A FORCING EXTENSION OF A SIMPLIFIED $(\omega_2, 1)$ MORASS WITH NO SIMPLIFIED $(\omega_2, 1)$ MORASS WITH LINEAR LIMITS

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Old statement:using supercompact cardinals

- Proof

New statement:using strongly unfoldable cardinals

- Evidence
- Proof

old statement

$$\text{Con}(\mathit{ZFC} + \exists\, \kappa\, \text{supercompact cardinal}) \Longrightarrow$$

$$\text{Con}(\mathit{ZFC} + \exists(\omega_2, 1)\text{morass} + \neg\exists(\omega_2, 1) - \text{morass with linear limits})$$

(Stanley)

Supercompact cardinals

Definition

κ is a θ-supercompact cardinal iff there exists $j : V \to M$ such that $cp(j) = \kappa$ and $M^\theta \subseteq M$.

κ is supercompact iff for all $\theta \in On$, κ is θ-supercompact.

Supercompactness $\Longrightarrow V \neq L$.

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Simplified morasses

Definition

κ regular cardinal. A simplified $(\kappa, 1)$ morass is a sequence $\langle \varphi_\xi; \mathcal{G}_{\xi\tau} : \xi < \tau \leq \kappa \rangle$ where

$$\mathcal{G}_{\xi\tau} = \{b : \varphi_\xi \to \varphi_\tau \mid b \text{ order preserving}\}$$

such that:

- $\varphi_\xi < \kappa$ and $|\mathcal{G}_{\xi\tau}| < \kappa$ for $\xi < \tau < \kappa$ and $\varphi_\kappa = \kappa^+$.
- Coherence.
- $\mathcal{G}_{\xi\xi+1} = \{id, f\}$ where f is a split function.
- If $\lim(\xi)$ $\varphi_\xi = \bigcup_{\eta<\xi}\{b''\varphi_\eta \mid b \in \mathcal{G}_{\eta\xi}\}$.

Facts about morasses

- Simplified $(\kappa, 1)$ morasses implies the gap 2 cardinal theorem.
- There are simplified $(\omega, 1)$ morasses.
- If $V = L$ then for κ regular cardinal there are simplified $(\kappa, 1)$ morasses.
- Simplified $(\kappa, 1)$ morass implies $\square_{\kappa,\kappa}$.
- Simplified $(\kappa, 1)$ morass with linear limits implies $\square_{\kappa}$.

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Proof using a supercompact cardinal

- Laver: If κ supercompact cardinal, then there is a forcing extension such that κ is still supercompact and it is indestructible under κ-directed closed forcings.
- The forcing which adds a simplified $(\kappa, 1)$ morass is κ- closed.
- Collapse κ to ω_2.

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$V = L$

- For any κ cardinal, $\Box_\kappa$.
- For κ regular cardinal, there are $(\kappa, 1)$-morasses (Jensen).
- Weakly compact cardinals, (strongly) unfoldable cardinals relativized to L.
- κ is weakly compact iff there is no $(\kappa, 1)$-morass with linear limits (Donder).

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strongly unfoldable cardinals

Definition

Let κ be an inaccessible cardinal, M is a κ-model iff M is a transitive, $M \models ZF^-$, $|M| = \kappa$ with $\kappa \in M$ and $M^{<\kappa} \subseteq M$.

Definition

κ is θ-strongly unfoldable cardinal iff $\forall M\ (M\ \kappa - \text{model} \Longrightarrow \exists j, N[N\text{transitive},\ V_\theta \subseteq N,\ j : M \to N, cp(j) = \kappa, j(\kappa) \geq \theta])$. (Villaveces)

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κ is strongly unfoldable iff for all $\theta \in On$, κ is a θ-strongly unfoldable cardinal.

Fact: κ is weakly compact cardinal iff κ is κ-unfoldable cardinal.

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Laver's preparation for other large cardinals

For κ strong, strongly compact, measurable and strongly unfoldable cardinals (Hamkins):

In all cases: lottery preparation relative to a function $f : \kappa \longrightarrow \kappa$ such that $j(f)(\kappa)$ is an ordinal arbitrary high below $j(\kappa)$.

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κ-properness forcing or preserving κ^+

If κ is strongly unfoldable cardinal, after the lottery preparation relative to f, κ strongly unfoldability is preserved by any $\mathbb{P}$ $< \kappa$-closed, κ-proper forcing (Hamkins, Johnstone)

($2^{<\kappa} = \kappa$) The forcing which adds a $(\kappa, 1)$ morass is κ-closed and κ^+-c.c.

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New statement using unfoldable cardinals

$$\text{Con}(ZFC + \exists\,\kappa\,\text{strongly unfoldable cardinal}) \Longrightarrow$$

$$\text{Con}(ZFC + \exists(\omega_2, 1)\text{morass} + \neg\exists(\omega_2, 1) - \text{morass with linear limits})$$

Proof:

- Let κ be strongly unfoldable cardinal and M a κ-model, there exists an embedding $j : M \to N$ with $cp(j) = \kappa$ and...
- Find a function $f : \kappa \to \kappa$ such that $j(f)(\kappa)$ guess any value below $j(\kappa)$ (for free).
- Apply the lottery preparation to κ using f.
- Add the simplified $(\kappa, 1)$ morass. κ is still strongly unfoldable cardinal.
- Collapse κ to ω_2.
- There is a simplified $(\omega_2, 1)$ morass but it is false $\square_{\omega_2}$.

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Thanks!