Computational Complexity of NL1 with Assumptions

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Introduction

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- Lambek Calculus (associative and non-associative) was introduced by Lambek in 1958 in order to consider formal grammars as deductive systems.
- The P-TIME decidability for Classical Non-associative Lambek Calculus (NL) was established by de Groote and Lamarche in 2002.
- Buszkowski in 2005 showed that systems of Non-associative Lambek Calculus with finitely many nonlogical axioms are decidable in polynomial time and grammars based on these systems generate context-free languages.

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- We consider Non-associative Lambek Calculus with identity and a finite set of nonlogical axioms and prove that such system is decidable in polynomial time.
- To obtain this result the method used by Buszkowski in (2005) was adapted.

The formalism of NL1

Types of NL1:

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- $\mathrm{At} = \{\mathrm{p}, \mathrm{q}, \mathrm{r}, \ldots\}$ - the denumerable set of atoms (also called primitive types)
- $\mathrm{Tp1}$ - the set of formulas (also called types):
  - $\mathbf{1} \in \mathrm{Tp1}$,
  - $\mathrm{At} \subseteq \mathrm{Tp1}$,
  - if $A, B \in \mathrm{Tp1}$, then $(A \bullet B) \in \mathrm{Tp1}, (A/B) \in \mathrm{Tp1}, (A \backslash B) \in \mathrm{Tp1}$, where binary connectives $\backslash$, $/$, $\bullet$, are called *left residuation, right residuation*, and *product*, respectively.

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We set $(X \circ \Lambda) = (\Lambda \circ X) = X$.
Notations:

- $X[Y]$ - a formula structure X with a distinguished substructure Y
- $X[Z]$ - the substitution of Z for Y in X

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Gentzen-style axiomatization of NL1.
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For any system S we write $\mathrm{S} \vdash X \to A$ if the sequent $X \to A$ is derivable in S.

NL1 with assumptions

- By $\mathrm{NL1}(\Gamma)$ we denote the calculus $\mathrm{NL1}$ with additional set Γ of assumptions, where Γ is a finite set of sequents of the form $A \rightarrow B$, and $A, B \in \mathrm{Tp1}$.

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- We use in Γ sequents of the form $A \rightarrow B$ for simplicity, but the set Γ may consist of arbitrary sequents.
- It is easy to show that for any finite set of sequents Γ there is a set Γ' of sequents of the form $A \rightarrow B$ such that systems $\mathrm{NL1}(\Gamma)$ and $\mathrm{NL1}(\Gamma')$ are equivalent.

Remarks

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- The decidable procedure for NL1 rely on cut elimination which yields the subformula property.
- For the case of $\mathrm{NL1}(\Gamma\ )$ cut elimination is not possible, hence for this system subformula property is established in a different way.

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- T-sequent - a sequent $X \to A$ such that A and all formulas appearing in X belong to T.
- We write: $\mathrm{NL1}(\Gamma) \vdash X \to_T A$ if a sequent $X \to A$ has a proof in $\mathrm{NL1}(\Gamma)$ consisting of T-sequents only.

Subformula property for $\mathrm{NL1}(\Gamma)$

Lemma 1

For every T-sequents $X \rightarrow A$,
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- The most general algebraic models of NL1: residuated groupoids with identity.
- The model used in the proof of lemma 1: The residuated groupoid with identity of cones over the given preordered groupoid with identity.

Remarks to the proof of lemma 1

The preordered groupoid considered in the proof is a structure $(M, \leq, \circ, \Lambda)$, where

- M is a set of all formula structures all of whose atomic substructures belong to T and $\Lambda \in M$
- Preordering $\leq$ is a reflexive and transitive closure of the relation $\leq_b$ defined as follows:
  - $Y[Z] \leq_b Y[\Lambda] \quad \text{if} \quad Z \rightarrow_T \mathbf{1}$,
  - $Y[Z] \leq_b Y[A] \quad \text{if} \quad Z \rightarrow_T A$,
  - $Y[A \bullet B] \leq_b Y[A \circ B] \quad \text{if} \quad A \bullet B \in T$.

In the proof we use the fact, that every sequent provable in $\mathrm{NL1}(\Gamma)$ is true in the model $(\mathcal{C}(M), \mu)$, where

- $\mathcal{C}(M)$ is the residuated groupoid of cones with identity over preordered groupoid $(M, \leq, \circ, \Lambda)$ defined above,
- An assignment μ on $\mathcal{C}(M)$ is defined by setting:

$$\mu(p) = \{X \in M : X \rightarrow_T p\},$$

for all atoms p.

Basic sequents

A sequent is said to be *basic* if it is a T-sequent of the form $\Lambda \rightarrow A$, $A \rightarrow B$, $A \circ B \rightarrow C$.

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- We remaind that T is a finite set of formulas, closed under subformulas and such that $\mathbf{1} \in T$ and T contains all formulas appearing in Γ.
- For such T we shall describe an effective procedure which produces the set S^T consists of all basic sequents derivable in $\mathrm{NL1}(\Gamma)$.

Construction of the set S^T

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- $\Lambda \rightarrow \mathbf{1}$
- all T-sequents of the form (Id)
- all sequents from Γ
- all T-sequents of the form:
  - $\mathbf{1} \circ A \rightarrow A$, $A \circ \mathbf{1} \rightarrow A$,
  - $A \circ B \rightarrow A \bullet B$,
  - $A \circ (A \backslash B) \rightarrow B$, $(A / B) \circ B \rightarrow A$.

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Assume S_n has already been defined.
S_{n+1} is S_n enriched with sequents resulting from the following rules:

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Assume S_n has already been defined.
S_{n+1} is S_n enriched with sequents resulting from the following rules:

(S1) if $(A \circ B \to C) \in S_n$ and $(A \bullet B) \in T$, then $(A \bullet B \to C) \in S_{n+1}$,

(S2) if $(A \circ X \to C) \in S_n$ and $(A \backslash C) \in T$, then $(X \to A \backslash C) \in S_{n+1}$,

(S3) if $(X \circ B \to C) \in S_n$ and $(C / B) \in T$, then $(X \to C / B) \in S_{n+1}$,

(S4) if $(\Lambda \to A) \in S_n$ and $(A \circ X \to C) \in S_n$, then $(X \to C) \in S_{n+1}$, (S5) if $(\Lambda \rightarrow A) \in S_n$ and $(X \circ A \rightarrow C) \in S_n$, then $(X \rightarrow C) \in S_{n+1}$,

(S6) if $(A \rightarrow B) \in S_n$ and $(B \circ X \rightarrow C) \in S_n$, then $(A \circ X \rightarrow C) \in S_{n+1}$,

(S7) if $(A \rightarrow B) \in S_n$ and $(X \circ B \rightarrow C) \in S_n$, then $(X \circ A \rightarrow C) \in S_{n+1}$,

(S8) if $(A \circ B \rightarrow C) \in S_n$ and $(C \rightarrow D) \in S_n$, then $(A \circ B \rightarrow D) \in S_{n+1}$.

(S5) if $(\Lambda \rightarrow A) \in S_n$ and $(X \circ A \rightarrow C) \in S_n$, then $(X \rightarrow C) \in S_{n+1}$,

(S6) if $(A \rightarrow B) \in S_n$ and $(B \circ X \rightarrow C) \in S_n$, then $(A \circ X \rightarrow C) \in S_{n+1}$,

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(S8) if $(A \circ B \rightarrow C) \in S_n$ and $(C \rightarrow D) \in S_n$, then $(A \circ B \rightarrow D) \in S_{n+1}$.

Clearly, $S_n \subseteq S_{n+1}$ for all $n \geq 0$.
We define S^T as the join of this chain.

Properties of the set S^T

$$S^T = \bigcup_{n=0}^{\infty} S_n$$

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- S^T is a set of basic sequents, hence it must be finite.
- It yields $S^T = S_{k+1}$, for the least k such that $S_k = S_{k+1}$, and this k is not greater then the number of basic sequents.

P-TIME decidability of S^T

Fact

The set S^T can be constructed in polynomial time.

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- There are n, n^2 and n^3 basic sequents of the form $\Lambda \rightarrow A$, $A \rightarrow B$ and $A \circ B \rightarrow C$, respectively.
- Hence, we have $m = n^3 + n^2 + n$ basic sequents.

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- The set S_0 can be constructed in time $0(n^2)$.

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- Hence, we have $m = n^3 + n^2 + n$ basic sequents.
- The set S_0 can be constructed in time $0(n^2)$.
- To get S_{i+1} from S_i we must close S_i under the rules (S1)-(S8) which can be done in at most m^3 steps for each rule.

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- The set S_0 can be constructed in time $0(n^2)$.
- To get S_{i+1} from S_i we must close S_i under the rules (S1)-(S8) which can be done in at most m^3 steps for each rule.
- The least k such that $S^T = S_k$ is at most m.

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- The least k such that $S^T = S_k$ is at most m.
- Then finely, we can construct S^T from T in time $0(m^4) = 0(n^{12})$.

Auxiliary systems

Now we take into consideration two auxiliary systems.

System $S(T)$:

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Lemma 2

For any sequent $X \rightarrow A$:

$$S(T) \vdash X \rightarrow A \quad \text{iff} \quad S(T)^- \vdash X \rightarrow A.$$

Interpolation for $S(T)$

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If $S(T) \vdash X[Y] \to A$, then there exists $D \in T$ such that

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Lemma 4

For any T-sequent $X \to A$:

$$\mathrm{NL1}(\Gamma) \vdash X \to_T A \quad \text{iff} \quad S(T) \vdash X \to A.$$

P-TIME decidbility of $\mathrm{NL1}(\Gamma)$

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- As T we choose the set of all subformulas of formulas appearing in $X \to A$, formulas appearing in Γ and $\mathbf{1} \in T$.
- Hence, T has n elements and we can construct it in time $0(n^2)$.

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- By lemma 1 and 4 we have:
  $\mathrm{NL1}(\Gamma) \vdash X \rightarrow A \quad \text{iff} \quad X \rightarrow_T A$,
  $X \rightarrow_T A \quad \text{iff} \quad S(T) \vdash X \rightarrow A$.

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- Checking derivability in context-free grammars is P-TIME decidable. For example, by known CYK algorithm, it can be done in time not exceed $k \cdot n^3$, where k is the size of S^T.

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- Checking derivability in context-free grammars is P-TIME decidable. For example, by known CYK algorithm, it can be done in time not exceed $k \cdot n^3$, where k is the size of S^T.
- The size of S^T is at most $0(n^3)$ and S^T can be constructed in $0(n^{12})$.

Hence, the total time is $0(n^{12})$, i.e. $\mathrm{NL1}(\Gamma)$ is P-TIME decidable.

Further results

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- $\mathrm{NL1P}(\Gamma)$ - $\mathrm{NL1}(\Gamma)$ with the permutation rule

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Theorem 1 can also be proven for systems:

- $\mathrm{NL1P}(\Gamma)$ - $\mathrm{NL1}(\Gamma)$ with the permutation rule
- $\mathrm{GLC}(\Gamma)$ - Generalized Lambek Calculus with assumptions enriched with the permutation rule and/or identity for some product symbols

Main bibliography

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