Partitioning κ-fold covers into κ many subcovers

Márton Elekes
emarci@renyi.hu
www.renyi.hu/˜emarci

Rényi Institute, Budapest, Hungary

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Joint work with Tamás Mátrai and Lajos Soukup.

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Outline

1. Introduction
   - The problem
   - Motivation
   - Two easy examples
2. New results
   - Convex bodies
   - Closed sets
   - Arbitrary sets
   - Graphs
3. Open problems

Definition

Let X be a set and κ be a cardinal (usually infinite). We say that $\mathcal{H} \subset P(X)$ is a κ-fold cover of X if each $x \in X$ is covered at least κ times.

Question

(Main question) *Under what assumptions can we decompose a κ-fold cover into κ many disjoint subcovers?*

An equivalent formulation:

Definition

Let $\mathcal{H} \subset P(X)$. We say that $c : \mathcal{H} \to \kappa$ is a good colouring with κ colours, (or a good κ-colouring), if $\forall x \in X$ and $\forall \alpha < \kappa$ $\exists H \in \mathcal{H}$ such that $x \in H$ and $c(H) = \alpha$.

Fact

$\mathcal{H}$ has a good κ-colouring iff it can be decomposed into κ many disjoint subcovers.

Remark

It would also be natural (and useful) to define these notions relative to a set $Y \subset X$, but for the sake of simplicity we stick to $Y = X$ in this talk.

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Theorem

(Mani-Pach, unpublished, more than 20 years old, ca. 100 pages) Every 33-fold cover of $\mathbb{R}^2$ with unit discs has a good 2-colouring.

Theorem

(Tardos-Tóth) Every 43-fold cover of $\mathbb{R}^2$ with translates of a triangle has a good 2-colouring.

Theorem

(Tóth, ???) For every convex polygon there exists $n \in \mathbb{N}$ so that every n-fold cover of $\mathbb{R}^2$ with translates of the polygon has a good 2-colouring.

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(Pach-Tardos-Tóth) For every $n \in \mathbb{N}$ there is an n-fold cover of $\mathbb{R}^2$ with axis-parallel rectangles or with translates of a suitable concave quadrilateral that has no good 2-colouring.

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Theorem

(Aharoni-Hajnal-Milner) Let κ be a cardinal (finite or infinite) and L be a linearly ordered set. Then every κ-fold cover of L consisting of convex sets has a good κ-colouring.

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Two easy examples

Statement

Let κ be infinite and X be a set with $|X| \leq \kappa$. Then every κ-fold cover of X has a good κ-colouring.

Proof Trivial transfinite recursion. Let $\{x_\alpha : \alpha < \kappa\}$ be so that each $x \in X$ occurs κ times. When x shows up for the α's time, there is an uncoloured H containing x, give it colour α. $\square$

Statement

Let κ be infinite and X be a set with $|X| \geq 2^\kappa$. Then there is a κ-fold cover of X that has not even a good 2-colouring.

Proof We may assume $X = [\kappa]^\kappa$. The cover $\mathcal{H}$ will be of the form $\{H_\alpha : \alpha < \kappa\}$. The idea is that for every $A \in [\kappa]^\kappa$ there will be an $x \in X$ so that $x \in H_\alpha \iff \alpha \in A$. But this is easily achieved by choosing $x = A$, that is, by setting $H_\alpha = \{A \in [\kappa]^\kappa : \alpha \in A\}$. $\square$

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Convex bodies

The case $\kappa < \omega$ is very well studied by geometers.

For $\kappa = \omega$ there are many counterexamples.

Theorem

There is an ω-fold cover of $\mathbb{R}^2$ by axis-parallel closed rectangles that has no good 2-colouring.

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Hence under *CH* everything is clear.
The next slide summarises what we know if we do not assume *CH*.

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Let $\kappa \geq \omega_1$. Then every κ-fold cover of $\mathbb{R}^2$ by closed discs has a good κ-colouring.

But!

Theorem

Assume $MA_\kappa(\sigma\text{-centered})$. Then there exists a κ-fold cover of $\mathbb{R}^2$ by isometric copies of a strictly convex compact set that has no good 2-colouring.

We do not now if the isometries can be replaced by translations.

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Closed sets

Let first $\kappa \leq \omega$.

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There exists an ω-fold cover of $\mathbb{R}^2$ with translates of a fixed compact set that has no good 2-colouring.

Let now κ be uncountable.
As mentioned above, if *CH* holds then all κ-fold covers have good κ-colourings for every $\kappa \geq \omega_1$.
The next theorem shows that this positive statement is also consistent with an arbitrarily large continuum. More precisely, we can add an arbitrary number of Cohen reals to a suitable model of *ZFC*.

Theorem

Let λ be a cardinal and V be a model of ZFC satisfying GCH $+ \square_\mu$ for every $\omega = cf(\mu) < \mu \leq \lambda$. Denote by $V^{\mathcal{C}_\lambda}$ the model obtained by adding λ Cohen reals. Then in $V^{\mathcal{C}_\lambda}$ for every $\kappa \geq \omega_1$ every κ-fold cover of $\mathbb{R}^2$ consisting of closed sets has a good κ-colouring.

How about the negative consistency?

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Let λ be a cardinal and V be a model of ZFC satisfying GCH $+ \square_\mu$ for every $\omega = cf(\mu) < \mu \leq \lambda$. Denote by $V^{\mathcal{C}_\lambda}$ the model obtained by adding λ Cohen reals. Then in $V^{\mathcal{C}_\lambda}$ for every $\kappa \geq \omega_1$ every κ-fold cover of $\mathbb{R}^2$ consisting of closed sets has a good κ-colouring.

How about the negative consistency?

Closed sets

Let first $\kappa \leq \omega$.

Theorem

There exists an ω-fold cover of $\mathbb{R}^2$ with translates of a fixed compact set that has no good 2-colouring.

Let now κ be uncountable.
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Assume $MA_\kappa(\sigma\text{-centered})$. Then there exists a κ-fold cover of $\mathbb{R}^2$ by translates of a compact set that has a no good 2-colouring.

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Actually, the $\kappa = \omega$ result is a consequence of this one, as $MA_\omega(\sigma\text{-centered})$ is true.

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Arbitrary sets

We look for 'an optimal bound for the size of elements of the κ-fold cover $\mathcal{H}$'. The right notion turns out to be the following.

Definition

Let $\mathcal{S}(\kappa)$ be the minimal cardinal such that for every $\lambda < \mathcal{S}(\kappa)$ every κ-fold cover $\mathcal{H}$ with $|H| < \lambda$ $(\forall H \in \mathcal{H})$ has a good κ-colouring.

Theorem

$\kappa^{++} \le \mathcal{S}(\kappa) \le (2^\kappa)^+$ for every $\kappa \ge \omega$.

Corollary

Assume GCH. Then $\mathcal{S}(\kappa) = \kappa^{++} = (2^\kappa)^+$ for every $\kappa \ge \omega$.

The next slide shows that neither value is 'correct'.

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Let $\kappa \geq \omega$. Then $\mathcal{S}(\kappa) = (2^\kappa)^+$ can fail, since $\dagger_{\kappa^+} + 2^\kappa > \kappa^+$ is consistent.

Theorem

Assume MA$(\mathrm{countable})$. *Then* $\mathcal{S}(\omega) = (2^\omega)^+$.

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This shows that $\mathcal{S}(\omega) = \omega^{++}$ can fail, since *MA*$(\mathrm{countable}) + \neg$*CH* is consistent.

So far we can only push this one cardinal higher.

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Assume Baumgartner's Axiom $+$*CH*. *Then* $\mathcal{S}(\omega_1) > \omega_1^{++}$.

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Remark

By a simple argument all result of this section can be translated to the language of Bernstein property of families of sets.

Graphs

As this is a very special case, we are more ambitious here. We look for complete characterisations of good κ-colourable graphs.
The case of infinite κ is completely solved.

Theorem

Let $\kappa \geq \omega$ and $G = (V, E)$ be a graph such that each vertex is of degree at least κ. Then E has a good κ-colouring, that is, the edges can be coloured by κ colours so that every vertex is covered by edges of all colours.

$\kappa = 2$ is also solved ($\kappa < 2$ is trivial).

Theorem

Let $G = (V, E)$ be graph such that each vertex is of degree at least 2. Then E has a good κ-colouring iff no connected component of G is an odd cycle.

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For $3 \leq \kappa < \omega$ such a characterisation seems to be difficult. Indeed, even for finite 3-regular graphs this is *NP*-complete.

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Open problems

Question

Let $\mathcal{H}$ be an ω_1-fold cover of $\mathbb{R}^2$ by closed sets such that $|\mathcal{H}| = \omega_1$. Does it have a good ω_1-colouring?

This follows from *CH*, but is this true in *ZFC*?

Question

Is there an ω-fold cover of $\mathbb{R}^2$ by translates of a compact convex set that has no a good ω-colouring?

There are so many more! See the preprint that is going to be available soon at

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