# Higher-Order Reverse Topology

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# Outline

# Review of second-order reverse math

Traditional reverse math studies subsystems of *second-order arithmetic*.

- **Language:** Number (type-0) and set (type-1) variables; $\{0, +, \cdot, \text{etc.}\}$; "$=_0$" for numbers (but not sets); "$\in$" relates numbers and sets.
- Base theory, $\mathbf{RCA}_0$: Axioms for number-theoretic $\mathbb{N}$; induction schema for $\Sigma^0_1$ formulas; comprehension schema for $\Delta^0_1$ formulas.
    - The second-order part of the minimal $\omega$-model of $\mathrm{RCA}_0$ consists of all computable (recursive) sets.
    - The first-order part of the theory $\mathrm{RCA}_0$ is $\Sigma^0_1$-PA [4].
- A stronger theory, $\mathbf{ACA}_0$: Axioms for $\mathrm{RCA}_0$; comprehension schema for arithmetical (or "$\Pi^0_\infty$") formulas.
    - The second-order part of the minimal $\omega$-model of $\mathrm{ACA}_0$ consists of all arithmetical sets.
    - The first-order part of the theory $\mathrm{ACA}_0$ is PA [4].

# Finite types

**Definition**

*The* **finite types** *are defined inductively:*

- 0 *is a type.*
- *If* $\sigma$ *and* $\tau$ *are types then* $(\sigma \to \tau)$ *is a type.*

0 is the type of natural numbers; $(\sigma \to \tau)$ is the type of a functional mapping type-$\sigma$ elements to type-$\tau$ elements.

**Definition**

*The* **standard types** *are defined inductively:*

- 0 *is a standard type.*
- *If* $n$ *is a standard type, then* $n + 1 := (n \to 0)$ *is a standard type.*

**Example:** reals are of type 1.

# Higher-order reverse math

The language of second-order arithmetic may be too restrictive. In a recent paper [3], Kohlenbach described a base theory in a more-flexible, higher-order language.

- **Language:** Variables of all finite types; $\{0, +, \cdot, \text{etc.}\}$ as before; "$=_0$" only for (type-0) numbers, as before; plus—
  - **Combinators** $\Pi_{\rho,\tau}$, $\Sigma_{\sigma,\rho,\tau}$ (for $\lambda$-**abstraction**);
  - Some symbol for **application**, not shown here; and
  - Symbol $R_0$, for **primitive recursion**.
- Base theory, $\mathbf{RCA}_0^\omega$ ($=$ E-PRA$^\omega$ $+$ QF-AC$^{1,0}$): Axioms for number-theoretic $\mathbb{N}$, as before; induction schema for quantifier-free formulas; axioms defining $R_0$, the $\Pi_{\rho,\tau}$'s, and the $\Sigma_{\sigma,\rho,\tau}$'s; extensionality axioms; and QF-AC$^{1,0}$:

$$\forall x^1 \exists n^0 (\Phi(x, n)) \to \exists F^{(1\to 0)} \forall x^1 (\Phi(x, F(x))),$$

where $\Phi$ is a quantifier-free formula.

# The axioms $(E_1)$ and $(E_2)$

**Definition**

*The axiom $(E_1)$ is the statement:*

$$\exists E_1{}^2 \left[\forall x^1 (E_1(x) =_0 1) \leftrightarrow \exists n^0 (x(n) \neq_0 0)\right].$$

**Definition**

*The axiom $(E_2)$ is the statement:*

$$\exists E_2{}^3 \left[\forall X^2 (E_2(X) =_0 1) \leftrightarrow \exists x^1 (X(x) \neq_0 0)\right].$$

- Higher-order equality is defined inductively. E.g., $x^1 =_1 y^1 \iff \forall n^0 (x(n) =_0 y(n))$.
- Think of $E_1$ as a functional determining type-1 equality:

$$x^1 =_1 y^1 \iff E_1(\lambda n^0.(x(n) - y(n))) =_0 0.$$

# Conservation results

**Proposition (Kohlenbach [3])**

*$RCA_0^\omega$ is conservative over and implies $RCA_0$.*

**Proposition (H.)**

1. *$RCA_0^\omega + (E_1)$ is conservative over and implies $ACA_0$.*
2. *$RCA_0^\omega + QF\text{-}AC^{0,1}$ is conservative over and implies $\Sigma^1_1\text{-}AC_0$.*
3. *$RCA_0^\omega + (E_2)$ is conservative over and implies $\Pi^1_\infty\text{-}CA_0$.*
4. *Etc.*

The proof of the second proposition uses term models and is analogous to the proof of the first.

# Sets, families

## Definition

1. *A* **real** *is a (type-1) function.*
2. *A* **set** *is a (type-2) functional $X$ such that* $\forall x^1 (X(x) =_0 0 \vee X(x) =_0 1)$.
3. *A* **family** *is a (type-3) functional $\mathcal{F}$ such that* $\forall X^2 (\mathcal{F}(x) =_0 0 \vee \mathcal{F}(X) =_0 1)$.

*(We write "$x \in X$" as shorthand for "$X(x) =_0 1$.")*

We consider only topologies on the reals.

# Topologies

**Definition**

*A family $\mathcal{F}$ is a topology iff:*

1. $\emptyset := (\lambda x^1.0) \in \mathcal{F}$;
2. ${}^{\mathbb{N}}\mathbb{N} := (\lambda x^1.1) \in \mathcal{F}$;
3. *if* $X \in \mathcal{F}$ *and* $Y \in \mathcal{F}$ *then* $X \cap Y := (\lambda x. \min(X(x), Y(x))) \in \mathcal{F}$; *and*
4. *if* $\mathcal{G} \subseteq_2 \mathcal{F}$ *and* $\bigcup \mathcal{G} := \{x : \exists X^2 \in \mathcal{G}\,(x \in X)\}$ *exists, then* $\bigcup \mathcal{G} \in \mathcal{F}$.

**Examples:** The indiscrete topology is $\{\emptyset, {}^{\mathbb{N}}\mathbb{N}\}$, and the discrete topology is $(\lambda X^2.1)$.

# Simple equivalences

## Proposition (H. and folklore)

*Over $RCA_0^\omega$, we have the following equivalences:*

1. $(E_2) \iff$ *there is a topology for a connected space (i.e., only $\emptyset$ and ${}^{\mathbb{N}}\mathbb{N}$ are clopen).*
2. $(E_2) \iff$ *there is a topology with a dense, nowhere-dense set.*
3. $(E_2) \iff$ *there is a topology generated by a countable enumeration for a basis.*

A consequence of (3) is that any topological statement examined in second-order reverse math follows, in higher-order reverse math, from the existence of such a **formal** topology. So second-order reverse topology does not carry over nicely to higher-order theories.

# More simple equivalences

## Proposition (H. and folklore)

*Over* $RCA_0^\omega + (E_1)$, *we have the following equivalences:*

1. $(E_2) \iff$ *there is a topology with a countable dense set.*
2. $(E_2) \iff$ *there is a topology for a space that is the countable union of nowhere-dense sets (i.e., is of first category).*

# Topology in $\mathrm{RCA}_0^\omega + (E_1)$

**Proposition (H.)**

*If $\mathcal{T}$ is a topology existing in a minimal term model of $\mathrm{RCA}_0^\omega + (E_1)$ then $\mathcal{T}$ is equivalent to $T \times \mathcal{P}(^{\mathbb{N}}\mathbb{N} \setminus X)$, where $X = \{x_0, x_1, \dots\}$ is a countable set and $T$ is a topology on $X$.*

(In other words $\mathcal{T}$ is essentially just a topology on $\mathbb{N}$.)

# Open questions

Over $\mathsf{RCA}_0^\omega + (E_2)$:

- $\mathsf{QF\text{-}AC}^{1,2} \implies$ "every $T_2$ space has a witnessing functional" $\implies \mathsf{QF\text{-}AC}^{1,1}$.
- "Every $T_2$ space has a witnessing functional" $\implies$:
  - every compact $T_2$ space is $T_4$.
  - every compact $T_2$ space has a basis of size $\leq 2^{\aleph_0}$.
- The principle $(E_3) \implies$ that every compact, $T_2$ space is $T_4$.

Open question: what about reversals?

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