Maximality Principles for Closed Forcings

Gunter Fuchs
Institut für Mathematische Logik und Grundlagenforschung
Westfälische Wilhelms-Universität Münster

Logic Colloquium 2007, Wroclaw

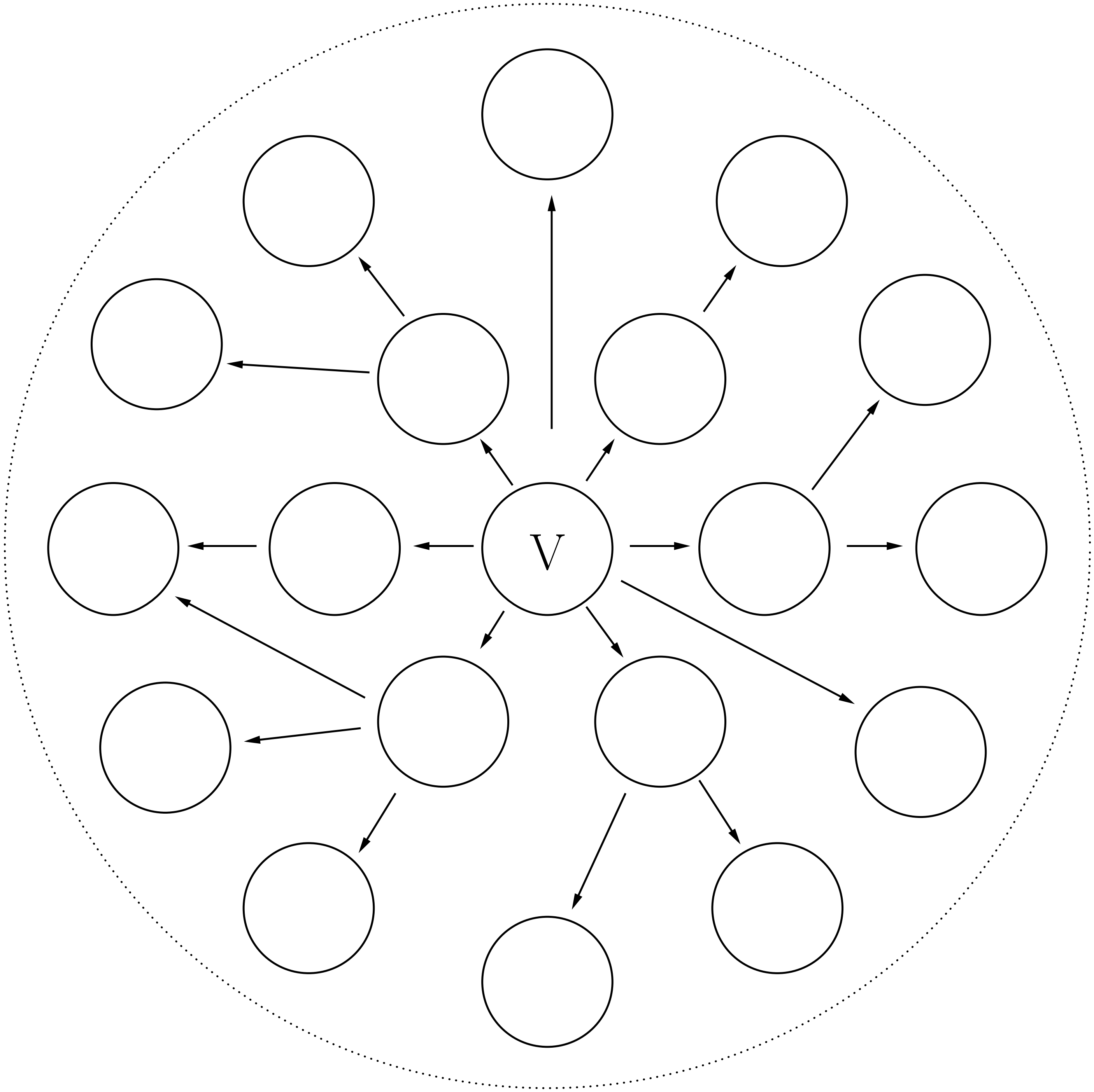
July 18, 2007

This is the first of a two-part talk on closed maximality principles.

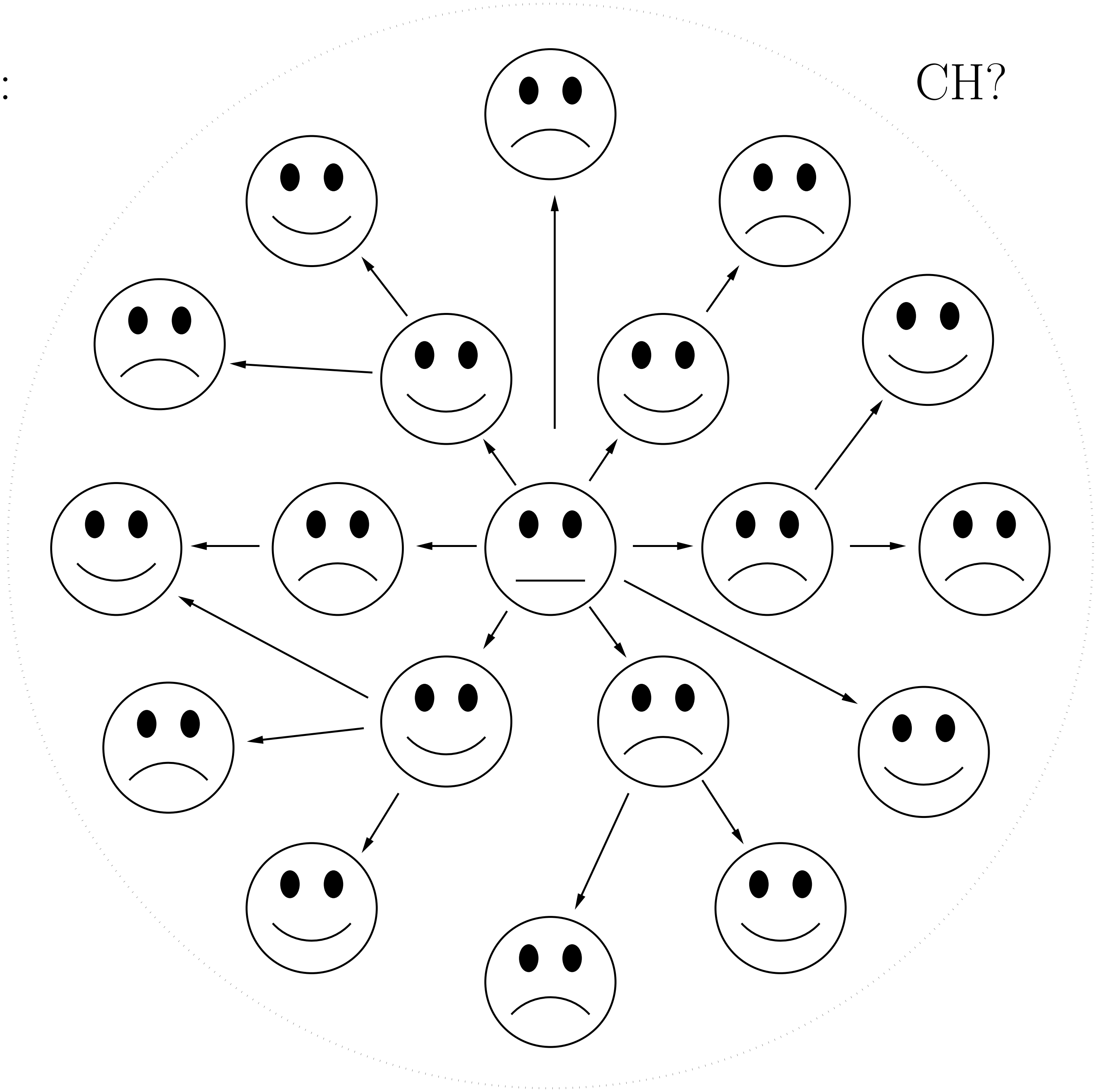
This is the first of a two-part talk on closed maximality principles.

I gave the second part last week at the First European Set Theory meeting in Bedlewo, and I apologize to those who attended that talk for some overlaps between the talks.

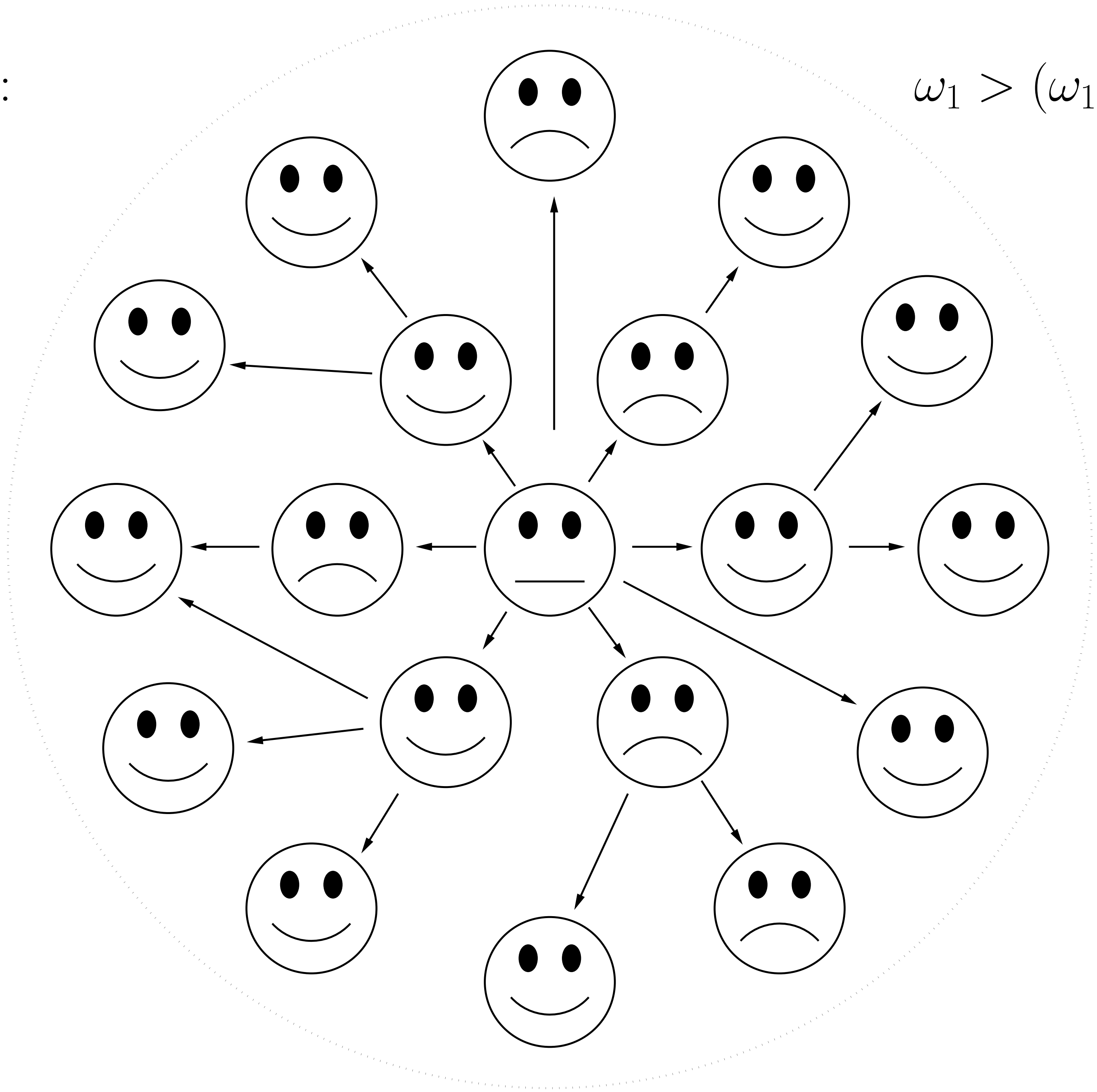
Let's view the universe
and its possible generic extensions
as a Kripke model
for modal logic.


V

Question: CH?

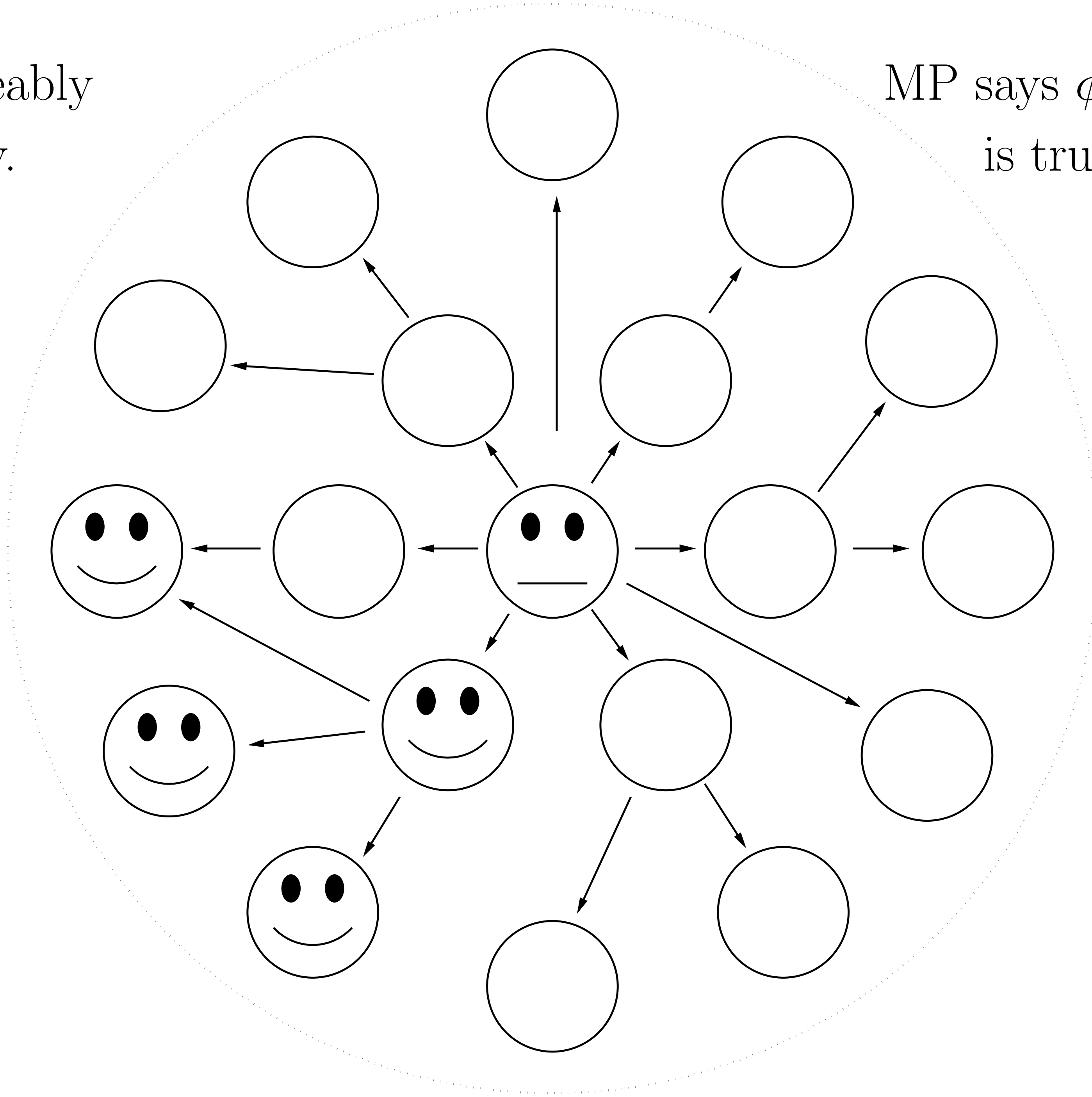


Question: $\omega_1 > (\omega_1)^L$?

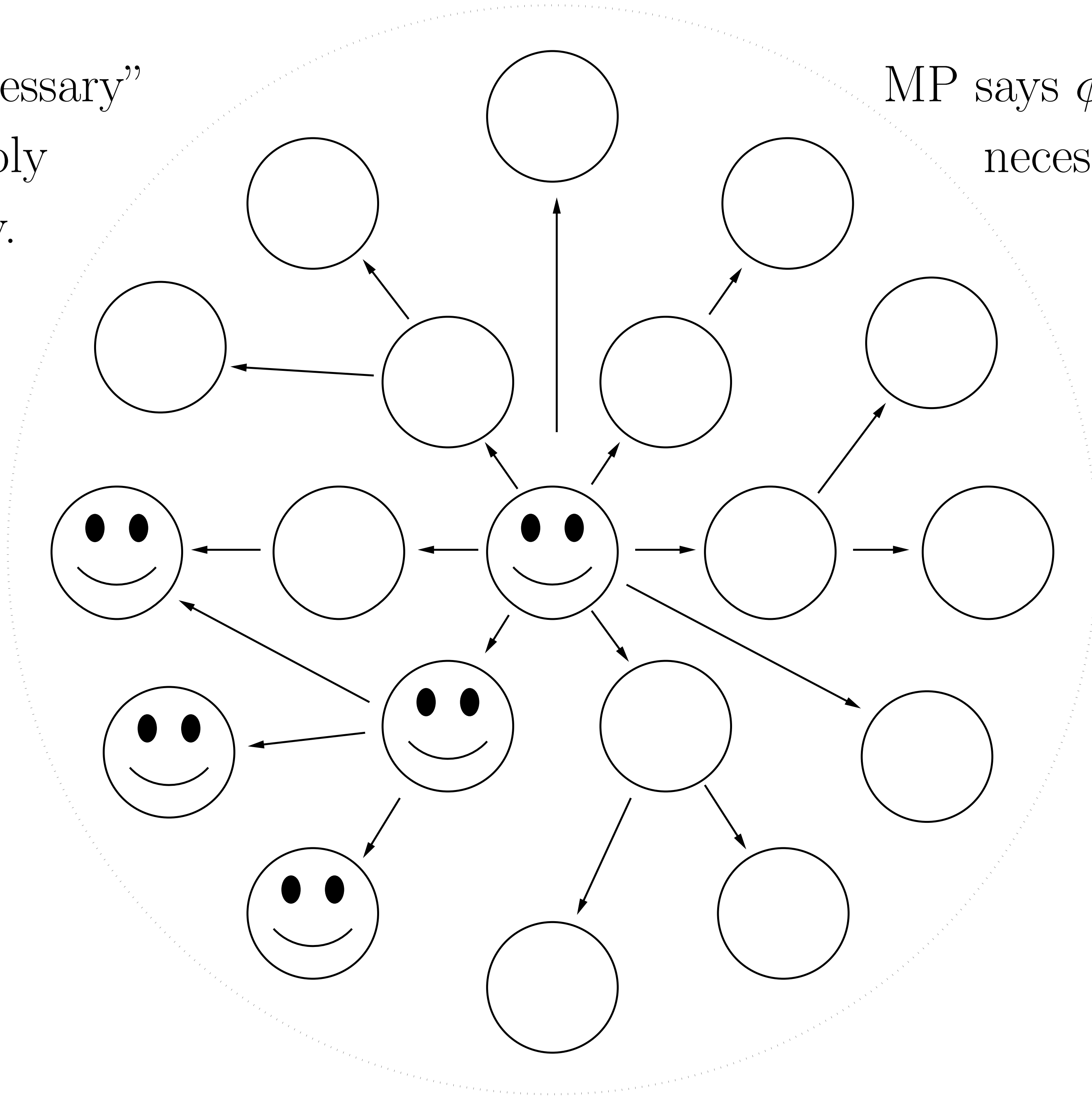


ϕ is forceably necessary.

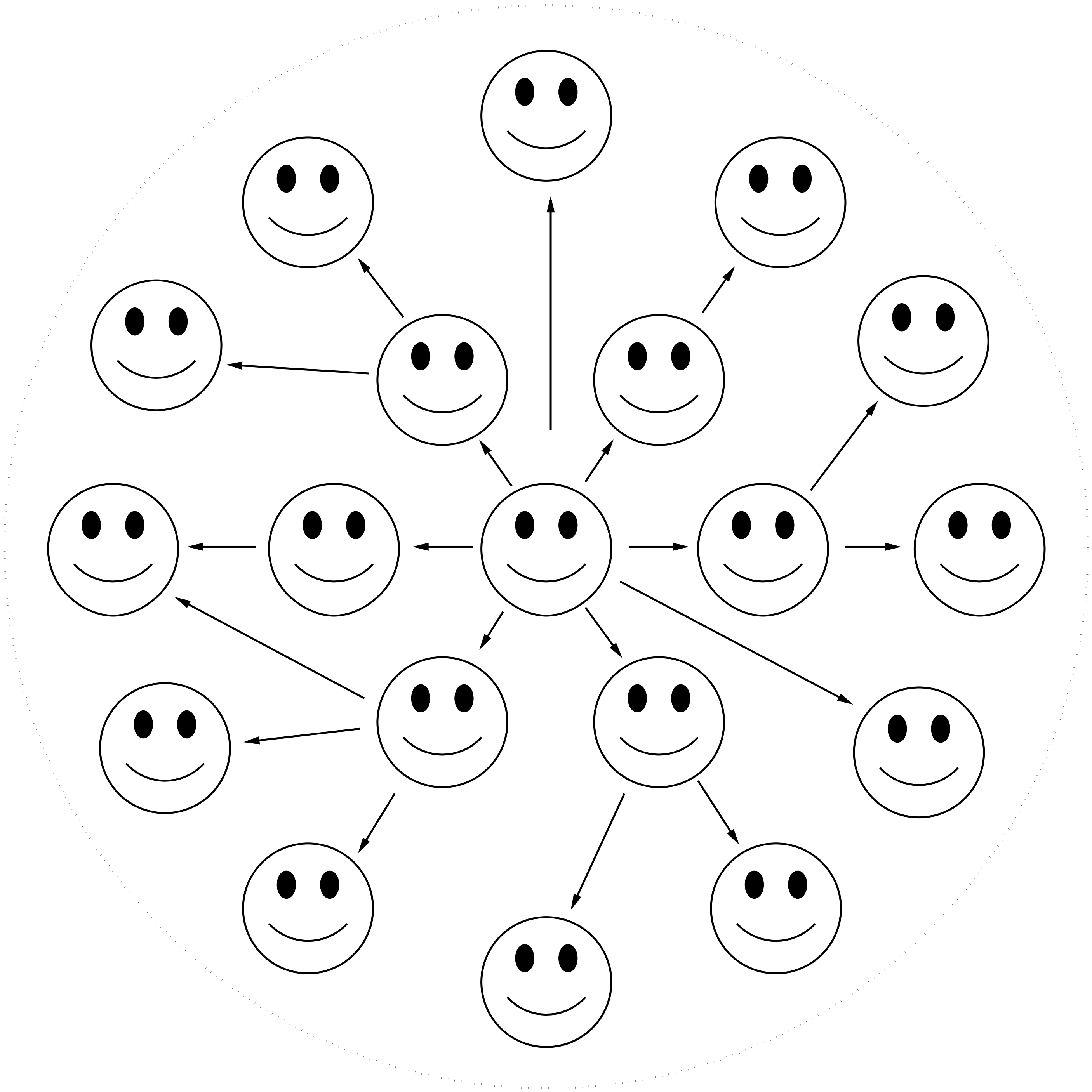
MP says ϕ is true.



"ϕ is necessary" is forceably necessary.



MP says ϕ is necessary.



Write $\Diamond\varphi$ to express that φ holds in a forcing extension (φ is forceable).

Note: This is the first order statement $\exists \mathbb{P} \quad \mathbb{P} \Vdash \varphi$.

Write $\Diamond\varphi$ to express that φ holds in a forcing extension (φ is forceable).

Note: This is the first order statement $\exists \mathbb{P} \quad \mathbb{P} \Vdash \varphi$.

$\Box\varphi$ means that φ holds in every forcing extension (φ is necessary).

This is again a first order statement.

Write $\Diamond\varphi$ to express that φ holds in a forcing extension (φ is forceable).

Note: This is the first order statement $\exists \mathbb{P} \quad \mathbb{P} \Vdash \varphi$.

$\Box\varphi$ means that φ holds in every forcing extension (φ is necessary).

This is again a first order statement.

So the statement $\Diamond(\Box\varphi)$ makes sense.

It expresses that it is forceable that φ is necessary, or in short, that φ is forceably necessary.

Write $\Diamond\varphi$ to express that φ holds in a forcing extension (φ is forceable).

Note: This is the first order statement $\exists\mathbb{P} \quad \mathbb{P} \Vdash \varphi$.

$\Box\varphi$ means that φ holds in every forcing extension (φ is necessary).

This is again a first order statement.

So the statement $\Diamond(\Box\varphi)$ makes sense.

It expresses that it is forceable that φ is necessary, or in short, that φ is forceably necessary.

The Maximality Principle MP is the scheme consisting of the formulae

$$(\Diamond\Box\varphi) \implies \varphi,$$

for every sentence φ. It was introduced in a slightly different formulation in 1977 here at the Logic Colloquium by Stavi and Väänänen, and then rediscovered independently by Hamkins, as stated.

Possible modifications of MP:

Possible modifications of MP:

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, . . .

Possible modifications of MP**:**

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, . . .

2. Allow parameters in the scheme $\Diamond\Box\varphi \implies \varphi$, i.e., boldface versions of the principles.

Possible modifications of MP**:**

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, . . .

2. Allow parameters in the scheme $\Diamond\Box\varphi \implies \varphi$, i.e., boldface versions of the principles.

3. Necessary forms of the boldface principles.

Possible modifications of MP**:**

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, . . .

2. Allow parameters in the scheme $\Diamond\Box\varphi \implies \varphi$, i.e., boldface versions of the principles.

3. Necessary forms of the boldface principles.

4. (Restrict to a subclass of formulae.)

Possible modifications of MP**:**

1. Restrict to certain classes of forcings, such as: Proper, ccc, stationary-preserving, ...
2. Allow parameters in the scheme $\Diamond\Box\varphi \implies \varphi$, i.e., boldface versions of the principles.
3. Necessary forms of the boldface principles.
4. (Restrict to a subclass of formulae.)

General form of the principle:

$$\mathsf{MP}_{\Gamma}(X),$$

where Γ is a class of partial orders and X is the parameter set.

I looked at the case where Γ is one of the following, for some fixed regular cardinal κ.

I looked at the case where Γ is one of the following, for some fixed regular cardinal κ.

1. The class of all $<\kappa$-closed forcings,

I looked at the case where Γ is one of the following, for some fixed regular cardinal κ.

1. The class of all $<\kappa$-closed forcings,

2. the class of all $<\kappa$-directed-closed forcings,

I looked at the case where Γ is one of the following, for some fixed regular cardinal κ.

1. The class of all $<\kappa$-closed forcings,
2. the class of all $<\kappa$-directed-closed forcings,
3. the class of all forcings of the form $\mathrm{Col}(\kappa, \lambda)$ or $\mathrm{Col}(\kappa, < \lambda)$, for some λ. Call the class $\mathrm{Col}(\kappa)$.

I looked at the case where Γ is one of the following, for some fixed regular cardinal κ.

1. The class of all $<\kappa$-closed forcings,
2. the class of all $<\kappa$-directed-closed forcings,
3. the class of all forcings of the form $\mathrm{Col}(\kappa, \lambda)$ or $\mathrm{Col}(\kappa, < \lambda)$, for some λ. Call the class $\mathrm{Col}(\kappa)$.

Note: $\kappa = \omega$ is allowed!

The corresponding parameter set will usually be one of the following:

$$\emptyset,\ H_\kappa \cup \{\kappa\},\ H_{\kappa^+}.$$

Overview

1. Relationships between versions of the maximality principles.

Overview

1. Relationships between versions of the maximality principles.

2. Consistency Investigations:

Overview

1. Relationships between versions of the maximality principles.

2. Consistency Investigations:

   Consistency strengths,

Overview

1. Relationships between versions of the maximality principles.

2. Consistency Investigations:

   Consistency strengths,
   Compatibility with large cardinals.

Overview

1. Relationships between versions of the maximality principles.

2. Consistency Investigations:

   Consistency strengths,
   Compatibility with large cardinals.

3. Implications.

Overview

1. Relationships between versions of the maximality principles.

2. Consistency Investigations:

   Consistency strengths,
   Compatibility with large cardinals.

3. Implications.

4. Separations.

Overview

1. Relationships between versions of the maximality principles.

2. Consistency Investigations:

   Consistency strengths,
   Compatibility with large cardinals.

3. Implications.

4. Separations.

5. Combinations.

Overview

1. Relationships between versions of the maximality principles.

2. Consistency Investigations:

   Consistency strengths,
   Compatibility with large cardinals.

3. Implications.

4. Separations.

5. Combinations.

6. Limitations.

Overview

1. Relationships between versions of the maximality principles.

2. Consistency Investigations:

   Consistency strengths,
   Compatibility with large cardinals.

3. Implications.

4. Separations.

5. Combinations.

6. Limitations.

The last two points were already covered in the second part of the talk.

Relationships between versions of the maximality principles

Note the following folkloristic fact:

Lemma 1. *Let* κ *be a regular cardinal and* $\lambda > \kappa$ *a cardinal with* $\lambda = \lambda^{<\kappa}$. *Then there is a dense subset* Δ *of* $\mathrm{Col}(\kappa, \lambda)$ *such that if* $\mathbb{P}$ *is a separative* $<\kappa$*-closed partial order with* $\overline{\overline{\mathbb{P}}} = \lambda$ *and* $\mathbb{1} \Vdash_{\mathbb{P}} (\overline{\overline{\lambda}} = \kappa)$, *then there is a dense subset* D *of* $\mathbb{P}$ *with* $\mathrm{Col}(\kappa, \lambda) \restriction \Delta \cong \mathbb{P} \restriction D$, *i.e.,* $\mathrm{Col}(\kappa, \lambda)$ *and* $\mathbb{P}$ *are forcing-equivalent.*

Relationships between versions of the maximality principles

Note the following folkloristic fact:

Lemma 1. *Let κ be a regular cardinal and $\lambda > \kappa$ a cardinal with $\lambda = \lambda^{<\kappa}$. Then there is a dense subset Δ of $\mathrm{Col}(\kappa, \lambda)$ such that if $\mathbb{P}$ is a separative $<\kappa$-closed partial order with $\overline{\overline{\mathbb{P}}} = \lambda$ and $\mathbb{1} \Vdash_{\mathbb{P}} (\overline{\overline{\lambda}} = \kappa)$, then there is a dense subset D of $\mathbb{P}$ with $\mathrm{Col}(\kappa, \lambda) \restriction \Delta \cong \mathbb{P} \restriction D$, i.e., $\mathrm{Col}(\kappa, \lambda)$ and $\mathbb{P}$ are forcing-equivalent.*

Corollary 2. *Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where κ is regular. Then if $\lambda \geq \overline{\overline{\mathbb{P}}}$ and $\lambda^{<\kappa} = \lambda$,*

$$(\mathbb{P} \times \mathrm{Col}(\kappa, \lambda)) \restriction D \cong \mathrm{Col}(\kappa, \lambda) \restriction \Delta,$$

for some dense set D and the dense set Δ from Lemma 1.

So $\mathrm{Col}(\kappa)$ absorbs any $<\kappa$-closed forcing.

Lemma 3.

$$\begin{aligned} & \mathsf{ZFC} + \mathsf{MP}_{\mathrm{Col}(\dot{\kappa})}(X) \\ \vdash \quad & \mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(X) \\ \vdash \quad & \mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(X). \end{aligned}$$

Lemma 3.

$$\begin{aligned} & \mathsf{ZFC} + \mathsf{MP}_{\mathrm{Col}(\dot\kappa)}(X) \\ \vdash\ & \mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(X) \\ \vdash\ & \mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(X). \end{aligned}$$

Proof. Let φ be a statement with parameters from X.

Lemma 3.

$$\begin{aligned} & \mathsf{ZFC} + \mathsf{MP}_{\mathrm{Col}(\dot\kappa)}(X) \\ \vdash\ & \mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(X) \\ \vdash\ & \mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(X). \end{aligned}$$

Proof. Let φ be a statement with parameters from X. To show

$$\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(X) \implies \mathsf{MP}_{<\kappa-\mathrm{closed}}(X),$$

it suffices to show:

Lemma 3.

$$\begin{aligned} & \mathsf{ZFC} + \mathsf{MP}_{\mathrm{Col}(\dot\kappa)}(X) \\ \vdash\ & \mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(X) \\ \vdash\ & \mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(X). \end{aligned}$$

Proof. Let φ be a statement with parameters from X. To show

$$\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(X) \implies \mathsf{MP}_{<\kappa-\mathrm{closed}}(X),$$

it suffices to show:

φ is $<\kappa$-closed-forceably necessary

$$\implies$$

φ is $<\kappa$-directed-closed-forceably necessary.

This can be seen as follows:

This can be seen as follows:

- Let $\mathbb{P}$ be a $<\kappa$-closed poset making φ $<\kappa$-closed-necessary.

This can be seen as follows:

- Let $\mathbb{P}$ be a $<\kappa$-closed poset making φ $<\kappa$-closed-necessary.
- $\mathbb{P}$ forces that it is $<\kappa$-closed-necessary that φ is $<\kappa$-closed-necessary.

This can be seen as follows:

- Let $\mathbb{P}$ be a $<\kappa$-closed poset making φ $<\kappa$-closed-necessary.

- $\mathbb{P}$ forces that it is $<\kappa$-closed-necessary that φ is $<\kappa$-closed-necessary.

- Let $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)$, where θ is sufficiently closed and large.

This can be seen as follows:

- Let $\mathbb{P}$ be a $<\kappa$-closed poset making φ $<\kappa$-closed-necessary.
- $\mathbb{P}$ forces that it is $<\kappa$-closed-necessary that φ is $<\kappa$-closed-necessary.
- Let $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)$, where θ is sufficiently closed and large.
- Note: $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)^{\mathrm{V}^{\mathbb{P}}}$.

This can be seen as follows:

- Let $\mathbb{P}$ be a $<\kappa$-closed poset making φ $<\kappa$-closed-necessary.
- $\mathbb{P}$ forces that it is $<\kappa$-closed-necessary that φ is $<\kappa$-closed-necessary.
- Let $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)$, where θ is sufficiently closed and large.
- Note: $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)^{\mathrm{V}^{\mathbb{P}}}$.
- φ is $<\kappa$-closed-necessary in $\mathrm{V}^{\mathbb{P}\times\mathbb{Q}}$.

This can be seen as follows:

- Let $\mathbb{P}$ be a $<\kappa$-closed poset making φ $<\kappa$-closed-necessary.
- $\mathbb{P}$ forces that it is $<\kappa$-closed-necessary that φ is $<\kappa$-closed-necessary.
- Let $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)$, where θ is sufficiently closed and large.
- Note: $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)^{\mathrm{V}^{\mathbb{P}}}$.
- φ is $<\kappa$-closed-necessary in $\mathrm{V}^{\mathbb{P}\times\mathbb{Q}}$.
- $\mathbb{P} \times \mathbb{Q}$ is forcing equivalent to $\mathbb{Q}$.

This can be seen as follows:

- Let $\mathbb{P}$ be a $<\kappa$-closed poset making φ $<\kappa$-closed-necessary.
- $\mathbb{P}$ forces that it is $<\kappa$-closed-necessary that φ is $<\kappa$-closed-necessary.
- Let $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)$, where θ is sufficiently closed and large.
- Note: $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)^{\mathrm{V}^{\mathbb{P}}}$.
- φ is $<\kappa$-closed-necessary in $\mathrm{V}^{\mathbb{P}\times\mathbb{Q}}$.
- $\mathbb{P} \times \mathbb{Q}$ is forcing equivalent to $\mathbb{Q}$.
- $\mathbb{Q}$ is $<\kappa$-directed-closed.

This can be seen as follows:

- Let $\mathbb{P}$ be a $<\kappa$-closed poset making φ $<\kappa$-closed-necessary.
- $\mathbb{P}$ forces that it is $<\kappa$-closed-necessary that φ is $<\kappa$-closed-necessary.
- Let $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)$, where θ is sufficiently closed and large.
- Note: $\mathbb{Q} = \mathrm{Col}(\kappa, \theta)^{\mathrm{V}^{\mathbb{P}}}$.
- φ is $<\kappa$-closed-necessary in $\mathrm{V}^{\mathbb{P}\times\mathbb{Q}}$.
- $\mathbb{P} \times \mathbb{Q}$ is forcing equivalent to $\mathbb{Q}$.
- $\mathbb{Q}$ is $<\kappa$-directed-closed.

The other statement is proven analogously. $\square$

$$
\begin{array}{ccc}
\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_\kappa \cup \{\kappa\}) & \Longleftarrow & \mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+}) \\
\Downarrow & & \Downarrow \\
\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_\kappa \cup \{\kappa\}) & \Longleftarrow & \mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+}) \\
\Downarrow & & \Downarrow \\
\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\}) & \Longleftarrow & \mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})
\end{array}
$$

Consistency

Theorem 4. *Assume* $\kappa < \delta$, $\mathrm{V}_\delta \prec \mathrm{V}$ *and* κ, *as well as* δ, *are regular. Then* $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$ *holds in* $\mathrm{V}[G]$, *where* G *is* V*-generic for* $\mathbb{P} = \mathrm{Col}(\kappa, {<}\delta)$.

Implications

Implications

Lemma 5. *Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where κ is regular, and let G be $\mathbb{P}$-generic over V.*

Implications

Lemma 5. *Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where κ is regular, and let G be $\mathbb{P}$-generic over V.*

1. (Jensen) If $\vec{S}$ is a $\diamondsuit_\kappa$-sequence, then $\mathrm{V}[G] \models$ "$\check{\vec{S}}$ is a $\diamondsuit_{\check{\kappa}}$-sequence".

Implications

Lemma 5. *Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where κ is regular, and let G be $\mathbb{P}$-generic over V.*

1. *(Jensen) If $\vec{S}$ is a $\diamondsuit_\kappa$-sequence, then $\mathrm{V}[G] \models$ "$\check{\vec{S}}$ is a $\diamondsuit_{\check{\kappa}}$-sequence".*
2. *(Silver for $\kappa = \omega_1$) Let T be a slim κ-tree ($\kappa > \omega$). Then $[T] = [T]^{\mathrm{V}[G]}$.*

Implications

Lemma 5. *Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where κ is regular, and let G be $\mathbb{P}$-generic over V.*

1. *(Jensen) If $\vec{S}$ is a $\diamondsuit_\kappa$-sequence, then $\mathrm{V}[G] \models$ "$\check{\vec{S}}$ is a $\diamondsuit_{\check{\kappa}}$-sequence".*

2. *(Silver for $\kappa = \omega_1$) Let T be a slim κ-tree ($\kappa > \omega$). Then $[T] = [T]^{\mathrm{V}[G]}$.*

3. *If S and T are normal κ-trees s.t. $\texttt{Iso}(S, T)$ has cardinality less than 2^κ, then $\texttt{Iso}(S, T) = (\texttt{Iso}(S, T))^{\mathrm{V}[G]}$.*

Implications

Lemma 5. *Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where κ is regular, and let G be $\mathbb{P}$-generic over V.*

1. *(Jensen) If $\vec{S}$ is a $\diamondsuit_\kappa$-sequence, then $\mathrm{V}[G] \models$ "$\check{\vec{S}}$ is a $\diamondsuit_{\check{\kappa}}$-sequence".*
2. *(Silver for $\kappa = \omega_1$) Let T be a slim κ-tree ($\kappa > \omega$). Then $[T] = [T]^{\mathrm{V}[G]}$.*
3. *If S and T are normal κ-trees s.t. $\mathtt{Iso}(S,T)$ has cardinality less than 2^κ, then $\mathtt{Iso}(S,T) = (\mathtt{Iso}(S,T))^{\mathrm{V}[G]}$.*
4. *If φ is a Σ^1_1-sentence and $A \subseteq \kappa$, then*

$$\langle \kappa, <, A \rangle \models \varphi \iff (\langle \kappa, <, A \rangle \models \varphi)^{\mathrm{V}[G]}.$$

   Note that this remains true even for Σ^1_2-sentences, if $\kappa = \omega$, by Shoenfield absoluteness.

Implications

Lemma 5. *Let $\mathbb{P}$ be a $<\kappa$-closed notion of forcing, where κ is regular, and let G be $\mathbb{P}$-generic over V.*

1. *(Jensen) If $\vec{S}$ is a $\diamondsuit_\kappa$-sequence, then $\mathrm{V}[G] \models$ "$\check{\vec{S}}$ is a $\diamondsuit_{\check{\kappa}}$-sequence".*
2. *(Silver for $\kappa = \omega_1$) Let T be a slim κ-tree ($\kappa > \omega$). Then $[T] = [T]^{\mathrm{V}[G]}$.*
3. *If S and T are normal κ-trees s.t. $\mathtt{Iso}(S,T)$ has cardinality less than 2^κ, then $\mathtt{Iso}(S,T) = (\mathtt{Iso}(S,T))^{\mathrm{V}[G]}$.*
4. *If φ is a Σ^1_1-sentence and $A \subseteq \kappa$, then*

$$\langle \kappa, <, A\rangle \models \varphi \iff (\langle \kappa, <, A\rangle \models \varphi)^{\mathrm{V}[G]}.$$

   Note that this remains true even for Σ^1_2-sentences, if $\kappa = \omega$, by Shoenfield absoluteness.

5. *If T is a κ-Souslin tree, then $\mathrm{V}[G] \models$ 'T is a κ-Souslin tree."*

Corollary 6. *The following statements, if true, are* $<\kappa$*-closed-necessary.*

Corollary 6. *The following statements, if true, are $<\kappa$-closed-necessary.*

1. $\diamondsuit_\kappa$.

Corollary 6. *The following statements, if true, are $<\kappa$-closed-necessary.*

1. $\diamondsuit_\kappa$*.*

2. S *and* T *are non-isomorphic normal* κ*-trees.*

Corollary 6. *The following statements, if true, are* $<\kappa$*-closed-necessary.*

1. $\diamondsuit_\kappa$.

2. S *and* T *are non-isomorphic normal* κ*-trees.*

3. T *is a rigid normal* κ*-tree.*

Corollary 6. *The following statements, if true, are $<\kappa$-closed-necessary.*

1. $\diamondsuit_\kappa$.
2. S *and* T *are non-isomorphic normal* κ*-trees.*
3. T *is a rigid normal* κ*-tree.*
4. T *is a* κ*-Aronszajn tree.*

Corollary 6. *The following statements, if true, are $<\kappa$-closed-necessary.*

1. $\Diamond_\kappa$.

2. S *and* T *are non-isomorphic normal* κ*-trees.*

3. T *is a rigid normal* κ*-tree.*

4. T *is a* κ*-Aronszajn tree.*

5. T *is a slim* κ*-tree which is not Kurepa.*

Corollary 6. *The following statements, if true, are $<\kappa$-closed-necessary.*

1. $\diamondsuit_\kappa$.

2. S *and* T *are non-isomorphic normal* κ*-trees.*

3. T *is a rigid normal* κ*-tree.*

4. T *is a* κ*-Aronszajn tree.*

5. T *is a slim* κ*-tree which is not Kurepa.*

6. T *is a* κ*-Souslin tree.*

Corollary 6. *The following statements, if true, are $<\kappa$-closed-necessary.*

1. $\diamondsuit_\kappa$.
2. S *and* T *are non-isomorphic normal* κ*-trees.*
3. T *is a rigid normal* κ*-tree.*
4. T *is a* κ*-Aronszajn tree.*
5. T *is a slim* κ*-tree which is not Kurepa.*
6. T *is a* κ*-Souslin tree.*
7. $\langle \kappa, <, A \rangle \models \varphi$*, where* φ *is a* Σ^1_2 *sentence and* A *is a subset of* κ^n*, for some* $n < \omega$*. If* $\kappa = \omega$*, then* Σ^1_2 *can be replaced by* Σ^1_3*.*

Theorem 7. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(S \cup \{\kappa\})$*. Then*

Theorem 7. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(S \cup \{\kappa\})$. *Then*

1. *If* $\kappa > \omega$, *then* $\diamondsuit_\kappa$ *holds.*

Theorem 7. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(S \cup \{\kappa\})$. *Then*

1. *If* $\kappa > \omega$, *then* $\diamondsuit_\kappa$ *holds.*

2. *If* κ *is the successor of the regular cardinal* $\bar{\kappa}$ *and* $\bar{\kappa}^{<\bar{\kappa}} = \bar{\kappa}$, *then there is a* κ*-Souslin tree. In particular, this is true for* $\kappa = \omega_1$.

Theorem 7. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(S \cup \{\kappa\})$. *Then*

1. *If* $\kappa > \omega$, *then* $\diamondsuit_\kappa$ *holds.*

2. *If* κ *is the successor of the regular cardinal* $\bar{\kappa}$ *and* $\bar{\kappa}^{<\bar{\kappa}} = \bar{\kappa}$, *then there is a* κ*-Souslin tree. In particular, this is true for* $\kappa = \omega_1$.

3. *For any* $A \subseteq H_\kappa$ *with* $A \in S$, *any* Σ^1_2*-sentence* φ *and any* $<\kappa$*-closed notion of forcing* $\mathbb{P}$, *it follows that*

$$\langle H_\kappa, \in, A\rangle \models \varphi \iff \mathbb{1} \Vdash_{\mathbb{P}} (\langle H_{\check{\kappa}}, \in, \check{A}\rangle \models \varphi).$$

So $<\kappa$*-closed-generic* $\mathbf{\Sigma}^1_2$*-absoluteness over* H_κ *holds.*

Theorem 7. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(S \cup \{\kappa\})$. *Then*

1. *If* $\kappa > \omega$, *then* $\diamondsuit_\kappa$ *holds.*

2. *If* κ *is the successor of the regular cardinal* $\bar{\kappa}$ *and* $\bar{\kappa}^{<\bar{\kappa}} = \bar{\kappa}$, *then there is a* κ*-Souslin tree. In particular, this is true for* $\kappa = \omega_1$.

3. *For any* $A \subseteq H_\kappa$ *with* $A \in S$, *any* Σ^1_2*-sentence* φ *and any* $<\kappa$*-closed notion of forcing* $\mathbb{P}$, *it follows that*

$$\langle H_\kappa, \in, A \rangle \models \varphi \iff \mathbb{1} \Vdash_{\mathbb{P}} (\langle H_{\check{\kappa}}, \in, \check{A} \rangle \models \varphi).$$

So $<\kappa$*-closed-generic* $\boldsymbol{\Sigma}^1_2$*-absoluteness over* H_κ *holds.*

In case $\kappa = \omega$, *generic* Σ^1_3*-absoluteness in parameters from* $S \cap \mathcal{P}(\omega)$ *follows.*

Theorem 7. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(S \cup \{\kappa\})$*. Then*

1. *If* $\kappa > \omega$*, then* $\diamondsuit_\kappa$ *holds.*

2. *If* κ *is the successor of the regular cardinal* $\bar{\kappa}$ *and* $\bar{\kappa}^{<\bar{\kappa}} = \bar{\kappa}$*, then there is a* κ*-Souslin tree. In particular, this is true for* $\kappa = \omega_1$*.*

3. *For any* $A \subseteq H_\kappa$ *with* $A \in S$*, any* Σ^1_2*-sentence* φ *and any* $<\kappa$*-closed notion of forcing* $\mathbb{P}$*, it follows that*

$$\langle H_\kappa, \in, A \rangle \models \varphi \iff \mathbb{1} \Vdash_{\mathbb{P}} (\langle H_{\check{\kappa}}, \in, \check{A} \rangle \models \varphi).$$

   So $<\kappa$*-closed-generic* $\mathbf{\Sigma}^1_2$*-absoluteness over* H_κ *holds.*

   In case $\kappa = \omega$*, generic* Σ^1_3*-absoluteness in parameters from* $S \cap \mathcal{P}(\omega)$ *follows.*

   So if $S = H_{\kappa^+}$*, boldface* $<\kappa$*-closed-generic* $\Sigma^1_2(H_\kappa)$*-absoluteness follows in case* $\kappa > \omega$*, and boldface generic* Σ^1_3*-absoluteness in case* $\kappa = \omega$*.*

From now on, assume that $S = H_{\kappa^+}$.

From now on, assume that $S = H_{\kappa^+}$.

4. If $\kappa > \omega$, then there is no slim κ-Kurepa tree.

From now on, assume that $S = H_{\kappa^+}$.

4. *If $\kappa > \omega$, then there is no slim κ-Kurepa tree.*

5. *κ^+ is inaccessible in L.*

From now on, assume that $S = H_{\kappa^+}$.

4. *If $\kappa > \omega$, then there is no slim κ-Kurepa tree.*

5. *κ^+ is inaccessible in L.*

6. *$L_{\kappa^+} \prec L$. So L is a model of T_{κ,κ^+}.*

From now on, assume that $S = H_{\kappa^+}$.

4. *If $\kappa > \omega$, then there is no slim κ-Kurepa tree.*

5. *κ^+ is inaccessible in L.*

6. *$L_{\kappa^+} \prec L$. So L is a model of T_{κ,κ^+}.*

Proof. Generic Σ^1_2-Absoluteness:

From now on, assume that $S = H_{\kappa^+}$.

4. If $\kappa > \omega$, then there is no slim κ-Kurepa tree.

5. κ^+ is inaccessible in L.

6. $L_{\kappa^+} \prec L$. So L is a model of T_{κ,κ^+}.

Proof. Generic Σ^1_2-Absoluteness:

- $2^{<\kappa} = \kappa = \overline{\overline{H_\kappa}}$, by $\diamondsuit_\kappa$.

From now on, assume that $S = H_{\kappa^+}$.

4. *If* $\kappa > \omega$, *then there is no slim* κ*-Kurepa tree.*

5. κ^+ *is inaccessible in* L.

6. $L_{\kappa^+} \prec L$. *So* L *is a model of* T_{κ,κ^+}.

Proof. Generic Σ^1_2-Absoluteness:

- $2^{<\kappa} = \kappa = \overline{\overline{H_\kappa}}$, by $\diamondsuit_\kappa$.

- If $\psi(A) =$ "$\langle H_\kappa, \in, A\rangle \models \varphi$" holds in V, then this is necessary.

From now on, assume that $S = H_{\kappa^+}$.

4. *If $\kappa > \omega$, then there is no slim κ-Kurepa tree.*

5. *κ^+ is inaccessible in L.*

6. *$L_{\kappa^+} \prec L$. So L is a model of T_{κ,κ^+}.*

Proof. Generic Σ^1_2-Absoluteness:

- $2^{<\kappa} = \kappa = \overline{\overline{H_\kappa}}$, by $\diamondsuit_\kappa$.
- If $\psi(A) =$ "$\langle H_\kappa, \in, A\rangle \models \varphi$" holds in V, then this is necessary.
- If $\psi(A)$ holds in $\mathrm{V}[G]$, then this is necessary. So $\psi(A)$ is forceably necessary, and hence true in V.

No slim Kurepa tree:

No slim Kurepa tree: Assume T were Kurepa.

No slim Kurepa tree: Assume T were Kurepa.

- $\mathrm{Col}(\kappa, [T])$, yields an extension in which T ceases to be Kurepa.

No slim Kurepa tree: Assume T were Kurepa.

- $\mathrm{Col}(\kappa, [T])$, yields an extension in which T ceases to be Kurepa.
- No branches can subsequently be added to T.

No slim Kurepa tree: Assume T were Kurepa.

- $\mathrm{Col}(\kappa, [T])$, yields an extension in which T ceases to be Kurepa.
- No branches can subsequently be added to T.
- So T is forceably necessarily not Kurepa.

No slim Kurepa tree: Assume T were Kurepa.

- $\mathrm{Col}(\kappa, [T])$, yields an extension in which T ceases to be Kurepa.
- No branches can subsequently be added to T.
- So T is forceably necessarily not Kurepa.

$L_{\kappa^+} \prec L$:

No slim Kurepa tree: Assume T were Kurepa.

- $\mathrm{Col}(\kappa, [T])$, yields an extension in which T ceases to be Kurepa.
- No branches can subsequently be added to T.
- So T is forceably necessarily not Kurepa.

$L_{\kappa^+} \prec L$: Tarski-Vaught criterion. $\square$

Equiconsistencies

Lemma 8. *Let* M *be a model of* $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$. *Let* δ *be the supremum of the ordinals that are definable over* L^M *in the parameter* κ. *Then* $L_\delta \prec L$.

Equiconsistencies

Lemma 8. *Let* M *be a model of* $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$*. Let* δ *be the supremum of the ordinals that are definable over* L^M *in the parameter* κ*. Then* $L_\delta \prec L$*.*

Proof.

- $\delta \leq (\kappa^+)^M$, by $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$,

Lemma 8. *Let M be a model of* $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$*. Let δ be the supremum of the ordinals that are definable over L^M in the parameter κ. Then $L_\delta \prec L$.*

Proof.

- $\delta \leq (\kappa^+)^M$, by $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$,
- then verify the Tarski-Vaught criterion.

□

Summarizing, we have shown:

Summarizing, we have shown:

Corollary 9. *The following equiconsistencies hold:*

Summarizing, we have shown:

Corollary 9. *The following equiconsistencies hold:*

1. *The theory* $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ *is transitive model equiconsistent to*

$$\mathsf{ZFC} + \kappa \text{ is regular } + \kappa < \delta + \mathrm{V}_\delta \prec \mathrm{V},$$

locally in κ.

Summarizing, we have shown:

Corollary 9. *The following equiconsistencies hold:*

1. *The theory* $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ *is transitive model equiconsistent to*

$$\mathsf{ZFC} + \kappa \text{ is regular } + \kappa < \delta + \mathrm{V}_\delta \prec \mathrm{V},$$

   locally in κ.

2. *The theory* $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+}) + \delta = \kappa^+$ *is transitive model equiconsistent to the theory*

$$\mathsf{ZFC} + \kappa \text{ is regular } + \kappa < \delta + \delta \text{ is inaccessible } + \mathrm{V}_\delta \prec \mathrm{V},$$

   locally in κ *and* δ.

Compatibility of the closed maximality principles at κ with κ being a large cardinal

Lemma 10. *Let $\varphi(\kappa)$ express one of the following statements about κ: κ is inaccessible, Mahlo, subtle, Woodin.*

Compatibility of the closed maximality principles at κ with κ being a large cardinal

Lemma 10. *Let $\varphi(\kappa)$ express one of the following statements about κ: κ is inaccessible, Mahlo, subtle, Woodin.*

1. *The theory* ZFC + $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\}) + \varphi(\kappa)$ *is transitive model equiconsistent to* "ZFC $+\ \kappa$ *is regular* $+\ \kappa < \delta\ +\ \mathrm{V}_\delta \prec \mathrm{V}\ + \varphi(\kappa)$", *locally in κ.*

Compatibility of the closed maximality principles at κ with κ being a large cardinal

Lemma 10. *Let $\varphi(\kappa)$ express one of the following statements about κ: κ is inaccessible, Mahlo, subtle, Woodin.*

1. *The theory* $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\}) + \varphi(\kappa)$ *is transitive model equiconsistent to* '$\mathsf{ZFC} + \kappa$ *is regular* $+ \kappa < \delta + \mathrm{V}_\delta \prec \mathrm{V} + \varphi(\kappa)$", *locally in* κ.

2. *The theory* $\mathsf{ZFC} + \mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+}) + \delta = \kappa^+ + \varphi(\kappa)$ *is transitive model equiconsistent to the theory* '$\mathsf{ZFC} + \kappa$ *and* δ *are regular* $+\kappa < \delta + \mathrm{V}_\delta \prec \mathrm{V}$", *locally in* κ *and* δ.

A weak version of the following Lemma was independently proven by Leibman.

Lemma 11. *Suppose* κ *is supercompact and* $\kappa < \delta$*, where* δ *is an inaccessible cardinal such that* $\mathrm{V}_\delta \prec \mathrm{V}$*. Then there is a forcing extension* $\mathrm{V}[G]$ *of* V *in which* $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$ *holds and in which* κ *is still supercompact.*

A weak version of the following Lemma was independently proven by Leibman.

Lemma 11. *Suppose κ is supercompact and $\kappa < \delta$, where δ is an inaccessible cardinal such that $\mathrm{V}_\delta \prec \mathrm{V}$. Then there is a forcing extension $\mathrm{V}[G]$ of V in which $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$ holds and in which κ is still supercompact.*

Proof.

- Force to make κ Laver indestructible,

A weak version of the following Lemma was independently proven by Leibman.

Lemma 11. *Suppose κ is supercompact and $\kappa < \delta$, where δ is an inaccessible cardinal such that $\mathrm{V}_\delta \prec \mathrm{V}$. Then there is a forcing extension $\mathrm{V}[G]$ of V in which $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$ holds and in which κ is still supercompact.*

Proof.

- Force to make κ Laver indestructible,
- then force $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$.

□

A related Question

What is the consistency strength of a weakly compact κ such that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})/\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds?

A related Question

What is the consistency strength of a weakly compact κ such that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})/\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds?

The following is worthwhile to note in this context:

Observation 12. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\}) + \kappa$ *is weakly compact. Then the weak compactness of* κ *is indestructible under* $<\kappa$*-closed forcing.*

A related Question

What is the consistency strength of a weakly compact κ such that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})/\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds?

The following is worthwhile to note in this context:

Observation 12. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\}) + \kappa$ *is weakly compact. Then the weak compactness of* κ *is indestructible under* $<\kappa$*-closed forcing.*

Proof. That κ is weakly compact is expressed by a Π^1_2-formula over H_κ. $\square$

A Digression: The strength of an indestructibly weakly compact cardinal

A Digression: The strength of an indestructibly weakly compact cardinal

Apter and Hamkins: If κ is weakly compact, and its weak compactness is indestructible by $<\kappa$-directed-closed forcing, and this indestructibility was achieved by forcing that has a closure point below κ, then κ was supercompact in the ground model.

A Digression: The strength of an indestructibly weakly compact cardinal

Apter and Hamkins: If κ is weakly compact, and its weak compactness is indestructible by $<\kappa$-directed-closed forcing, and this indestructibility was achieved by forcing that has a closure point below κ, then κ was supercompact in the ground model.

Schimmerling and Steel: If K exists and κ is weakly compact, then κ is weakly compact in K and ${\kappa^+}^K = \kappa^+$.

A Digression: The strength of an indestructibly weakly compact cardinal

Apter and Hamkins: If κ is weakly compact, and its weak compactness is indestructible by $<\kappa$-directed-closed forcing, and this indestructibility was achieved by forcing that has a closure point below κ, then κ was supercompact in the ground model.

Schimmerling and Steel: If K exists and κ is weakly compact, then κ is weakly compact in K and ${\kappa^+}^K = \kappa^+$.

Fuchs and Schindler: Obtain a non-domestic mouse.

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in H_{κ^+}.

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in H_{κ^+}.

Let $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ be the principle stating that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with H_{κ^+} interpreted in the extension).

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in H_{κ^+}.

Let $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ be the principle stating that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with H_{κ^+} interpreted in the extension).

Theorem 13. [Fuchs/Hamkins] $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *is inconsistent with* ZFC, *if* $\kappa > \omega$.

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in H_{κ^+}.

Let $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ be the principle stating that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with H_{κ^+} interpreted in the extension).

Theorem 13. [Fuchs/Hamkins] $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *is inconsistent with* ZFC, *if* $\kappa > \omega$.

Proof. Assume ZFC$+\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$.

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in H_{κ^+}.

Let $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ be the principle stating that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with H_{κ^+} interpreted in the extension).

Theorem 13. [Fuchs/Hamkins] $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *is inconsistent with* ZFC, *if* $\kappa > \omega$.

Proof. Assume ZFC$+\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$. Force to add a slim κ-Kurepa tree.

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\mathsf{MP}_{<\kappa-\text{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in H_{κ^+}.

Let $\Box\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ be the principle stating that $\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with H_{κ^+} interpreted in the extension).

Theorem 13. [Fuchs/Hamkins] $\Box\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ *is inconsistent with* ZFC, *if* $\kappa > \omega$.

Proof. Assume ZFC+$\Box\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$. Force to add a slim κ-Kurepa tree. Contradiction. □

Compare this with the following:

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in H_{κ^+}.

Let $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ be the principle stating that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with H_{κ^+} interpreted in the extension).

Theorem 13. [Fuchs/Hamkins] $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *is inconsistent with* ZFC, *if* $\kappa > \omega$.

Proof. Assume ZFC+$\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$. Force to add a slim κ-Kurepa tree. Contradiction. □

Compare this with the following:

Theorem 14. [Woodin] $\Box\mathsf{MP}(\mathbb{R})$ *is consistent, assuming strong axioms of infinity.*

Impossible strengthenings of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$

Note: $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{\kappa\})$ cannot be consistently strengthened by allowing for parameters which are not in H_{κ^+}.

Let $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ be the principle stating that $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ holds in every forcing extension obtained by $<\kappa$-closed forcing (with H_{κ^+} interpreted in the extension).

Theorem 13. [Fuchs/Hamkins] $\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *is inconsistent with* ZFC*, if* $\kappa > \omega$*.*

Proof. Assume ZFC$+\Box\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$. Force to add a slim κ-Kurepa tree. Contradiction. □

Compare this with the following:

Theorem 14. [Woodin] $\Box\mathsf{MP}(\mathbb{R})$ *is consistent, assuming strong axioms of infinity.*

Theorem 15. [Hamkins/Woodin] $\Box\mathsf{MP}_{ccc}(\mathbb{R})$ *is equiconsistent with the existence of a weakly compact cardinal.*

The same proof shows that the principle $\Box\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+})$ is inconsistent.

The same proof shows that the principle $\Box\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+})$ is inconsistent.

Note that it is not the case that the stronger a principle is, the stronger its necessary form is! Indeed, the following questions arise:

The same proof shows that the principle $\Box\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+})$ is inconsistent.

Note that it is not the case that the stronger a principle is, the stronger its necessary form is! Indeed, the following questions arise:

Question 16. *Is $\Box\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$ consistent?*

Is $\Box\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+})$ consistent?

Separating the principles

Separating the principles

Recall the relationships between the principles:

$$\begin{array}{ccc}
\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_\kappa \cup \{\kappa\}) & \Longleftarrow & \mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+}) \\
\Downarrow & & \Downarrow \\
\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_\kappa \cup \{\kappa\}) & \Longleftarrow & \mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+}) \\
\Downarrow & & \Downarrow \\
\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\}) & \Longleftarrow & \mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})
\end{array}$$

Can any of these implications be reversed?

Producing other models of closed maximality principles

Producing other models of closed maximality principles

Observation 17. $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$, *if true, is* $<\kappa$*-closed-necessary. Actually,* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{a\})$ *persists to* $<\kappa$*-closed extensions, for any* a*.*

The analogous statements apply to the maximality principles for $<\kappa$*-directed-closed forcings and forcings from* $\mathrm{Col}(\kappa)$ *as well.*

Producing other models of closed maximality principles

Observation 17. $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$*, if true, is* $<\kappa$*-closed-necessary. Actually,* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{a\})$ *persists to* $<\kappa$*-closed extensions, for any* a*.*

The analogous statements apply to the maximality principles for $<\kappa$*-directed-closed forcings and forcings from* $\mathrm{Col}(\kappa)$ *as well.*

For the boldface versions of the maximality principles for $<\kappa$-closed or $<\kappa$-directed-closed forcing, there is the following Lemma:

Lemma 18. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$*. Let* $\mathbb{P}$ *be a* $<\kappa^+$*-closed notion of forcing. If* G *is* $\mathbb{P}$*-generic, then in* $\mathrm{V}[G]$*,* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *continues to hold. This remains true if "*$<\kappa$*-closed" is replaced with "*$<\kappa$*-directed-closed".*

Producing other models of closed maximality principles

Observation 17. $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$*, if true, is* $<\kappa$*-closed-necessary. Actually,* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{a\})$ *persists to* $<\kappa$*-closed extensions, for any* a*.*

The analogous statements apply to the maximality principles for $<\kappa$*-directed-closed forcings and forcings from* $\mathrm{Col}(\kappa)$ *as well.*

For the boldface versions of the maximality principles for $<\kappa$-closed or $<\kappa$-directed-closed forcing, there is the following Lemma:

Lemma 18. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$*. Let* $\mathbb{P}$ *be a* $<\kappa^+$*-closed notion of forcing. If* G *is* $\mathbb{P}$*-generic, then in* $\mathrm{V}[G]$, $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *continues to hold. This remains true if "*$<\kappa$*-closed" is replaced with "*$<\kappa$*-directed-closed".*

Note: Why is a version of the previous lemma for $\mathrm{Col}(\kappa)$ and $\mathrm{Col}(\kappa^+)$ missing?

Producing other models of closed maximality principles

Observation 17. $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$*, if true, is* $<\kappa$*-closed-necessary. Actually,* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\{a\})$ *persists to* $<\kappa$*-closed extensions, for any* a*.*

The analogous statements apply to the maximality principles for $<\kappa$*-directed-closed forcings and forcings from* $\mathrm{Col}(\kappa)$ *as well.*

For the boldface versions of the maximality principles for $<\kappa$-closed or $<\kappa$-directed-closed forcing, there is the following Lemma:

Lemma 18. *Assume* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$*. Let* $\mathbb{P}$ *be a* $<\kappa^+$*-closed notion of forcing. If* G *is* $\mathbb{P}$*-generic, then in* $\mathrm{V}[G]$*,* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *continues to hold. This remains true if "*$<\kappa$*-closed" is replaced with "*$<\kappa$*-directed-closed".*

Note: Why is a version of the previous lemma for $\mathrm{Col}(\kappa)$ and $\mathrm{Col}(\kappa^+)$ missing? Because there is none.

Separating $\mathsf{MP}_{<\kappa-\text{closed}}$ from $\mathsf{MP}_{<\kappa-\text{dir. cl.}}$

Separating $\mathsf{MP}_{<\kappa-\text{closed}}$ from $\mathsf{MP}_{<\kappa-\text{dir. cl.}}$

Lemma 19. *Assuming κ is supercompact, $\kappa < \delta$ and $\mathrm{V}_\delta \prec \mathrm{V}$, there is a model in which $\mathsf{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$ holds, but $\mathsf{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\})$ does not.*

If moreover δ is inaccessible, then there is a model in which $\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ holds, but $\mathsf{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\})$ does not.

Proof. Focus on the boldface part.

Proof. Focus on the boldface part.

- Do the Laver preparation.

Proof. Focus on the boldface part.

- Do the Laver preparation.
- Force $\mathrm{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa^+})$. Call the resulting model M.

Proof. Focus on the boldface part.

- Do the Laver preparation.
- Force $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+})$. Call the resulting model M.
- Force over M to add a κ^+-regressive κ^+-Kurepa tree.

Proof. Focus on the boldface part.

- Do the Laver preparation.
- Force $\mathrm{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+})$. Call the resulting model M.
- Force over M to add a κ^+-regressive κ^+-Kurepa tree.

  The forcing is $<\kappa^+$-closed and destroys κ's supercompactness (König-Yoshinobu). Call the model N.

Proof. Focus on the boldface part.

- Do the Laver preparation.

- Force $\mathsf{MP}_{<\kappa-\text{dir. cl.}}(H_{\kappa^+})$. Call the resulting model M.

- Force over M to add a κ^+-regressive κ^+-Kurepa tree.

  The forcing is $<\kappa^+$-closed and destroys κ's supercompactness (König-Yoshinobu). Call the model N.

- N is a model of $\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$.

Proof. Focus on the boldface part.

- Do the Laver preparation.
- Force $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+})$. Call the resulting model M.
- Force over M to add a κ^+-regressive κ^+-Kurepa tree.

  The forcing is $<\kappa^+$-closed and destroys κ's supercompactness (König-Yoshinobu). Call the model N.
- N is a model of $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$.
- N is not a model of $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(\{\kappa\})$.

$\square$

Separating $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}$ from $\mathsf{MP}_{\mathrm{Col}(\dot\kappa)}$

Separating $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}$ **from** $\mathsf{MP}_{\mathrm{Col}(\dot\kappa)}$

Lemma 20.

1. $\mathsf{MP}_{\mathrm{Col}(\kappa)}(\emptyset)$ *implies that* $\mathrm{V} \neq \mathsf{HOD}$*.*

Separating $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}$ from $\mathsf{MP}_{\mathrm{Col}(\dot\kappa)}$

Lemma 20.

1. $\mathsf{MP}_{\mathrm{Col}(\kappa)}(\emptyset)$ *implies that* $\mathrm{V} \neq \mathsf{HOD}$.

2. $\mathsf{MP}_{<\kappa-\mathrm{closed}}(\emptyset)$ *implies that there is a forcing extension of an initial segment of* L *in which* $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_\kappa \cup \{\kappa\}) + \mathrm{V} = \mathsf{HOD}$ *holds. Analogously,* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *implies that there is a forcing extension of* L *in which* $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_{\kappa^+}) + \mathrm{V} = \mathsf{HOD}$ *holds.*

Proof. Part 1:

Proof. Part 1: "$\mathrm{V} \neq$ HOD" is $\mathrm{Col}(\kappa)$-forceably necessary.

Proof. Part 1: "$V \neq$ HOD" is $\mathrm{Col}(\kappa)$-forceably necessary.

Part 2: Focus on the boldface claim. Let $\delta = (\kappa^+)$.

Proof. Part 1: "$\mathrm{V} \neq$ HOD" is $\mathrm{Col}(\kappa)$-forceably necessary.

Part 2: Focus on the boldface claim. Let $\delta = (\kappa^+)$.

- $L_\delta \prec L$.

Proof. Part 1: "$\mathrm{V} \neq$ HOD" is $\mathrm{Col}(\kappa)$-forceably necessary.

Part 2: Focus on the boldface claim. Let $\delta = (\kappa^+)$.

- $L_\delta \prec L$.
- Let G be $\mathrm{Col}(\kappa, <\delta)$-generic over L. So $L[G]$ is a model of $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$.

Proof. Part 1: "$\mathrm{V} \neq$ HOD" is $\mathrm{Col}(\kappa)$-forceably necessary.

Part 2: Focus on the boldface claim. Let $\delta = (\kappa^+)$.

- $L_\delta \prec L$.
- Let G be $\mathrm{Col}(\kappa, <\delta)$-generic over L. So $L[G]$ is a model of $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$.
- Force to code G into the continuum function well above δ.

Proof. Part 1: "$\mathrm{V} \neq$ HOD" is $\mathrm{Col}(\kappa)$-forceably necessary.

Part 2: Focus on the boldface claim. Let $\delta = (\kappa^+)$.

- $L_\delta \prec L$.
- Let G be $\mathrm{Col}(\kappa, < \delta)$-generic over L. So $L[G]$ is a model of $\mathrm{MP}_{\mathrm{Col}(\kappa)}(H_{\kappa^+})$.
- Force to code G into the continuum function well above δ.
- The result is a model of $\mathrm{V} =$ HOD, where $\mathrm{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ still holds, because the forcing was $<\kappa^+$-closed.

Boldface vs. lightface Principles

Boldface vs. lightface Principles

Lemma 21.

1. *Assuming* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$*, there is a forcing extension in which* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$ *holds but* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *fails.*

Boldface vs. lightface Principles

Lemma 21.

1. *Assuming* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$, *there is a forcing extension in which* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$ *holds but* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *fails.*

2. *Assuming* $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_\kappa \cup \{\kappa\})$, *there is forcing extension in which* $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_\kappa \cup \{\kappa\})$ *holds but* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *fails.*

Boldface vs. lightface Principles

Lemma 21.

1. *Assuming* $\mathsf{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$*, there is a forcing extension in which* $\mathsf{MP}_{<\kappa-\text{closed}}(H_\kappa \cup \{\kappa\})$ *holds but* $\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ *fails.*

2. *Assuming* $\mathsf{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\})$*, there is forcing extension in which* $\mathsf{MP}_{<\kappa-\text{dir. cl.}}(H_\kappa \cup \{\kappa\})$ *holds but* $\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ *fails.*

3. *Assuming* $\mathsf{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$*, there is a model of* $\mathsf{MP}_{\text{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$ *in which* $\mathsf{MP}_{<\kappa-\text{closed}}(H_{\kappa^+})$ *is false.*

Boldface vs. lightface Principles

Lemma 21.

1. *Assuming* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$, *there is a forcing extension in which* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_\kappa \cup \{\kappa\})$ *holds but* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *fails.*

2. *Assuming* $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_\kappa \cup \{\kappa\})$, *there is forcing extension in which* $\mathsf{MP}_{<\kappa-\mathrm{dir.\ cl.}}(H_\kappa \cup \{\kappa\})$ *holds but* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *fails.*

3. *Assuming* $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$, *there is a model of* $\mathsf{MP}_{\mathrm{Col}(\kappa)}(H_\kappa \cup \{\kappa\})$ *in which* $\mathsf{MP}_{<\kappa-\mathrm{closed}}(H_{\kappa^+})$ *is false.*

So in general,

none of the implications

shown in the figure

can be reversed.