Uncountably categorical approximations and Gromov-Hausdorff limits

Assaf Hasson   Boris Zilber

Mathematical Institute
University of Oxford

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3. Zilber's reformulation: Uncountably categorical structures have analytic prototypes.

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$$\mathrm{tr.d}(\mathbb{Q}(x_1, \ldots, x_n, e^{x_1} \ldots, e^{x_n})) - \mathrm{lin.d}(x_1, \ldots, x_n) \geq 0$$

for all $x_1, \ldots x_n \in \mathbb{C}$.

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Leading to Zilber's bold conjecture:
$\mathbb{C}_{\mathrm{exp}} := (\mathbb{C}, 0, 1, +, \cdot, \exp)$ satisfies a non first order version of Hrushovski's free construction.

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But what are metric approximations of model theoretic structures ?

A tentative definition

Definition

Given two metric $\mathcal{L}$-structures $\mathcal{M}_1$ and $\mathcal{M}_2$ embedded in a common metric space

$$\mathrm{d}(M_1, M_2) = \sup\{\mathrm{d}_{\mathrm{H}}(R(\mathcal{M}_1), R(\mathcal{M}_2)) : R \in \mathcal{L} \text{ a basic relation}\}$$

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Given an index set I, and ultrafilter $\mathcal{D}$ on I a collection of $\mathcal{L}$-substructures $\{\mathcal{M}_i\}_{i \in I}$ of $\mathcal{M}$ *metrically approximate* $\mathcal{M}$ if

$$\lim_{\mathcal{D}} \mathrm{d}(\mathcal{M}_i, \mathcal{M}) = 0$$

Some examples

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- What is the connection between the metric and the logic in such approximations. Is the property of being metrically approximable model theoretically meaningful.
- What should be the model theoretic restrictions on metric approximations.
- Are there more interesting, more natural, examples of metric approximations.
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