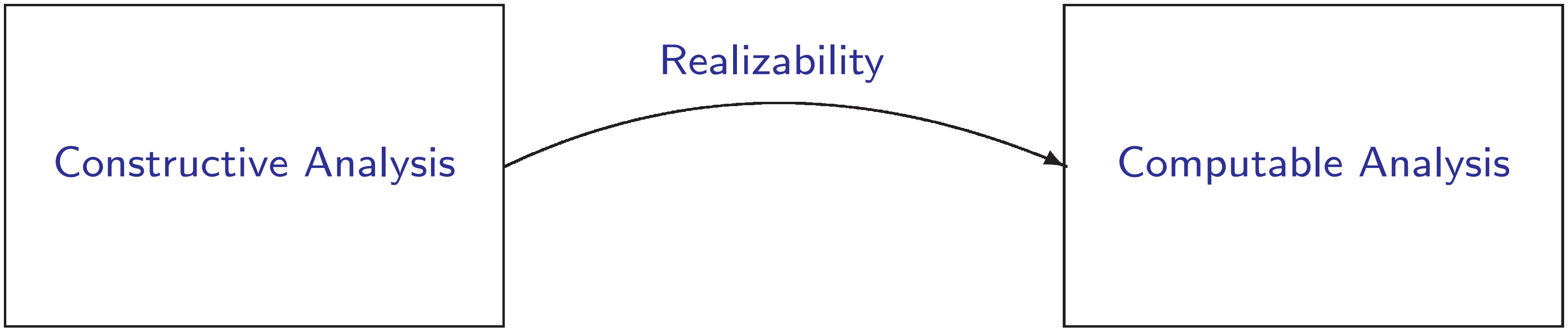
Constructive and Computable Mathematics



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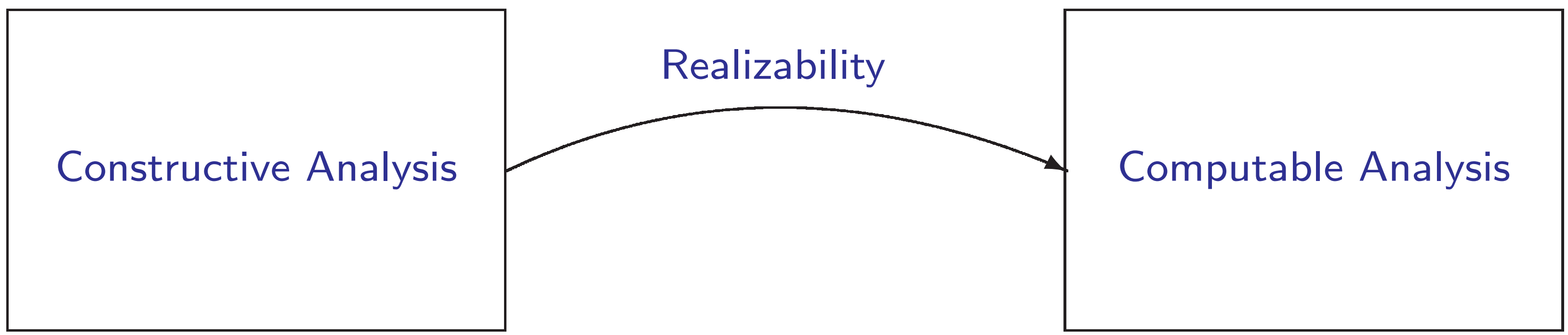

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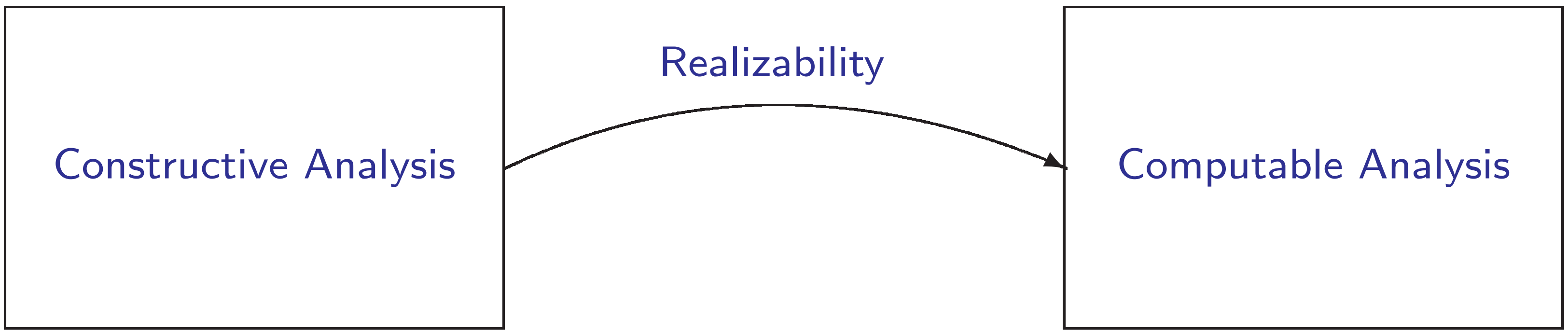

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- Some Theorems in Constructive Analysis, if interpreted via realizability, lead to tautologies in Computable Analysis.
  Example: Banach's Inverse Mapping Theorem.

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