Structural Completeness for Fuzzy Logics

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Outline

- basic definitions
- passive structural completeness
- hereditary $\mathcal{SC}$ and deduction theorem
- results in particular fuzzy logics

Basic definitions

Rule: pair $T \triangleright \varphi$, where T is a finite set of formulas and φ a formula

Logic $\mathbf{L}$: a structural finitary consequence relation

set of rules closed under substitutions and Tarski's conditions

Extension of logic $\mathbf{L}$: any *logic* containing $\mathbf{L}$

Definition a logic is $\mathcal{SC}$ iff each of its extensions has new theorems

Basic definitions

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Definition a logic is $\mathcal{SC}$ iff each of its extensions has new theorems

Derivable rule: a rule $T \rhd \varphi$ is derivable in $\mathbf{L}$ iff $T \vdash_{\mathbf{L}} \varphi$

Admissible rule: a rule $T \rhd \varphi$ is admissible in $\mathbf{L}$ iff for each substitution σ if $\vdash_{\mathbf{L}} \sigma(T)$ then $\vdash_{\mathbf{L}} \sigma(\varphi)$

Equivalent def. a logic is $\mathcal{SC}$ iff each admissible rule is derivable

Passive structural completeness

Admissible rule: a rule $T \rhd \varphi$ is admissible in $\mathbf{L}$ iff for each substitution σ: (there is $\psi \in T$ s.t. $\nvdash_{\mathbf{L}} \sigma(\psi)$) OR ($\vdash_{\mathbf{L}} \sigma(\varphi)$)

Passive rule: a rule $T \rhd \varphi$ is passive in $\mathbf{L}$ iff for each substitution σ: there is $\psi \in T$ s.t. $\nvdash_{\mathbf{L}} \sigma(\psi)$

Setting: assume from now on that $\mathbf{L}$ is consistent

Observation: $T \rhd \varphi$ is passive iff the rule $T \rhd v$ is admissible

assuming that v does not occur in T

Convention: call rule $T \vdash v$ a *rule with inconsistent conclusion*—RIC

Definition: a logic is $\mathcal{PSC}$ iff each admissible RIC is derivable

Observation: a logic is $\mathcal{PSC}$ iff each passive rule is derivable

$\mathcal{PSC}$ upwards and an example

Theorem Any extension of a logic with $\mathcal{PSC}$ is $\mathcal{PSC}$

$\mathcal{PSC}$ upwards and an example

Theorem Any extension of a logic with $\mathcal{PSC}$ is $\mathcal{PSC}$

Rule $v \leftrightarrow \neg v \vdash p$ is passive in Ł$_3$

it is passive already in *classical* logic

Rule $v \leftrightarrow \neg v \vdash p$ is not derivable in Ł$_3$

evaluate both v and p by $\frac{1}{2}$

Conclusion: Ł$_3$ is not $\mathcal{PSC}$

and so it also in not $\mathcal{SC}$

Corollary: Any logic in language of Ł$_3$ weaker than Ł$_3$ is not $\mathcal{PSC}$

and so it also in not $\mathcal{SC}$

Corollary: the following logics lack $\mathcal{SC}$: FL_{ew}, AMALL, MTL, IMTL, BL, Ł.

$\mathcal{PSC}$ downwards

Ugly assumption Let $\mathcal{L}' \subseteq \mathcal{L}$ be languages and $\mathbf{L}$ a logic $\mathcal{L}$. $\mathbf{L}$ is *$\mathcal{L}'$-substitution friendly* if for each set of $\mathcal{L}'$-formulas T and each $\mathcal{L}$-substitution σ such that $\vdash_{\mathbf{L}} \sigma(T)$ there is an $\mathcal{L}'$-substitution σ' such that $\vdash_{\mathbf{L}} \sigma'(T)$.

Theorem Let $\mathbf{L}$ be an $\mathcal{L}'$-substitution friendly logic. If $\mathbf{L}$ is $\mathcal{PSC}$ then so is $\mathbf{L}{\restriction}\mathcal{L}'$.

Combining $\mathcal{PSC}$ downwards and upwards

Theorem Let $\mathbf{L}$ be a $\mathcal{L}'$-substitution friendly logic. If $\mathbf{L}$ is $\mathcal{PSC}$ then so is any logic extending $\mathbf{L}\restriction\mathcal{L}'$.

Corollary Let $\mathbf{L}$ be a logic in the language $\mathcal{L}$. If there a language $\mathcal{L}' \subseteq \mathcal{L}$ such that $\mathbf{L}$ is $\mathcal{L}'$-substitution friendly and there is a logic $\mathbf{L}'$ extending $\mathbf{L}\restriction\mathcal{L}'$ which is not $\mathcal{PSC}$, then $\mathbf{L}$ is not (passively) $\mathcal{SC}$.

Substitution friendliness

Setting $\mathbf{L}$ is a weakly implicative logic and $\{\to\} \subseteq \mathcal{L}' \subseteq \mathcal{L}$.

Theorem $\mathbf{L}$ is $\mathcal{L}'$-substitution friendly if one of the following holds:

- for each set $\mathcal{L}$-formulas $\varphi_1, \dots, \varphi_n, \dots$ there is $\mathcal{L}$-substitution σ and $\mathcal{L}'$-formulas $\psi_1, \dots, \psi_n, \dots$ such that $\sigma(\varphi_i) \rightleftarrows \psi_i$ are theorems of $\mathbf{L}$ for each i.
- there is $\mathcal{L}$-substitution σ such that for each $\mathcal{L}$-formula φ there is an $\mathcal{L}'$-formula ψ such that $\sigma(\varphi) \rightleftarrows \psi$ are theorems of $\mathbf{L}$.
- there is a set of $\mathcal{L}'$-formulas Ψ, such that for each n-ary connective $c \in \mathcal{L}$ and formulas $\psi_1, \dots, \psi_n \in \Psi$ there is $\psi \in \Psi$ such that $c(\psi_1, \dots, \psi_n) \rightleftarrows \psi$ are theorems of $\mathbf{L}$.

Corollary Let $\{\to\} \subseteq \mathcal{L}' \subseteq \mathcal{L} \subseteq \mathcal{L}_{\mathsf{FL}}$, $\mathbf{L}$ be an implicative logic extending $\mathsf{FL}_w \upharpoonright \mathcal{L}$, and $\bot$ is definable in $\mathbf{L} \upharpoonright \mathcal{L}'$. Then $\mathbf{L}$ is $\mathcal{L}'$-substitution friendly.

Application(s)

Lemma n-valued Łukasiwicz logic is not $\mathcal{PSC}$

Corollary Let $\mathbf{L}$ be an implicative logic in a language $\{\to\} \subseteq \mathcal{L} \subseteq \mathcal{L}_{\mathsf{FL}}$. Further assume that

- $\bot$ is definable in $\mathbf{L}{\restriction}\mathcal{L}$
- $\mathbf{L}$ is an extension of $\mathsf{FL}_w{\restriction}\mathcal{L}$
- there is a natural $n \geq 3$ such that n-valued Łukasiwicz logic is an extension of $\mathbf{L}{\restriction}\{\to, \bot\}$.

Then $\mathbf{L}$ is not (passively) $\mathcal{SC}$.

Corollary: the following logics lack $\mathcal{SC}$: FL_{ew}, AMALL, $\mathsf{S}_n\mathsf{FL}_{ew}$, $\mathsf{C}_n\mathsf{FL}_{ew}$, MTL, S_nMTL, C_nMTL, IMTL, S_nIMTL, C_nIMTL, BL, S_nBL, C_nBL, Ł.

Hereditary $\mathcal{SC}$ and $\mathcal{LDT}$

Definition: logic is $\mathcal{HSC}$ if all its extension are $\mathcal{SC}$.

Nice equivalences: $\mathbf{L}$ is $\mathcal{HSC}$ iff all its *axiomatic* extensions are $\mathcal{SC}$ iff all its extensions are *axiomatic*

Local deduction theorem: $\mathbf{L}$ has $\mathcal{LDT}$ if for each theory T and formulas φ, ψ there is a finite set of formulas $\Delta^{\mathbf{L}}_{T,\varphi,\psi}$ in two variables s.t. $T, \varphi \vdash \psi$ iff $T \vdash \Delta^{\mathbf{L}}_{T,\varphi,\psi}(\varphi, \psi)$. $\mathbf{L}$ has *normal* deduction theorem if furthermore $\Delta^{\mathbf{L}}_{T,\varphi,\psi}(\varphi, \psi), \varphi \vdash_{\mathbf{L}} \psi$

Global deduction theorem: $\mathbf{L}$ has $\mathcal{GDT}$ there is a finite set of formulas $\Delta^{\mathbf{L}}$ in two variables s.t. $T, \varphi \vdash \psi$ iff $T \vdash \Delta^{\mathbf{L}}_{T,\varphi,\psi}(\varphi, \psi)$

Hereditary $\mathcal{LDT}$: $\mathbf{L}$ has $\mathcal{HLDT}$ if each extension $\mathbf{L}'$ has $\mathcal{LDT}$ and $\Delta^{\mathbf{L}'}_{T,\varphi,\psi}(\varphi, \psi), \varphi \vdash_{\mathbf{L}} \psi$

Theorem and its applications

Theorem Let $\mathbf{L}$ be a logic with normal $\mathcal{LDT}$. Then $\mathbf{L}$ has $\mathcal{HLDT}$ iff $\mathbf{L}$ is $\mathcal{HSC}$.

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Corollary The following logics are $\mathcal{HSC}$:

- $\mathsf{C}_n\mathsf{FL}_{ew}\restriction\mathcal{L}$ for $\{\rightarrow\} \subseteq \mathcal{L} \subseteq \{\rightarrow, \wedge\}$

- $\mathsf{C}_n\mathsf{MTL}\restriction\mathcal{L}$ for $\{\rightarrow\} \subseteq \mathcal{L} \subseteq \{\rightarrow, \wedge, \vee\}$

- $\mathsf{C}_n\mathsf{BL}\restriction\mathcal{L}$ for $\{\rightarrow\} \subseteq \mathcal{L} \subseteq \{\rightarrow, \wedge, \vee, \&\}$

The following are provable in $C_{n+1}FL_{ew}$:

1. $(\varphi \to^n (\psi \to \chi)) \rightleftarrows ((\varphi \to^n \psi) \to (\varphi \to^n \chi))$

2. $(\varphi \to^n (\psi \wedge \chi)) \rightleftarrows ((\varphi \to^n \psi) \wedge (\varphi \to^n \chi))$

The following are provable in $C_{n+1}MTL$:

4. $(\varphi \to^n (\psi \vee \chi)) \rightleftarrows ((\varphi \to^n \psi) \vee (\varphi \to^n \chi))$

The following are provable in $C_{n+1}BL$:

5. $(\varphi \to^n (\psi \mathbin{\&} \chi)) \rightleftarrows ((\varphi \to^n \psi) \mathbin{\&} (\varphi \to^n \chi))$

Example of particular results in fuzzy logics

Theorem Any fragment of Cancellative hoop logic where t and $\odot$ are definable is structurally complete.

Suppose that $T \not\vdash \varphi$. Then there is a valuation v for $\mathbb{Z}^-$ such that $v(A) = 0$ for all $\psi \in T$ and $v(\varphi) < 0$. Let q be a propositional variable not occurring in Γ or B and define the substitution:

$$\sigma(p) = q^{|v(p)|}$$

Claim. $\vdash \sigma(\psi) \leftrightarrow q^{|v(\psi)|}$.

From the claim we get $\vdash \sigma(\psi)$ for all $\psi \in \Gamma$, and $\not\vdash \sigma(\varphi)$.

Fragments with $\to$ and without 0

Logic	$\to$	$\to, \wedge, \vee$	$\to, \vee$	$\to, \&$	$\to, \&, \wedge, \vee$
MTL = IMTL = SMTL	?	?	?	?	?
C_nMTL = C_nIMTL	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	?	?
CHL	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$
ΠMTL	?	?	?	?	?
BL = SBL	?	?	?	?	?
C_nBL	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$
G	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$
Ł	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$	$\mathcal{SC}$
Π	?	?	?	$\mathcal{HSC}$	$\mathcal{HSC}$

Fragments with $\rightarrow, \mathbf{0}$

Logic	$\rightarrow, \mathbf{0}$	$\rightarrow, \wedge, \vee, \mathbf{0}$	$\rightarrow, \vee, \mathbf{0}$	$\rightarrow, \&, \mathbf{0}$	$\rightarrow, \&, \mathbf{0}, \wedge, \vee$
MTL	No	No	No	No	No
C_nMTL	No	No	No	No	No
S_nMTL	No	No	No	No	No
IMTL	No	No	No	No	No
SMTL	?	?	?	?	?
ΠMTL	?	?	?	?	?
BL	No	No	No	No	No
C_nBL	No	No	No	No	No
S_nBL	No	No	No	No	No
SBL	?	?	?	?	?
G= C_2MTL	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$
G_n	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$	$\mathcal{HSC}$
Ł	No	No	No	No	No
Ł$_n$ = S_nŁ= C_nŁ	No	No	No	No	No
Π	?	?	?	$\mathcal{HSC}$	$\mathcal{HSC}$

Thank you for your attention