Cardinal Structure Under Determinacy

We assume throughout $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$, and develop the cardinal structure or "cardinal arithmetic" as far as possible.

We describe:

- The "global" results, those that hold throughout the entire Wadge hierarchy but are of a more general nature.
- The "local" results which require a more detailed inductive analysis, but which provide a detailed understanding of the cardinal structure. These results are currently only known to extend through a comparatively small initial segment of the Wadge hierarchy.

As an application we present a result which links the cardinal structure of $L(\mathbb{R})$ to that of V.

Basic Concepts

AD: Every two player integer game is determined.
For $A \subseteq \omega^\omega$, we have the game G_A:

$$\begin{array}{llllll} \mathrm{I} & x(0) & & x(2) & & x(4) & & \ldots \\ \mathrm{II} & & x(1) & & x(3) & & x(5) & & \ldots \end{array}$$

I wins the run iff $x \in A$, where

$$x = (x(0), x(1), x(2), \ldots).$$

A *strategy* (for an integer game) for I (II) is a function σ from the sequences $s \in \omega^{<\omega}$ of even (odd) length to ω. We say σ is a *winning strategy* for I (II) if for all runs $x \in \omega^\omega$ of the game where I (II) has followed σ, we have $x \in A$ ($x \notin A$).

If σ is a strategy for I (or II), and $x = (x(1), x(3), \ldots) \in \omega^\omega$, let $\sigma(x) \in \omega^\omega$ be the result of following σ against II's play of x. Note that $\sigma[\omega^\omega]$ is a $\mathbf{\Sigma}^1_1$ subset of ω^ω.

We employ variations of this notation, e.g., describe a game by saying I plays out x, II plays out y. Here we might use $\sigma(y)$ to denote just I's response x following σ against II plays of y. Meaning should be clear from context.

Concepts generalize in natural ways to games on sets X other than ω. If X cannot be wellordered in ZF then we usually use *quaistrategies* (which assign a non-empty set $\sigma(s) \subseteq X$ to $s \in X^{<\omega}$ of appropriate parity length).

If $\mathbf{\Gamma}$ is a collection of subsets of ω^ω, we write $\det(\mathbf{\Gamma})$ to denote that G_A is determined for all $A \in \mathbf{\Gamma}$.

DC: If $R \subseteq X^{<\omega}$ and

$$\forall (x_0, \ldots x_{n-1}) \in R\ \exists x_n\ (x_1, \ldots, x_{n-1}, x_n) \in R,$$

then $\exists \vec{x} \in X^\omega\ \forall n\ (\vec{x} \restriction n \in R)$.

Equivalent to: Every illfounded relation has an infinite descending sequence.

$L(\mathbb{R})$

$L(\mathbb{R})$ is the smallest inner-model (i.e., transitive, proper class model) containing the reals $\mathbb{R}$ (which we often identify with ω^ω).

It is defined through a hierarchy similar to L.

$$\begin{aligned} J_0(\mathbb{R}) &= V_{\omega+1} \\ J_\alpha(\mathbb{R}) &= \bigcup_{\beta<\alpha} J_\beta(\mathbb{R}) \text{ for } \alpha \text{ limit.} \\ J_{\alpha+1}(\mathbb{R}) &= \mathrm{rud}(J_\alpha(\mathbb{R}) \cup \{J_\alpha(\mathbb{R})\}) \end{aligned}$$

$J_{\alpha+1}(\mathbb{R}) = \bigcup_n S^n(J_\alpha(\mathbb{R}))$, where $S(X) = \bigcup_{i=1}^{11} F_i[X \cup \{X\}]$.

Each $S^n(J_\alpha(\mathbb{R}))$ is transitive. Every set in $L(\mathbb{R})$ is ordinal definable from a real. In fact, there is uniformly in α a $\Sigma_1(J_\alpha(\mathbb{R}))$ map from $\omega\alpha^{<\omega} \times \mathbb{R}$ onto $J_\alpha(\mathbb{R})$.

Some facts:

(1) AD contradicts ZFC but is consistent with restricted forms of determinacy, e.g., projective determinacy PD, or the determinacy of games in $L(\mathbb{R})$.
(2) AD is equivalent to AD_X, where X is any countable set with at least two elements.
(3) AD_{ω_1} is inconsistent.
(4) $\mathsf{AD}_{\mathbb{R}}$ is a (presumably) consistent strengthening of AD. It is equivalent (Martin-Woodin) to $\mathsf{AD}+$ every set has a scale.
(5) $\mathsf{AD} \Rightarrow \mathsf{AD}^{L(\mathbb{R})}$.
(6) DC is independent of even $\mathsf{AD}_{\mathbb{R}}$ (Solovay), but on the other hand $\mathsf{AD} \Rightarrow \mathsf{DC}^{L(\mathbb{R})}$ (Kechris).
(7) AD implies regularity properties for sets of reals, e.g., every set of reals has the perfect set property, the Baire property, is measurable, Ramsey. The determinacy needed is local, e.g., $\mathbf{\Pi}^1_1$-det implies perfect set property for $\mathbf{\Sigma}^1_2$.
(8) Under AD, successor cardinals need not be regular (but $\mathrm{cof}(\kappa^+) > \omega$).

Global Results-First Pass: Separation, Reduction, Prewellordering

Definition. For $A, B \subseteq \omega^\omega$, we say that A is *Wadge reducible* to B, $A \leq_w B$, if there is a continuous function $f\colon \omega^\omega \to \omega^\omega$ such that $A = f^{-1}(B)$, i.e., $x \in A$ iff $f(x) \in B$.

We say A is *Lipschitz reducible* to B, $A \leq_\ell B$ if there is a Lipschitz continuous f (i.e., a strategy for II) such that $A = f^{-1}(B)$.

For $A, B \subseteq \omega^\omega$, we have the basic (Lipschitz) Wadge game $G_\ell(A, B)$: I plays out x, II plays out y, and II wins the run iff $(x \in A \leftrightarrow y \in B)$.

If II has a winning strategy then $A \leq_\ell B$. If I has a winning strategy then $B \leq_\ell A^c$.

We consider pairs $\{A, A^c\}$. We say $\{A, A^c\} \leq \{B, B^c\}$ if $A \leq B$ or $A \leq B^c$ ($\leq$ means $\leq_\ell$ or $\leq_w$).

Theorem (Martin-Monk)**.** *$\leq_\ell$ is a wellordering (and hence also $\leq_w$).*

We say A is Lipschitz selfdual if $A \leq_\ell A^c$, and likewise for Wadge selfdual. A theorem of Steel says A is Lipschitz selfdual iff A is Wadge selfdual. We write $\{A\}$ in this case.

Definition. If $A \subseteq \omega^\omega$, then $o_\ell(A)$ denotes the rank of $\{A, A^c\}$ in $\leq_\ell$. $o(A) = o_w(A)$ denotes the rank in $\leq_w$.

The following results suffice to give a complete picture of the ℓ and w degrees.

- If A is non-selfdual, then $A \oplus A^c$ is selfdual and is the next ℓ-degree after A.

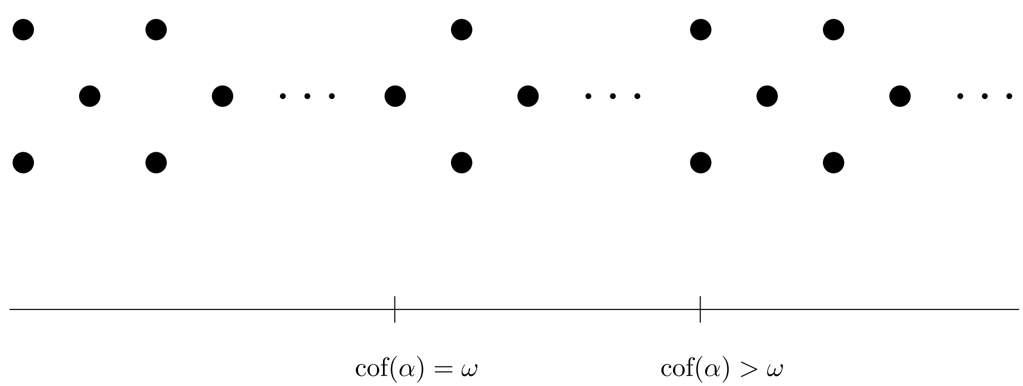
  Here $A \oplus B$ denotes the join

  $$\begin{aligned} A \oplus B = \{x\colon & (x(0) \text{ is even } \wedge x' \in A) \\ & \vee (x(0) \text{ is odd } \wedge x' \in B)\}, \end{aligned}$$

  where $x'(k) = x(k+1)$.

- If A is selfdual, then the next ℓ-degree after A is selfdual and consists of $A' = \{0^\frown x \colon x \in A\}$.
- At limit ordinals α of cofinality omega there is a selfdual degree consisting of the countable join of sets of degrees cofinal in α.
- At limit ordinals of uncountable cofinality there is a non-selfdual degree.
- The next ω_1 ℓ-degrees after a selfdual degree are all w-equivalent.

Picture of the w-degrees



Definition. A pointclass is a collection $\mathbf{\Gamma} \subseteq \mathcal{P}(\omega^\omega)$ closed under Wadge reduction.

We let $o(\mathbf{\Gamma}) = \sup\{o(A) \colon A \in \mathbf{\Gamma}\}$.

We say $\mathbf{\Gamma}$ is selfdual if $A \in \mathbf{\Gamma} \Rightarrow A^c \in \mathbf{\Gamma}$. Otherwise $\mathbf{\Gamma}$ is non-selfdual.

If $\mathbf{\Gamma}$ is non-selfdual, then $\mathbf{\Gamma}$ has a universal set. Let $A \in \mathbf{\Gamma} - \check{\mathbf{\Gamma}}$. Define $U \subseteq \omega^\omega \times \omega^\omega$ by $U(x, y) \leftrightarrow \tau_x(y) \in A$.

Here every $x \in \omega^\omega$ is viewed as a strategy τ_x for II by: $\tau_x(s) = x(\langle s \rangle)$ where $s \mapsto \langle s \rangle$ is a reasonable bijection between $\omega^{<\omega}$ and ω.

Fact. From universal sets one can construct (in ZF) *good* universal sets, that is, universal sets which admit continuous s-m-n functions, and hence have the recursion theorem.

Definition. $\mathbf{\Gamma}$ has the *separation* property, sep($\mathbf{\Gamma}$), if whenever A, $B \in \mathbf{\Gamma}$ and $A \cap B = \emptyset$, then there is a $C \in \mathbf{\Delta} = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ with $A \subseteq C$, $B \cap C = \emptyset$.

Theorem (Steel-Van Wesep)**.** *For any non-selfdual* $\mathbf{\Gamma}$*, exactly one of sep*($\mathbf{\Gamma}$)*, sep*($\check{\mathbf{\Gamma}}$) *holds.*

Definition. A *norm* φ on a set $A \subseteq \omega^\omega$ is a map $\varphi\colon A \to \mathrm{On}$. We say φ is *regular* if $\mathrm{ran}(\varphi) \in \mathrm{On}$.

Norms φ on sets A can be identified with *prewellorderings* $\preceq$ of A (connected, reflexive, transitive, binary relations on A). The pwo induces a wellordering on the equivalence classes

$$[x] = \{y\colon x \preceq y \wedge y \preceq x\}.$$

The corresponding norm is $\varphi(x) =$ rank of $[x]$.

Definition. A $\mathbf{\Gamma}$-norm φ on $A \subseteq \omega^\omega$ is a norm such the relations

$$x <^* y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \varphi(x) < \varphi(y)))$$
$$x \leq^* y \leftrightarrow x \in A \wedge (y \notin A \vee (y \in A \wedge \varphi(x) \leq \varphi(y)))$$

are both in $\mathbf{\Gamma}$.

Definition. $\mathbf{\Gamma}$ has the *prewellordering property*, pwo($\mathbf{\Gamma}$), if every $A \in \mathbf{\Gamma}$ admits a $\mathbf{\Gamma}$-norm.

Let $\varphi\colon A \to \theta$ be a (regular) $\mathbf{\Gamma}$-norm on A. for $\alpha < \theta$, let $A_\alpha = \{x \in A\colon \varphi(x) = \alpha\}$. So, $A_\alpha = A_{\leq\alpha} - A_{<\alpha}$ (in natural notation).

If $x \in A$ and $\varphi(x) = \alpha$, then:

$$A_{\leq\alpha} = \{y\colon y \leq^* x\} = \{y\colon \neg(x <^* y)\}.$$

So, $A_{\leq\alpha} \in \mathbf{\Delta}$. Likewise $A_{<\alpha} \in \mathbf{\Delta}$, and so $A_\alpha \in \mathbf{\Delta}$. So, pwo($\mathbf{\Gamma}$) give an effective way of writing every $A \in \mathbf{\Gamma}$ as an increasing union of $\mathbf{\Delta}$ sets.

The initial segment $\prec_\alpha$ of the prewellordering can be computed as:

$$\begin{aligned} x \prec_\alpha y &\leftrightarrow x, y \in A_{\leq\alpha} \wedge (x \leq^* y) \\ &\leftrightarrow x, y \in A_{\leq\alpha} \wedge \neg(y <^* x) \end{aligned}$$

So, $\prec_\alpha \in \mathbf{\Delta}$ if $\mathbf{\Gamma}$ is closed under $\wedge$, $\vee$.

Definition. $\delta(\mathbf{\Gamma})$ = the supremum of the lengths of the $\mathbf{\Delta}$ prewellorderings.

So, if $\mathbf{\Gamma}$ is closed under $\wedge$, $\vee$, and φ is a $\mathbf{\Gamma}$-norm then $|\varphi| \leq \delta(\mathbf{\Gamma})$.

Fact. If $\mathbf{\Gamma}$ is closed under $\forall^{\omega^\omega}$, $\wedge$, $\vee$, and φ is a $\mathbf{\Gamma}$ norm on a $\mathbf{\Gamma}$-complete set, then $|\varphi| = \delta(\mathbf{\Gamma})$.

Levy Classes

Definition. $\mathbf{\Gamma}$ is a *Levy class* if it is a non-selfdual pointclass closed under either $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$ (or both).

We let $\mathbf{\Sigma}^1_\alpha$ enumerate the Levy classes closed under $\exists^{\omega^\omega}$ ($\mathbf{\Pi}^1_\alpha$ those closed under $\forall^{\omega^\omega}$).

$\mathbf{\Sigma}^1_0$ = open, $\mathbf{\Sigma}^1_1$ = analytic, etc.

Theorem (Steel)**.** *For every Levy class $\mathbf{\Gamma}$, either pwo$(\mathbf{\Gamma})$ or pwo$(\check{\mathbf{\Gamma}})$.*

Steel's analysis gives more information.

Let $C \subseteq \theta$ be the c.u.b. set of limit α such that $\Lambda_\alpha \doteq \{A\colon o(A) < \alpha\}$ is closed under quantifiers.

For $\mathbf{\Gamma}$ a Levy class, let α be the largest element of C such that $\Lambda_\alpha \subseteq \mathbf{\Gamma}$.

Then $\mathbf{\Gamma}$ is in the projective hierarchy over $\Lambda = \Lambda_\alpha$. This hierarchy can fall into one of several types.

Let $\mathbf{\Gamma}_0$ be the non-selfdual pointclass with $o(\mathbf{\Gamma}_0) = \alpha$ and w.l.o.g. sep($\check{\mathbf{\Gamma}}$). We call $\mathbf{\Gamma}_0$ a *Steel pointclass.* Steel showed $\mathbf{\Gamma}_0$ is closed under $\forall^{\omega^\omega}$.

Types of Projective Hierarchies

- Type 1.) $\mathrm{cof}(\alpha) = \omega$.
  Let $\mathbf{\Sigma}_0 = \bigcup_\omega \Lambda$. Then $\mathrm{pwo}(\mathbf{\Sigma}_0)$, and by periodicity $\mathrm{pwo}(\mathbf{\Pi}_{2n+1})$, $\mathrm{pwo}(\mathbf{\Sigma}_{2n+2})$.

- Type 2.) $\mathrm{cof}(\alpha) > \omega$ and $\mathbf{\Gamma}_0$ is not closed under $\vee$.
  Let $\mathbf{\Pi}_1 = \mathbf{\Gamma}_0$. Then $\mathrm{pwo}(\mathbf{\Pi}_{2n+1})$, $\mathrm{pwo}(\mathbf{\Sigma}_{2n+2})$.

- Type 3.) $\mathrm{cof}(\alpha) > \omega$, $\mathbf{\Gamma}_0$ is closed under $\vee$ but not $\exists^{\omega^\omega}$.
  Same conclusion as in type 2.

- Type 4.) $\mathbf{\Gamma}_0$ is closed under quantifiers.
  Then $\mathrm{pwo}(\mathbf{\Gamma}_0)$. Let
  $$\mathbf{\Sigma}_0 = \exists^{\omega^\omega}(\mathbf{\Gamma}_0 \wedge \check{\mathbf{\Gamma}}_0).$$
  Then $\mathrm{pwo}(\mathbf{\Sigma}_0)$, $\mathrm{pwo}(\mathbf{\Pi}_{2n+1})$, $\mathrm{pwo}(\mathbf{\Sigma}_{2n+2})$.

Second Pass: Scales and Suslin Cardinals

A tree on X is a subset of $X^{<\omega}$ closed under initial segment.

We identify trees on $X \times Y$ with subsets of $X^{<\omega} \times Y^{<\omega}$.

If T is a tree on X, then

$$[T] = \{f \in X^\omega \colon \forall n\ f \restriction n \in T\}.$$

If T is a tree on $X \times Y$ then

$$\begin{aligned} p[T] &= \{f \in X^\omega \colon \exists g \in Y^\omega\ (f, g) \in [T]\} \\ &= \{f \in X^\omega \colon T_f \text{ is illfounded}\}, \end{aligned}$$

where $T_f = \{s \in Y^{<\omega} \colon (f \restriction \mathrm{lh}(s), s) \in T\}$.

Definition. $A \subseteq \omega^\omega$ is κ-Suslin if $A = p[T]$ for some tree T on $\omega \times \kappa$. $S(\kappa)$ denotes the pointclass of κ-Suslin sets. κ is a *Suslin cardinal* if $S(\kappa) - \bigcup_{\lambda<\kappa} S(\lambda) \neq \emptyset$.

note: If makes sense to speak of κ-Suslin subsets of λ^ω for $\lambda \in \mathrm{On}$ as well.

Fact. For any Suslin cardinal κ, $S(\kappa)$ is closed under countable unions and intersection, $\exists^{\omega^\omega}$ and (Kechris) is non-selfdual.

Suslin representations $\approx$ Scales.

Definition. A *semi-scale* on $A \subseteq \omega^\omega$ is a sequence of norms $\{\varphi_n\}_{n\in\omega}$ such that if $x_n \in A$, $x_n \to x$, and for each n, $\varphi_n(x_m)$ is eventually equal to some λ_n, then $x \in A$.

If in addition we have $\varphi_n(x) \leq \lambda_n$ for all n, then the $\{\varphi_n\}$ is said to be a *scale*.

Fact. For all cardinals κ, A is κ-Suslin iff A admits a semi-scale with norms into κ iff A admits a scale with norms into κ.

If $\vec{\varphi}$ is a semi-scale, the corresponding Suslin representation is given by the tree

$$\begin{aligned} T_{\vec{\varphi}} = \{((a_0, \dots, a_{n-1}), (\alpha_0, \dots, \alpha_{n-1}))\colon \\ \exists x \in A\ (x \restriction n = (a_0, \dots, a_{n-1}) \\ \wedge \forall i < n\ \varphi_i(x) = \alpha_i\}. \end{aligned}$$

If $A = p[T]$, let for $x \in A$, $\varphi_n(x) = n^{\text{th}}$ coordinate of the leftmost branch of T_x.

Then $\vec{\varphi}$ is a semi-scale on A. Let $\psi_n = \langle \varphi_0(x), \dots \varphi_{n-1}(x) \rangle =$ rank of $(\varphi_0(x), \dots \varphi_{n-1}(x))$ in lexicographic ordering on κ^n. Then $\vec{\psi}$ is a scale on A. [with a little adjustment to T can make the ψ_i map into κ.]

<u>note</u>: If $\vec{\varphi}$ is a scale, then $\vec{\varphi}(T_{\vec{\varphi}}) = \vec{\varphi}$. But, not every tree is the tree of a scale, so we only have $T_{\vec{\varphi}(T)} \subseteq T$.

Theorem (Steel-Woodin)**.** *Assume* AD*. The Suslin cardinals are closed below their supremum.*

Some pointclass arguments together with an analysis of Martin for constructing the next Suslin gives a classification of the Suslin cardinals and scales from AD.

Suppose κ is a limit of Suslin cardinals, and κ is below the supremum of the Suslin cardinals (so κ is a Suslin cardinal). We have the following cases.

- $\mathrm{cof}(\kappa) = \omega$ (type I).
  Let $\mathbf{\Sigma}_0 = \bigcup_\omega S_{<\kappa}$. Then scale($\mathbf{\Sigma}_0$) with norms into κ, and scale($\mathbf{\Pi}_{2n+1}$), scale($\mathbf{\Sigma}_{2n+2}$) with norms into $\delta_{2n+1} \doteq \delta(\mathbf{\Pi}_{2n+1})$.
  $\delta_{2n+1} = (\lambda_{2n+1})^+$, where $\mathrm{cof}(\lambda_{2n+1}) = \omega$ ($\lambda_1 = \kappa$). $\delta_1, \lambda_3, \delta_3, \dots$ are the next ω Suslin cardinals after κ. $S(\lambda_{2n+1}) = \mathbf{\Sigma}_{2n+1}$, $S(\delta_{2n+1}) = \mathbf{\Sigma}_{2n+2}$.

- $\mathrm{cof}(\kappa) > \omega$ and $\mathbf{\Gamma}_0$ (Steel pointclass) not closed under $\exists^{\omega^\omega}$ (type II, III).
  Let $\mathbf{\Sigma}_0 = \exists^{\omega^\omega}\mathbf{\Gamma}_0$. Then scale($\mathbf{\Gamma}_0$), scale($\mathbf{\Sigma}_0$) with norms into κ and scale($\mathbf{\Pi}_{2n+1}$), scale($\mathbf{\Sigma}_{2n+2}$) with norms into $\delta_{2n+1} = \delta(\mathbf{\Pi}_{2n+1})$. $\lambda_1, \delta_1, \lambda_3, \ldots$ are the next ω Suslin cardinals after κ. $\delta_{2n+1} = (\lambda_{2n+1})^+$, where $\mathrm{cof}(\lambda_{2n+1}) = \omega$ and $S(\lambda_{2n+1}) = \mathbf{\Sigma}_{2n+1}$, $S(\delta_{2n+1}) = \mathbf{\Sigma}_{2n+2}$.

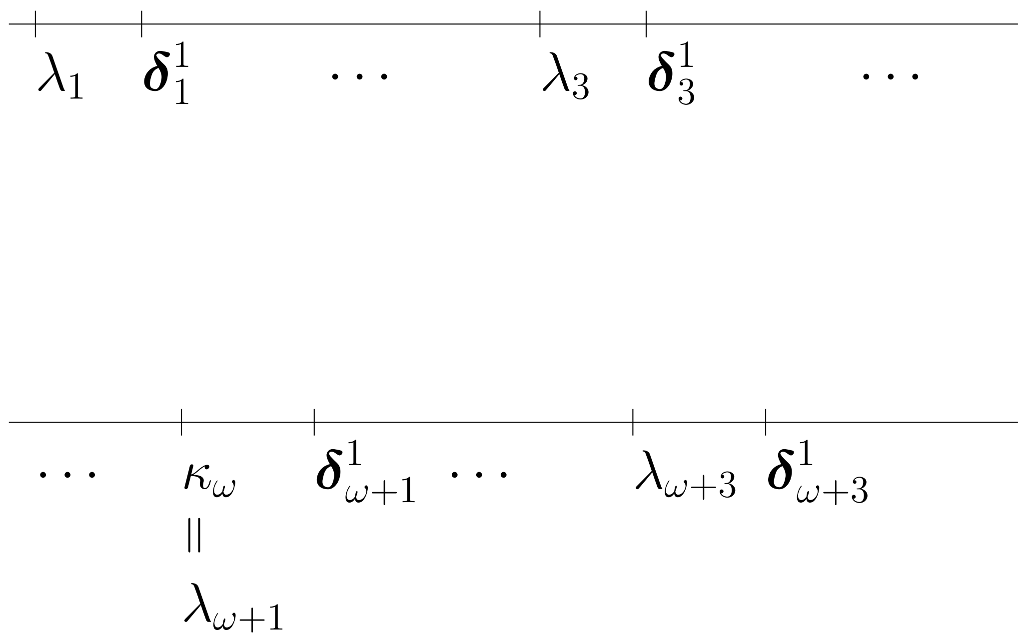
- $\mathrm{cof}(\kappa) > \omega$ and $\mathbf{\Gamma}_0$ is closed under $\exists^{\omega^\omega}$ (type IV).
  Then scale($\mathbf{\Gamma}_0$) with norms into κ. Let $\Lambda = \Lambda(\mathbf{\Gamma}_0, \kappa)$ be the Martin pointclass. Let $\mathbf{\Sigma}_0 = \bigcup_\omega \Lambda$. Let $\lambda_1 = o(\Lambda)$. Then scale($\mathbf{\Pi}_{2n+1}$), scale($\mathbf{\Sigma}_{2n+2}$) with norms into $\delta_{2n+1} = \delta(\mathbf{\Pi}_{2n+1})$. $\lambda_1, \delta_3, \lambda_3$ are the next ω Suslin cardinals after κ. $\delta_{2n+1} = (\lambda_{2n+1})^+$, where $\mathrm{cof}(\lambda_{2n+1}) = \omega$ and $S(\lambda_{2n+1}) = \mathbf{\Sigma}_{2n+1}$, $S(\delta_{2n+1}) = \mathbf{\Sigma}_{2n+2}$.

Steel gives a more detailed analysis assuming $V = L(\mathbb{R})$.

Consider the first ω_1 Suslin cardinals.

Let $\boldsymbol{\delta}^1_\alpha = \delta(\boldsymbol{\Sigma}^1_\alpha)$ $(\alpha \geq 1)$.

At limit stages we are in type I, so we have the following picture.



The First ω_1 Suslin Cardinals

- Compute the $\boldsymbol{\delta}^1_\alpha$.
- Use this analysis to determine the cardinal structure.

General Plan:

Establish the strong partition relation on the $\boldsymbol{\delta}^1_\alpha$, for α odd, and use this to describe the cardinal structure.

The combinatorics involved is given by the notion of a *description*.

To get the strong partition relation on the odd $\boldsymbol{\delta}^1_\alpha$, we analyze the measures on $\boldsymbol{\delta}^1_\alpha$, assuming such an analysis below $\boldsymbol{\delta}^1_\alpha$. This, by an argument of Kunen, gives an analysis of the subsets of $\boldsymbol{\delta}^1_\alpha$ which gives a coding of the subsets of $\boldsymbol{\delta}^1_\alpha$ sufficient to give the strong partition relation on $\boldsymbol{\delta}^1_\alpha$ (using a general theorem of Martin).

The measure analysis below $\boldsymbol{\delta}^1_\alpha$ is enough to get the weak partition relation on $\boldsymbol{\delta}^1_\alpha$ which is enough to do the description analysis at

$\boldsymbol{\delta}^1_\alpha$ which (1) computes the cardinal structure below $\lambda_{\alpha+2}$ and (2) analyzes the measures at $\boldsymbol{\delta}^1_\alpha$.

The arguments for (1) and (2) are very similar and use the same combinatorics (i.e., same descriptions).

For simplicity, we concentrate here on (1), and show how to generate the cardinal structure assuming the strong partition relations on the odd $\boldsymbol{\delta}^1_\alpha$. The argument for the measure analysis are similar.

Note that no new measures arise at limit stages since $\alpha < \omega_1$.

So, assume the $\boldsymbol{\delta}^1_\alpha$, α odd, have the strong partition property.

Partition Properties and Types

Recall the Erdös-Rado partition notation.

$\kappa \longrightarrow (\kappa)^\lambda$: For every partition $\mathcal{P}\colon \kappa^\lambda \longrightarrow \{0, 1\}$ there is a homogeneous set $H \subseteq \kappa$ of size κ.

We say κ has the *weak* partition property if $\kappa \longrightarrow (\kappa)^\lambda$ for all $\lambda < \kappa$ and κ has the *strong* partition property if $\kappa \longrightarrow (\kappa)^\kappa$.

More useful is the c.u.b. reformulation of the partition properties.

Definition. A function $f\colon \lambda \longrightarrow \mathrm{On}$ is of uniform cofinality ω if there is a function $g\colon \lambda \times \omega \longrightarrow \mathrm{On}$ which is increasing in the second argument and such that for all $\alpha < \lambda$:

$$f(\alpha) = \sup_n g(\alpha, n)$$

Definition. We say f is of the *correct type* if f is increasing, everywhere discontinuous ($f(\alpha) > \sup_{\beta<\alpha} f(\beta)$ for limit α), and of uniform cofinality ω.

We say $\kappa \xrightarrow{c.u.b.} (\kappa)^\lambda$ if for every partition $\mathcal{P}$ of the function $f\colon \lambda \to \kappa$ of the correct type, there is a c.u.b. $C \subseteq \kappa$ homogeneous for $\mathcal{P}$.

Fact.

$$\kappa \xrightarrow{c.u.b.} (\kappa)^\lambda \Rightarrow \kappa \to (\kappa)^\lambda$$

$$\kappa \to (\kappa)^{\omega\cdot\lambda} \Rightarrow \kappa \xrightarrow{c.u.b.} (\kappa)^\lambda$$

Proof. For the first fact, let $\mathcal{P}$ be a partition of the functions $g\colon \lambda \to \kappa$. In particular, $\mathcal{P}$ partitions functions of the correct type; let C be c.u.b. and homogeneous for this subpartition. Define $h\colon \kappa \to \kappa$ by $h(\alpha) = \omega^{\text{th}}$ element of C greater than $\sup_{\beta<\alpha} h(\beta)$. Let $H = \operatorname{ran}(h)$, then H is homogeneous for $\mathcal{P}$.

For the second fact, let $\mathcal{P}$ be a partition of the functions $f: \lambda \to \kappa$ of the correct type. Let $\mathcal{P}'$ be the partition of increasing $g: \omega{\cdot}\lambda \to \kappa$ given by $\mathcal{P}'(g) = \mathcal{P}(f)$ where $f(\alpha) = \sup_n g(\alpha, n)$. Let H be homogeneous for $\mathcal{P}'$ and let C be the set of limit points of H. C is homogeneous for $\mathcal{P}$. $\square$

More generally:

Definition. Let $g: \lambda \to \mathrm{On}$. We say $f: \lambda \to \mathrm{On}$ is of *uniform cofinality* g if there is a function

$$f': \{(\alpha, \beta): \alpha < \lambda \wedge \beta < g(\alpha)\} \to \mathrm{On}$$

which is increasing in the second argument and such that $f(\alpha) = \sup_\beta f'(\alpha, \beta)$.

If g is the constant ρ function, then we say f has uniform cofinality ρ.

If μ is a measure on λ, we have the notion of f being of uniform cofinality g almost everywhere w.r.t. μ.

If κ has strong partition property, we have partition property for functions $f\colon \kappa \longrightarrow \kappa$ of type g (increasing, discontinuous, of uniform cofinality g). Given a measure on κ, this defines a measure on $j_\mu(\kappa)$.

Analysis at $\boldsymbol{\delta}^1_1 = \omega_1$, Trivial Descriptions.

We start with weak partition property on ω_1. [Use simple coding of functions $f\colon \alpha \longrightarrow \omega_1$ for $\alpha < \omega_1$. Let $\pi\colon \omega \longrightarrow \alpha$ be a bijection. $x \in \omega^\omega$ codes partial function f_x given by $f_x(\pi(n)) = |x_n|$ if $x_n \in$ WO, undefined otherwise.]

From this we get that sets of the form C^n form a base for a measure on $(\omega_1)^n$. We denote this W_1^n. It is the n-fold product of the normal measure W_1^1 on ω_1.

Definition. A trivial description is an integer $d \in \omega$. The set $\mathcal{D}$ of trivial description is ordered in the usual ordering on ω. The *lowering operator* $\mathcal{L}$ is defined on all $d \in \mathcal{D}$ except the minimal description $d = 0$. $\mathcal{L}(d) = d - 1$.

Let $\mathcal{D}_n$ be those trivial descriptions $d < n$.

Interpretation: If $f : n \longrightarrow \omega_1$, and $d \in \mathcal{D}_n$, define $h(f; d) = f(d)$.

Let $(W_1^n; d) = [f \mapsto (f; d)]_{W_1^n}$.

If $g : \omega_1 \longrightarrow \omega_1$, let also:

$(g; f; d) = g(f(d))$, $(g; W_1^n; d) = [f \mapsto (g; f; d)]_{W_1^n}$.

Kunen Tree Aside from partition property of ω_1, we need the *Kunen Tree.*

For $x \in \omega^\omega$, let $<_x = \{(n,m) \colon \langle n,m \rangle) = 1\}$.

$\mathrm{LO} = \{x \colon \ <_x \text{ is a linear order}\}$, $\qquad \mathrm{WO} = \{x \colon \ <_x \text{ is a wellorder}\}$.

Let S be the Shoenfield tree on $\omega \times \omega_1$ with $\mathrm{WO} = p[S]$.

$(s, \vec{\alpha}) \in S$ iff s doesn't violate being in LO and $\forall i, j \leq |s| \, (s(\langle i, j \rangle) = 1 \rightarrow \alpha_i < \alpha_j)$.

With a small patch-up. can assume S is *homogeneous*, i.e., s determines the order-type of $\vec{\alpha}$. Also, $\alpha_0 > \alpha_1, \ldots, \alpha_{n-1}$.

Theorem (Kunen)**.** *There is a tree T on $\omega \times \omega_1$ with the following property. For any $f \colon \omega_1 \rightarrow \omega_1$ there is an $x \in \omega^\omega$ such that T_x is wellfounded and for all infinite ordinals $\alpha < \omega_1$ we have $f(\alpha) < |T_x \restriction \alpha|$.*

Proof. If $f\colon \omega_1 \longrightarrow \omega_1$, play the game where I plays out x, II plays out y, and II wins the run iff

$$x \in \mathrm{WO} \longrightarrow (y \in \mathrm{WO} \wedge |y| \leq f(|x|)).$$

II has a winning strategy by boundedness. This suggests the following definition.

V is the tree on $\omega \times \omega \times \omega_1 \times \omega \times \omega$ given by:

$$\begin{aligned}(s,t,\vec{\alpha},u,v) \in V \leftrightarrow \exists \sigma, x, y, z \text{ extending } s,t,u,v\\ [\sigma(x) = y \wedge (t,\vec{\alpha}) \in S \wedge\\ \forall i\ y(\langle z(i), z(i+1)\rangle) = 1]\end{aligned}$$

For f and σ as above, V_σ is wellfounded and for any infinite α, $|V_\sigma \restriction \alpha| \geq f(\alpha)$.

Can identify V with a tree on $\omega \times \omega_1$. $\square$

An easy partition argument shows:

If $f\colon (\omega_1)^n \longrightarrow \omega_1$ is such that $f(\vec{\alpha}) < \alpha_i$ for almost all $\vec{\alpha}$, then there is a c.u.b. $C \subseteq \omega_1$ such that $\forall^* \vec{\alpha}\ f(\vec{\alpha}) < N_C(\alpha_{i-1})$.

$N_C(\beta) =$ least element of C greater than β.

Translating we have:

Main Theorem: If $\alpha < (W_1^n; d)$, then there is a $g = N_C$ such that $\alpha < (g; W_1^n; \mathcal{L}(d))$.

Fix $x \in \omega^\omega$ such that $g(\beta) \leq |T_x \restriction \beta|$ for almost all β.

Then

$$\begin{aligned}(g; W_1^n; \mathcal{L}(d)) &\leq (\beta \mapsto |T_x \restriction \beta|; W_1^n; \mathcal{L}(d)) \\ &< (W_1^n; \mathcal{L}(d))^+\end{aligned}$$

So, $(W_1^n; d) \leq (W_1^n; \mathcal{L}(d))^+$.

Corollary. $j_{W_1^n}(\omega_1) \leq \omega_{n+1}$.

To get the lower bound we use:

Theorem (Martin)**.** *Assume $\kappa \to (\kappa)^\kappa$. Then for any measure μ on κ, $j_\mu(\kappa)$ is a cardinal.*

Also, if $m < n$ then $j_{W_1^m}(\omega_1) < j_{W_1^n}(\omega_1)$. In fact, $j_{W_1^m}(\omega_1)$ embeds into $(W_1^{m+1}; d)$, where $d = m+1$.

So we have $j_{W_1^n}(\omega_1) = \omega_{n+1}$.

Recall Shoenfield tree for a $\mathbf{\Sigma}^1_2$ set is weakly homogeneous with measures of the form W_1^n. Homogeneous tree construction gives that every $\mathbf{\Pi}^1_2$, and hence also every $\mathbf{\Sigma}^1_3$ set is λ-Suslin where $\lambda = \sup_n j_{W_1^n}(\omega_1) = \omega_\omega$.

So, $\lambda_3 \le \omega_\omega$ and $\boldsymbol{\delta}^1_3 \le \omega_{\omega+1}$.

Since $\mathrm{cof}(\lambda_3) = \omega$ we must have $\lambda_3 = \lambda = \omega_\omega$ and $\boldsymbol{\delta}^1_3 = \omega_{\omega+1}$.

Types of Functions on ω_1

Fix n, and a permutation π of $\{1, 2, , \dots, n\}$ beginning with n.

$$\pi = (n, i_2, \dots, i_n).$$

We say $f \colon (\omega_1)^n \to \omega_1$ is ordered by π if on a measure one set we have:

$$f(\alpha_1, \dots, \alpha_n) < f(\beta_1, \dots, \beta_n) \leftrightarrow$$
$$(\alpha_n, \alpha_{i_2}, \dots, \alpha_{i_n}) <_{\text{lex}} (\beta_n, \beta_{i_2}, \dots, \beta_{i_n}).$$

Fact. If f depends on all its arguments, then for some π, f is ordered by π.

If $f \colon (\omega_1)^n \to \omega_1$ then the type of f is determined by π and the possible uniform cofinalities:

- There is a measure one set on which f is continuous (i.e., $f \restriction C^n$ is continuous).
- f has uniform cofinality ω (of the correct type).
- $f(\alpha_1, \ldots, \alpha_n)$ has uniform cofinality α_i.

Example. Show that the uniform cofinalities $g(\vec{\alpha}) = \alpha_i$ and $g(\vec{\alpha}) = \alpha_j$ are distinct for $i \neq j$.

Proof. Suppose $f_1 \colon \{(\vec{\alpha}, \beta) \colon \beta < \alpha_i\} \to \omega_1$ induces f, and likewise for

$$f_2 \colon \{(\vec{\alpha}, \beta) \colon \beta < \alpha_j\}.$$

Consider the partition $\mathcal{P}$: partition

$$\alpha_1 < \cdots < \alpha_{i-1} < \beta_1 < \alpha_i < \cdots$$
$$< \alpha_{j-1} < \beta_2 < \alpha_j < \cdots < \alpha_n$$

according to whether $f_1(\vec{\alpha}, \beta_1) < f_2(\vec{\alpha}, \beta_2)$. Neither side can be homogeneous, a contradiction. $\square$

We introduce a canonical family of measures for each regular cardinal below $\aleph_{\omega_1}$.

Let π_n be the permutation

$$\pi_n = (n, 1, 2, \ldots, n-1).$$

Definition. S^n_1 is the measure on ω_{n+1} induced by functions $f : (\omega_1)^n \longrightarrow \omega_1$ ordered by π_n and of the correct type (and the measure W^n_1).

S^1_1 is the ω-cofinal normal measure on ω_2.

Fact. The family of measures S^n_1 dominate all the measures on ω_ω.

More precisely: For any measure μ on ω_ω there is an n and a c.u.b. $C \subseteq \boldsymbol{\delta}^1_3$ such that for all $\alpha \in C$ with $\mathrm{cof}(\alpha) = \omega_2$,

$$j_\mu(\alpha) \leq j_{S^n_1}(\alpha).$$

Example. We show this for $\mu = \nu \times \nu$, where ν is the measure corresponding to $\pi = (3,2,1)$.

Proof. We take $n = 3$. We define an auxiliary measure $\mathcal{M} = W_1^1 \times W_1^1 \times S_1^2$.

Given $f\colon\ <_3 \longrightarrow \omega_1$ of the correct type and given $\eta_1 < \eta_2 < \omega_1$ and h_3 representing $(\eta_1, \eta_2, [h_3]) \in \omega_1 \times \omega_1 \times \omega_3$, we define $g_1, g_2\colon (\omega_1)^3 \to \omega_1$ by:

$$g_i(\alpha_1, \alpha_2, \alpha_3) = f(\eta_i, h_3(\alpha_1, \alpha_2), \alpha_3).$$

Let $\alpha \in C$, the set of ordinals $< \boldsymbol{\delta}_3^1$ closed under ultrapowers by measures on ω_ω.

Define $\pi\colon j_\mu(\alpha) \to j_{S_1^3}(\alpha)$ as follows.

$\pi([F]_\mu) = [G]_{S_1^3}$.

$$G([f]_{W_1^3}) =$$
$$[(\eta_1, \eta_2, [h_3]) \mapsto G(f, \eta_1, \eta_2, h_3)]_{\mathcal{M}}.$$

$$G(f, \eta_1, \eta, h_3) = F([g_1], [g_2]), \text{ where } g_i = g_i(f, \eta_1, \eta_2, h_3) \text{ as above.}$$

The following observations complete the proof.

(1) For any fixed f of the correct type, fixed η_1, η_2, h_3 with $\eta_1 < \eta_2 < h_3(\gamma, \beta)$ for all $\gamma < \beta$ and all $\alpha_1 < \alpha_2 < \alpha_3$ in a c.u.b. set closed under $h_3(0)$ we have that g_i is welldefined and of order π.

Also, $g_1(\vec{\alpha}) < g_2(\vec{\beta})$ iff $\alpha_3 \leq \beta_3$ (the type of the product measure μ.

(2) For any fixed f, $[g_i]$ only depends on η_1, η_2, $[h_3]$.

(3) If $[f_1] = [f_2]$, then $\mathcal{M}$ almost all η_1, η_2, $[h_3]$ we have that $[g_i^1] = [g_i^2]$, where g_i^1 uses f_1 and likewise for g_i^2.

(4) If $\mu(A) = 1$, then
$$\forall^* f \ \forall^* \eta_1, \eta_2, h_3 \ ([g_1], [g_2]) \in A.$$

(5) $\forall^* f \ \exists (G_1, G_2) \ \forall^* \eta_1, \eta_2, h_3 \ (g_1 \leq G_1 \wedge g_2 \leq G_2)$.

(1)-(4) give that π is welldefined, (5) and $\alpha \in C$ give that $\pi([F]) < j_{S^3_1}(\alpha)$. $\square$

Notation Convention

Suppose $\theta \in \mathrm{On}$, $\mu_1, \ldots, \mu_n$ are measures, and $P \subseteq \mathrm{On}$. We write

$$\forall^*_{\mu_1}\alpha_1 \ \cdots \forall^*_{\mu_n}\alpha_n \ P(\theta(\alpha_1, \ldots, \alpha_n))$$

to abbreviate the following:

If we fix $\alpha_1 \mapsto \theta(\alpha_1)$ representing θ in the ultrapower by μ_1, then for μ_1 almost all α_1 it is the case that:

If we fix $\alpha_2 \mapsto \theta(\alpha_1, \alpha_2)$ representing $\theta(\alpha_1)$ with respect to μ_2, then for μ_2 almost all α_2 it is the case that:

$$\vdots$$

If we fix $\alpha_n \mapsto \theta(\alpha_1, \ldots, \alpha_n)$ representing $\theta(\alpha_1, \ldots, \alpha_{n-1})$ with respect to μ_n, then for μ_n almost all α_n we have $P(\theta(\alpha_1, \ldots, \alpha_n))$.

Example. Use trivial descriptions to compute $j_{S_1^m}(\omega_n) = \omega_k$ where

$$k = 1 + 2[\binom{n-1}{1} + \cdots + \binom{n-1}{m}].$$

We describe the cardinals below $j_{S_1^m}(\omega_n)$. We define a set $\mathcal{S}$ of special instances of non-trivial descriptions.

Definition. For $m, n \geq 1$, let $\mathcal{S}_{m,n}$ be the set of tuples of the form $d = (d_m, d_1, d_2, \ldots, d_\ell)$ or $d = (d_m, d_1, d_2, \ldots, d_\ell)^s$ where the d_i are trivial descriptions in $\mathcal{D}_{n-1}$ (i.e., $1 \leq d_i \leq n-1$), $\ell < m$, and $d_1 < d_2 < \cdots < d_\ell < d_m$.

We write $d = (d_m, d_1, d_2, \ldots, d_\ell)^{(s)}$ to denote that s may or may not appear.

Given $d \in \mathcal{S}_{m,n}$, we associate an ordinal $(d; S_1^m)$ to d defined as follow.

We represent $(d; S_1^m)$ with respect to the measure S_1^m by the function

$$[f] \mapsto (d; f),$$

for $f \colon (\omega_1)^m \longrightarrow \omega_1$ of the correct type.

$(d; f) < \omega_n$ is the ordinal represented with respect to W_1^{n-1} by the function

$$(\alpha_1, \ldots, \alpha_{n-1}) \mapsto (d; f)(\vec{\alpha}).$$

Finally,

$$(d; f)(\vec{\alpha}) = f(\ell + 1)(\alpha_{d_1}, \alpha_{d_2}, \ldots, \alpha_{d_\ell}, \alpha_{d_m}).$$

if s does not appear in d.

If s appears in d, let

$$
\begin{aligned}
(d; f)(\vec{\alpha}) &= f^s(\ell+1)(\alpha_{d_1}, \alpha_{d_2}, \ldots, \alpha_{d_\ell}, \alpha_{d_m}) \\
&= \sup_{\beta < \alpha_{d_\ell}} f(\ell+1)(\alpha_{d_1}, \alpha_{d_2}, \ldots, \beta, \alpha_{d_m}).
\end{aligned}
$$

There are $2[\binom{n-1}{1} + \cdots + \binom{n-1}{m}]$ many such ordinals $(d; S_1^m)$.

Claim. These are precisely the cardinals below $j_{S_1^m}(\omega_n)$.

The descriptions in $\mathcal{S}_{m,n}$ are ordered as follows.

Set $(d_m, d_1, \ldots, d_\ell)^{(s)} < (d'_m, d'_1, \ldots, d'_{\ell'})^{(s)}$ iff one of the following holds:

(1) There is a least place of disagreement in the description sequences, say $d_i \neq d'_i$, and $d_i < d'_i$.
(2) $\vec{d}$ is an initial segment of $\vec{d'}$, and s appears in $\vec{d}$.

(3) $\vec{d'}$ is an initial segment of $\vec{d}$, and s does not appear in $\vec{d'}$.

Let $\mathcal{L}(d_m, d_1, \ldots, d_\ell)^{(s)}$ be the largest sequence which is less than $(d_m, d_1, \ldots, d_\ell)^{(s)}$.

We describe the $\mathcal{L}$ operation explicitly:

- If $\ell = m - 1$ then $\mathcal{L}(d_m, d_1, \ldots, d_\ell) = (d_m, d_1, \ldots, d_\ell)^s$.
- If $\ell = m - 1$ then $\mathcal{L}(d_m, d_1, \ldots, d_\ell)^s = (d_m, d_1, \ldots, \mathcal{L}(d_\ell))$ if $\mathcal{L}(d_\ell) = d_\ell - 1$ is defined and $> d_{\ell-1}$ (if $\ell > 1$). Otherwise, $\mathcal{L}(d_m, d_1, \ldots, d_\ell)^s = (d_m, d_1, \ldots, d_{l-1})^s$.
- If $\ell < m - 1$, then $\mathcal{L}(d_m, d_1, \ldots, d_\ell) = (d_m, d_1, \ldots, d_\ell, d_{\ell+1})$ where $d_{\ell+1} = \mathcal{L}(d_m) = d_m - 1$ provided $d_m - 1 > d_\ell$ (if $\ell \geq 1$). Otherwise set $\mathcal{L}(d_m, d_1, \ldots, d_\ell) = (d_m, d_1, \ldots, d_\ell)^s$.

Example. $m = 3$, $n = 5$ (i.e., $j_{S_1^3}(\omega_5)$)

$$
\begin{array}{ccc}
d = (4) & \mathcal{L}(d) = (4,3) & \mathcal{L}^2(d) = (4,3)^s \\
\mathcal{L}^3(d) = (4,2) & \mathcal{L}^4(d) = (4,2,3) & \mathcal{L}^5(d) = (4,2,3)^s \\
\mathcal{L}^6(d) = (4,2)^s & \mathcal{L}^7(d) = (4,1) & \mathcal{L}^8(d) = (4,1,3) \\
\mathcal{L}^9(d) = (4,1,3)^s & \mathcal{L}^{10}(d) = (4,1,2) & \mathcal{L}^{11}(d) = (4,1,2)^s \\
\mathcal{L}^{12}(d) = (4,1)^s & \mathcal{L}^{13}(d) = (4)^s & \mathcal{L}^{14}(d) = (3) \\
\mathcal{L}^{15}(d) = (3,2) & \mathcal{L}^{16}(d) = (3,2)^s & \mathcal{L}^{17}(d) = (3,1) \\
\mathcal{L}^{18}(d) = (3,1,2) & \mathcal{L}^{19}(d) = (3,1,2)^s & \mathcal{L}^{20}(d) = (3,1)^s \\
\mathcal{L}^{21}(d) = (3)^s & \mathcal{L}^{22}(d) = (2) & \mathcal{L}^{23}(d) = (2,1) \\
\mathcal{L}^{24}(d) = (2,1)^s & \mathcal{L}^{25}(d) = (2)^s & \mathcal{L}^{26}(d) = (1) \\
\mathcal{L}^{27}(d) = (1)^s & &
\end{array}
$$

So, $j_{S_1^3}(\omega_5) = \omega_{29}$.

The upper bound follows from the following lemma.

Lemma. *For $d \in \mathcal{S}_{m,n}$ non-minimal, $(d; S^m_1) \leq (\mathcal{L}(d); S^m_1)^+$. If d is $\mathcal{L}$-minimal, then $(d; S^m_1) \leq \omega_1$.*

Proof. We consider the case $d = (4,1)$ in the above example. $\mathcal{L}(d) = (4,1,3)$.

Let $\theta < (d; S^m_1)$. Then,

$$\forall^*_{S^m_1}[f]\ \theta([f]) < (d; f).$$

Thus,

$$\begin{aligned}\forall^*_{S^m_1}[f]\ \forall^*_{W^{n-1}_1}\vec{\alpha}\ \ \theta(f)(\vec{\alpha}) < (d; f)(\vec{\alpha}) &= f(2)(\alpha_1, \alpha_4)\\ &= \sup_{\beta<\alpha_4} f(\alpha_2, \beta, \alpha_4).\end{aligned}$$

So,

$$\forall^*_{S_1^m}[f]\ \forall^*_{W_1^{n-1}}\vec{\alpha}\ \exists\beta<\alpha_4\ \ \theta(f)(\vec{\alpha})<f(\alpha_1,\beta,\alpha_4).$$

A partition argument shows that there is a $g\colon \omega_1\longrightarrow\omega_1$ such that

$$\forall^*_{S_1^m}[f]\ \forall^*_{W_1^{n-1}}\vec{\alpha}\ \ \theta(f)(\vec{\alpha})<f(\alpha_1,g(\alpha_3),\alpha_4).$$

Fix $x\in\omega^\omega$ such that $|T_x\restriction\alpha|>g(\alpha)$ almost everywhere.

Thus,

$$\forall^*_{S_1^m}[f]\ \forall^*_{W_1^{n-1}}\vec{\alpha}\ \ \theta(f)(\vec{\alpha})<f(\alpha_1,|T_x\restriction\alpha_3|,\alpha_4).$$

It follows that there a $\theta'<(\mathcal{L}(d);S_1^m)$ such that

$$\forall^*_{S_1^m}[f]\ \forall^*_{W_1^{n-1}}\vec{\alpha}\ \ \theta(f)(\vec{\alpha})=f(\alpha_1,|T_x\restriction\alpha_3(\theta'(f)(\vec{\alpha}))|,\alpha_4).$$

where $|T_x\restriction\alpha(\beta)|$ denotes the rank of β in $T_x\restriction\alpha$.

The map $\delta' \mapsto \delta$ defined by

$$\forall^*_{S_1^m}[f]\ \forall^*_{W_1^{n-1}}\vec{\alpha}$$
$$\delta(f)(\vec{\alpha}) = f(\alpha_1, |T_x \restriction \alpha_3(\delta'(f)(\vec{\alpha}))|, \alpha_4).$$

defines a map from $(\mathcal{L}(d); S_1^m)$ onto θ, so $\theta < (\mathcal{L}(d); S_1^m)^+$.

$\square$

The lower bound follows from the following lemma.

Lemma. $(d; S_1^m) \geq \omega_{|d|+1}$, *where* $|d|$ *denotes the rank of* d *in the* $\mathcal{L}$ *ordering.*

Proof. Let k be the rank of d in the $\mathcal{L}$-order. Thus there are k elements of $\mathcal{S}_{m,n}$ below d, say $d_1 < \cdots < d_k$.

We define an embedding π from $\omega_{k+1} = j_{W_1^k}(\omega_1)$ into (d, S_1^m).

For $g\colon (\omega_1)^k \longrightarrow \omega_1$, define $\pi([g]_{W_1^k}) = \theta$ where

$$\forall^*_{S_1^m} f\ \forall^*_{W_1^{n-1}} \vec{\alpha}$$
$$\theta(f)(\vec{\alpha}) = g((d_1; f)(\vec{\alpha}), \ldots, (d_k; f)(\vec{\alpha})).$$

That this works follows from the following observations.

(1) For fixed g, $\theta(f)$ depends only on $[f]_{W_1^m}$, since if $[f_1] = [f_2]$ then almost all $\vec{\alpha}$ will be in a c.u.b. C such that $f_1 \restriction C^{n-1} = f_2 \restriction C^{n-1}$.

(2) If $[g_1]_{W_1^k} = [g_2]_{W_1^k}$, and say $g_1 \restriction C^k = g_2 \restriction C^k$, then S_1^m almost all $[f]$ are represented by $f\colon\ <_m \longrightarrow C$ of the correct type. Thus, $\forall^*_{S_1^m} f\ \forall^*_{W_1^{n-1}} \vec{\alpha}$ we have

$$g_1((d_1; f)(\vec{\alpha}), \ldots, (d_k; f)(\vec{\alpha})) =$$
$$g_2((d_1; f)(\vec{\alpha}), \ldots, (d_k; f)(\vec{\alpha})).$$

Thus, θ depends only on $[g]_{W_1^k}$.

(3) For any fixed g, almost all f have range in a c.u.b. set closed under $g(0)$. Thus,

$$\theta(f)(\vec{\alpha}) = g((d_1; f)(\vec{\alpha}), \ldots, (d_k; f)(\vec{\alpha})) < (d; f)(\vec{\alpha}).$$

So, $\pi([g]) < (d; S_1^m)$.

(1) and (2) show π is well-defined, and (3) show π embeds $j_{W_1^k}$ into $(d; S_1^m)$. $\square$

Non-trivial Descriptions

To motivate non-trivial descriptions, we consider the following problem.

Problem.
Compute $j_{S_1^{m_1}} \circ j_{S_1^{m_2}} \circ \cdots \circ j_{S_1^{m_t}}(\omega_n)$.

On the one hand, our previous formula computes this value.

Example. $j_{S_1^2} \circ j_{S_1^2}(\omega_3) = j_{S_1^2}(\omega_7) = \omega_{43}$.

We wish, however, to analyze the iterated ultrapower directly. This leads to the general (next level) notion of description.

<ins>Set-Up</ins>: Given a finite sequence of measures $S_1^{m_1}, \ldots, S_1^{m_t}$ and an integer $n-1$ (corresponding to $j_{W_1^{n-1}}(\omega_1) = \omega_n$).

We will define for each such sequence of measures and $n-1$ a set of descriptions

$$\mathcal{D}_{n-1}(S_1^{m_1}, \ldots, S_1^{m_t}).$$

Slightly more generally, allow a finite sequence of measures $K_1, \ldots, K_t$ each of the form S_1^m or W_1^m. So, we define

$$\mathcal{D}_{n-1}(K_1, \ldots, K_t).$$

Such a d will give an ordinal $(d; K_1, \ldots, K_t)$ as follows.

$(d; K_1, \ldots, K_t)$ is represented w.r.t. K_1 by the function $[h_1] \mapsto (d; h_1; K_2, \ldots, K_t)$.

Here $h_1\colon <_{m_1} \to \omega_1$ of the correct type if $K_1 = S_1^{m_1}$, and $h_1\colon m_1 \to \omega_1$ if $K_1 = W_1^{m_1}$.

$(d; h_1; K_2, \ldots, K_t)$ is represented w.r.t. K_2 by the function $[h_2] \mapsto (d; h_1, h_2, K_3, \ldots, K_t)$.

Finally, $(d; h_1, \ldots, h_t) < \omega_n$ is represented with respect W_1^{n-1} by the function

$$(\alpha_1, \ldots, \alpha_{n-1}) \mapsto (d; h_1, \ldots, h_t)(\vec{\alpha}) < \omega_1.$$

It remains to define $\mathcal{D}$ and the interpretation $(d; h_1, \ldots, h_t)(\vec{\alpha})$.

Main Point: Allow composition of the h's.

Definition of $\mathcal{D} = \mathcal{D}_{n-1}(K_1, \ldots, K_t)$.

$\mathcal{D}$ is defined through the following cases. We also define a value $k(d) \in \{1, \ldots, t\} \cup \{\infty\}$.

- (1) We allow $d = \cdot_i$ for $1 \leq i \leq n-1$. Set $k(d) = \infty$.

Interpretation: $(d;\vec{h})(\vec{\alpha}) = \alpha_i$.

• (2) If $K_k = W_1^{m_k}$, we allow $d = (k;i)$ for $1 \leq i \leq m_k$. Set $k(d) = k$.
Interpretation: $(d;\vec{h})(\vec{\alpha}) = h_k(i)$.

(1) and (2) are called *basic descriptions*.

• (3) If $K_k = S_1^m$ we allow

$$d = (k; d_m, d_1, \ldots, d_\ell)^{(s)}$$

where $k(d_1), \ldots, k(d_\ell), k(d_m) > k$ and $d_1 < \cdots < d_\ell < d_m$ (defined below).

Interpretation:

$$(d;\vec{h})(\vec{\alpha}) = h_k^{(s)}(\ell+1)((d_1;\vec{h})(\vec{\alpha}), \ldots, (d_\ell;\vec{h})(\vec{\alpha}), (d_m;\vec{h})(\vec{\alpha})).$$

The descriptions from (3) are called *non-basic descriptions*.

Definition of Ordering:
$d < d'$ iff

$$\forall^* h_1, \ldots, h_t \; \forall^* \vec{\alpha} \; (d; \vec{h})(\vec{\alpha}) < (d'; \vec{h})(\vec{\alpha}).$$

Can describe the $<$ relation directly. It is also generated by a *lowering operator* $\mathcal{L}$ as before. We describe $\mathcal{L}$ directly.

We define for $k \in \{1, \ldots, t\} \cup \{\infty\}$ and $d \in \mathcal{D}$ with $k(d) \geq k$ a *partial lowering* $\mathcal{L}_k(d)$. We will take $\mathcal{L}(d) = \mathcal{L}_1(d)$.

Definition of $\mathcal{L}_k$ We define $\mathcal{L}_k$ through the following cases.

(1) $k = \infty$. In this case, $d = \cdot_i$. We define $\mathcal{L}_k(d) = i - 1$ if $i > 1$ and otherwise d is $\mathcal{L}_k$ minimal.

In the remaining cases, $1 \leq k \leq t$.

(2) $K_k = W_1^m$. If $k(d) = k$, then $d = (k, i)$ for some $1 \leq i \leq m$. We set $\mathcal{L}_k(d) = i - 1$ unless $i = 1$ in which case d is $\mathcal{L}_k$-minimal. If $k(d) > k$, then $\mathcal{L}_k(d) = \mathcal{L}_{k+1}(d)$ unless d in $\mathcal{L}_{k+1}$ minimal in which case $\mathcal{L}_k(d) = (k, m)$.

(3) $K_k = S_1^m$ and $k(d) > k$. If d is $\mathcal{L}_{k+1}$-minimal then d is also $\mathcal{L}_k$-minimal. Otherwise set $\mathcal{L}_k(d) = (k; \mathcal{L}_{k+1}(d))$.

(4) $K_k = S_1^m$ and $k(d) = k$. So, $d = (k; d_m, d_1, \ldots, d_\ell)^{(s)}$.

(a) $\ell = m-1$ and s does not appear. $\mathcal{L}_k(d) = (k; d_m, d_1, \ldots, d_\ell)^s$.

(b) $\ell = m-1$ and s does appears. $\mathcal{L}_k(d) = (k; d_m, d_1, \ldots, \mathcal{L}_{k+1}(d_\ell))$ if $\mathcal{L}_{k+1}(d_\ell)$ is defined and $> d_{\ell-1}$ (if $\ell > 1$). Otherwise $\mathcal{L}_k(d) = (k; d_m, d_1, \ldots, d_{\ell-1})^s$.

If $m = 1$ (so $\ell = 0$) or if $\ell = 1$ and $\mathcal{L}_{k+1}(d_\ell)$ is not defined, set $\mathcal{L}_k(d) = d_m$.

(c) $\ell < m-1$ and s does not appear. $\mathcal{L}_k(d) = (k; d_m, d_1, \ldots, d_\ell, \mathcal{L}_{k+1}(d_m))$ if $\mathcal{L}_{k+1}(d_m)$ is defined and $> d_\ell$ (if $\ell \geq 1$). Otherwise, $\mathcal{L}_k(d) = (k; d_m, d_1, \ldots, d_\ell)^s$.

(d) $\ell < m-1$ and s appears. $\mathcal{L}_k(d) = (k; d_m, d_1, \ldots, \mathcal{L}_{k+1}(d_\ell))^s$ if $\mathcal{L}_{k+1}(d_\ell)$ is defined and $> d_{\ell-1}$ (if $\ell > 1$). Otherwise, $\mathcal{L}_k(d) = (k; d_m, d_1, \ldots, d_{\ell-1})^s$ if $\ell > 1$ and $= d_m$ for $\ell = 1$.

Claim. The $(d; S_1^{m_1}, \ldots, S_k^{m_t})$ correspond to the cardinals below

$$j_{S_1^{m_1}} \circ j_{S_1^{m_2}} \circ \cdots \circ j_{S_1^{m_t}}(\omega_n).$$

(Where $d \in \mathcal{D}_{n-1}(S_1^{m_1}, \ldots S_t^{m_t})$)

Return to example $j_{S_1^2} \circ j_{S_1^2}(\omega_3)$.

$$d = (1; (2; \cdot_2))$$
$$\mathcal{L}(d) = (1; (2; \cdot_2); (2; \cdot_2, \cdot_1))$$
$$\mathcal{L}^2(d) = (1; (2; \cdot_2); (2; \cdot_2, \cdot_1))^s$$
$$\mathcal{L}^3(d) = (1; (2; \cdot_2); (2; \cdot_2, \cdot_1)^s)$$
$$\mathcal{L}^4(d) = (1; (2; \cdot_2); (2; \cdot_2, \cdot_1)^s)^s$$
$$\mathcal{L}^5(d) = (1; (2; \cdot_2); \cdot_2)$$
$$\mathcal{L}^6(d) = (1; (2; \cdot_2); \cdot_2)^s$$
$$\mathcal{L}^7(d) = (1; (2; \cdot_2); (2; \cdot_1))$$
$$\mathcal{L}^8(d) = (1; (2; \cdot_2); (2; \cdot_1))^s$$
$$\mathcal{L}^9(d) = (1; (2; \cdot_2); \cdot_1)$$
$$\mathcal{L}^{10}(d) = (1; (2; \cdot_2); \cdot_1)^s$$
$$\mathcal{L}^{11}(d) = (2; \cdot_2)$$
$$\mathcal{L}^{12}(d) = (1; (2; \cdot_2, \cdot_1))$$
$$\mathcal{L}^{13}(d) = (1; (2; \cdot_2, \cdot_1); (2, \cdot_2, \cdot_1)^s)$$
$$\mathcal{L}^{14}(d) = (1; (2; \cdot_2, \cdot_1); (2, \cdot_2, \cdot_1)^s)^s$$
$$\mathcal{L}^{15}(d) = (1; (2; \cdot_2, \cdot_1); \cdot_2)$$
$$\mathcal{L}^{16}(d) = (1; (2; \cdot_2, \cdot_1); \cdot_2)^s$$
$$\mathcal{L}^{17}(d) = (1; (2; \cdot_2, \cdot_1); (2; \cdot_1))$$
$$\mathcal{L}^{18}(d) = (1; (2; \cdot_2, \cdot_1); (2; \cdot_1))^s$$
$$\mathcal{L}^{19}(d) = (1; (2; \cdot_2, \cdot_1); \cdot_1)$$
$$\mathcal{L}^{20}(d) = (1; (2; \cdot_2, \cdot_1); \cdot_1)^s$$
$$\mathcal{L}^{21}(d) = (2; \cdot_2, \cdot_1)$$
$$\mathcal{L}^{22}(d) = (1; (2; \cdot_2, \cdot_1)^s)$$
$$\mathcal{L}^{23}(d) = (1; (2; \cdot_2, \cdot_1)^s, \cdot_2)$$
$$\mathcal{L}^{24}(d) = (1; (2; \cdot_2, \cdot_1)^s, \cdot_2)^s$$
$$\mathcal{L}^{25}(d) = (1; (2; \cdot_2, \cdot_1)^s, (2; \cdot_1))$$
$$\mathcal{L}^{26}(d) = (1; (2; \cdot_2, \cdot_1)^s, (2; \cdot_1))^s$$
$$\mathcal{L}^{27}(d) = (1; (2; \cdot_2, \cdot_1)^s, \cdot_1)$$
$$\mathcal{L}^{28}(d) = (1; (2; \cdot_2, \cdot_1)^s, \cdot_1)^s$$
$$\mathcal{L}^{29}(d) = (2; \cdot_2, \cdot_1)^s$$
$$\mathcal{L}^{30}(d) = (1; \cdot_2)$$
$$\mathcal{L}^{31}(d) = (1; \cdot_2, (2; \cdot_1))$$
$$\mathcal{L}^{32}(d) = (1; \cdot_2, (2; \cdot_1))^s$$
$$\mathcal{L}^{33}(d) = (1; \cdot_2, \cdot_1)$$
$$\mathcal{L}^{34}(d) = (1; \cdot_2, \cdot_1)^s$$
$$\mathcal{L}^{35}(d) = \cdot_2$$
$$\mathcal{L}^{36}(d) = (1; (2; \cdot_1))$$
$$\mathcal{L}^{37}(d) = (1; (2; \cdot_1), \cdot_1)$$
$$\mathcal{L}^{38}(d) = (1; (2; \cdot_1), \cdot_1)^s$$
$$\mathcal{L}^{39}(d) = (2; \cdot_1)$$
$$\mathcal{L}^{40}(d) = (1; \cdot_1)$$
$$\mathcal{L}^{41}(d) = \cdot_1$$

So, we again have $j_{S^2_1} \circ j_{S^2_1}(\omega_3) = \omega_{43}$.

Remark. The iterated ultrapower is not the same as the ultrapower by the product measure in the AD context. For example, $j_{S^2_1} \circ j_{S^2_1}(\omega_3) = \omega_{43}$ but $j_{S^2_1 \times S^2_1}(\omega_3) = \omega_{11}$.

The proof of the lower bound is exactly as before (using now the non-trivial descriptions).

The upperbound follows from the following lemma.

For $g\colon \omega_1 \to \omega_1$ define $\theta = (g; d; K_1, \ldots, K_t)$ by:

$$
\begin{aligned}
&\forall^* h_1, \ldots, h_t \; \forall^* \vec{\alpha} \\
&\qquad \theta([f_1], \ldots, [f_t])(\vec{\alpha}) = g((d; f_1, \ldots, f_t)(\vec{\alpha})).
\end{aligned}
$$

Lemma. *If* $\theta < (d; K_1, \ldots, K_t)$, *then there is a* $g\colon \omega_1 \to \omega_1$ *such that* $\theta < (g; \mathcal{L}(d); K_1, \ldots, K_t)$.

It follows that $(d; \vec{K}) \leq (\mathcal{L}(d); \vec{K})^+$.

Example of proof. Consider

$$d = \mathcal{L}^{16}(-) = (1; (2; \cdot_2, \cdot_1), \cdot_2)^s$$
$$\mathcal{L}(d) = \mathcal{L}^{17}(-) = (1; (2; \cdot_2, \cdot_1), (2; \cdot_1)).$$

from the above example.

Suppose $\theta < (d; S^2_1, S^2_1)$. Then,

$$\begin{aligned} &\forall^*_{S^2_1}[h_1]\ \forall^*_{S^2_1}[h_2]\ \forall^*_{W^2_1}\alpha_1, \alpha_2 \\ &\qquad \theta([h_1], [h_2])(\vec{\alpha}) < (d; h_1, h_2)(\vec{\alpha}) \\ &\qquad = \sup_{\beta < \alpha_2} h_1(\beta, h_2(\alpha_1, \alpha_2)) \end{aligned}$$

So,

$$\forall^*_{S^2_1}[h_1]\ \forall^*_{S^2_1}[h_2]\ \forall^*_{W^2_1}\alpha_1, \alpha_2 \exists \beta < \alpha_2$$
$$\theta([h_1],[h_2])(\vec{\alpha}) < h_1(\beta, h_2(\alpha_1, \alpha_2))$$

Hence,

$$\forall^*_{S^2_1}[h_1]\ \forall^*_{S^2_1}[h_2]\ \exists g \colon \omega_1 \to \omega_1\ \forall^*_{W^2_1}\alpha_1, \alpha_2$$
$$\theta([h_1],[h_2])(\vec{\alpha}) < h_1(g(\alpha_1), h_2(\alpha_1, \alpha_2))$$

For any h_1 and ordinal $\theta(h_1)$ such that $\forall^*_{S^2_1}[h_2]\ \exists g \cdots$, consider the partition

$\mathcal{P}(h_1)$: We partition pairs (h_2, g) where $h_2\colon\ <_2 \to \omega_1$ is of the correct type, g is of the correct type, and

$$h_2(0)(\gamma) < g(\gamma) < h_2(0, \gamma + 1)$$

for all γ according to whether

$$\forall^*_{S^2_1}[h_2]\ \forall^*_{W^2_1}\alpha_1, \alpha_2$$
$$\theta(h_1)(h_2)(\vec{\alpha}) < h_1(g(\alpha_1), h_2(\alpha_1, \alpha_2))$$

On the homogeneous side this must hold. This gives:

$$\forall^*_{S^2_1} h_1\ \exists \text{c.u.b.} C \subseteq \omega_1\ \forall^*_{S^2_1}[h_2]\ \forall^*_{W^2_1}\alpha_1, \alpha_2$$
$$\theta(h_1)(h_2)(\vec{\alpha}) < h_1(N_C(h_2(0)(\alpha_1)), h_2(\alpha_1, \alpha_2))$$

where $N_C(\beta) =$least element of C greater than β.

Next consider the partition

$\mathcal{P}$: We partition $h_1, g\colon <_2 \longrightarrow \omega_1$ of the correct type with $h_1(\alpha_1, \alpha_2) < g(\alpha_1, \alpha_2) < N_{h_1}(\alpha_1, \alpha_2)$ according to whether

$$\forall^*_{S^2_1}[h_2]\ \forall^*_{W^2_1}\alpha_1, \alpha_2$$
$$\theta(h_1)(h_2)(\vec{\alpha}) < g(h_2(0)(\alpha_1), h_2(\alpha_1, \alpha_2))$$

On the homogeneous side the stated property holds. Fixing $C \subseteq \omega_1$ homogeneous we get:

$$\forall^*_{S^2_1}[h_2]\ \forall^*_{W^2_1}\alpha_1, \alpha_2$$
$$\theta(h_1)(h_2)(\vec{\alpha}) < N_C(h_2(0)(\alpha_1), h_2(\alpha_1, \alpha_2))$$
$$= (N_C; \mathcal{L}(d); h_1, h_2)(\vec{\alpha})$$

and we are done.

Analysis between $\boldsymbol{\delta}^1_3$ and $\boldsymbol{\delta}^1_5$

We used trivial descriptions to analyze the cardinals below $\boldsymbol{\delta}^1_3$. Now we use the (non-trivial) descriptions to analyze the cardinal structure below $\boldsymbol{\delta}^1_5$.

Recall that for a sequence of measures $K_1, \dots, K_t$, with each $K_k = W_1^{m_k}$ or $S_1^{m_k}$, and each n, we have a set $\mathcal{D}_n(\vec{K})$ of descriptions defined.

Assume inductively the weak partition relation on $\boldsymbol{\delta}^1_3$ (i.e., analysis of measures below $\boldsymbol{\delta}^1_3$).

Definition. W_3^m is the measure on $\boldsymbol{\delta}^1_3$ induced by the weak partition relation of $\boldsymbol{\delta}^1_3$, functions $f\colon \omega_{m+1} \to \boldsymbol{\delta}^1_3$ of the correct type, and the measure S_1^m on ω_{m+1}.

The measure W_3^m on $\boldsymbol{\delta}_3^1$ dominate all of the measures on $\boldsymbol{\delta}_3^1$ in the following sense.

Fact. For any measure μ on $\boldsymbol{\delta}_3^1$, there is an m such that $j_\mu(\boldsymbol{\delta}_3^1) < j_{W_3^m}(\boldsymbol{\delta}_3^1)$.

The homogeneous tree construction show that $\lambda_5 = \sup_\mu(\boldsymbol{\delta}_3^1)$, and so $\lambda_5 = \sup_m j_{W_3^m}(\boldsymbol{\delta}_3^1)$.

We define an ordinal $(d; W_3^n; K_1, \ldots, K_t) < j_{W_3^n}(\boldsymbol{\delta}_3^1)$ for $d \in \mathcal{D}_n(\vec{K})$.

We define this in the usual iterated way, where for $f\colon \omega_{n+1} \to \boldsymbol{\delta}_3^1$, $h_1, \ldots, h_t$ ($h_i\colon <_i \to \omega_1$) we have:

$$(d; f; h_1, \ldots, h_t) = f((d; h_1, \ldots, h_t)).$$

We also consider objects of the form d^s and define:

$$(d^s; f; h_1, \ldots, h_t) = \sup\{f(\beta)\colon \beta < f((d; h_1, \ldots, h_t))\}.$$

These are well-defined provided $d^{(s)}$ satisfies the following.

Definition. $d \in \mathcal{D}_n(K_1, \ldots, K_t)$ satisfies condition C if $\forall^* \vec{h}\ (d; \vec{h})$ is almost everywhere of the correct type. d^s satisfies C if $\forall^* \vec{h}\ (d; \vec{h})$ is the supremum of ordinals represented by functions of the correct type.

We set $\mathcal{L}'((d)) = (d)^s$, and $\mathcal{L}'((d)^s) = (\mathcal{L}(d))$. Let $\mathcal{L}((d)^{(s)})$ to the least iterate of $\mathcal{L}'$ satisfying C.

Theorem. *The cardinals below λ_5 are precisely those of the form $(d^{(s)}; W_3^n, K_1, \ldots, K_t)$ for some $d \in \mathcal{D}_n(\vec{K})$ with $d^{(s)}$ satisfying condition C.*

The cardinal corresponding to $(d; W_3^m; \vec{K})$ is given by the rank of this tuple in the ordering generated by the relation:

$$((d)^{(s)}; W_3^m, K_1, \dots, K_t) < \\ (\mathcal{L}((d)^{(s)}); W_3^m; K_1, \dots, K_t, K_{t+1}).$$

The upper bound follows from the following theorem.

For $g\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$, define $(g; d; W_3^n, K_1, \dots, K_t)$ by:

$$\forall^*_{W_3^n}[f]\ \forall^* h_1, \dots h_t \\ (g; d; f, h_1, \dots, h_t) = g(f^{(s)}((d; h_1, \dots, h_t))).$$

Theorem. *If $\theta < (d; W_3^n, \vec{K})$, then there is a $g\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ such that $\theta < (g; d; W_3^n; \vec{K})$.*

The lower bound follows by embedding arguments roughly similar to those of the iterated ultrapower computation.

Example. Consider $((d); W_3^2; S_1^2, S_1^2)$ where $d = \mathcal{L}^{26}(1; (2; \cdot_2)) = \mathcal{L}^{26}(d_M) = (1; (2; \cdot_2, \cdot_1)^s, (2; \cdot_1))^s$ (considered previously).

We have:

$$
\begin{aligned}
&|((d); W_3^2, S_1^2, S_1^2)| = \sup_{K_3} |(\mathcal{L}^{26}(d_M))^s; W_3^2, S_1^2, S_1^2, K_3)| + 1 \\
&= \sup_{\vec{K}} |((\mathcal{L}^{27}(d_M)); W_3^2, S_1^2, S_1^2, \vec{K})| + \omega + 1 \\
&= \sup_{\vec{K}} |((\mathcal{L}^{29}(d_M)); W_3^2, S_1^2, S_1^2, \vec{K})| + \omega + \omega + 1 \\
&= \sup_{\vec{K}} |((\mathcal{L}^{30}(d_M)); W_3^2, S_1^2, S_1^2, \vec{K})| + \omega + \omega + \omega + 1 \\
&= \sup_{\vec{K}} |((\mathcal{L}^{31}(d_M)); W_3^2, S_1^2, S_1^2, \vec{K})| + \omega^\omega + \omega + \omega + \omega + 1 \\
&= \sup_{\vec{K}} |((\mathcal{L}^{33}(d_M)); W_3^2, S_1^2, S_1^2, \vec{K})| + \omega + \omega^\omega + \omega + \omega + \omega + 1 \\
&= \sup_{\vec{K}} |((\mathcal{L}^{36}(d_M)); W_3^2, S_1^2, S_1^2, \vec{K})| + \omega^\omega + \omega + \omega^\omega + \omega + \omega + \omega + 1 \\
&= \omega^\omega \cdot 2 + \omega \cdot 3 + 1
\end{aligned}
$$

That is, $((d); W_3^2, S_1^2, S_1^2) = \aleph_{\omega^\omega \cdot 2 + \omega \cdot 3 + 1}$.

More on the Cardinal Structure

The three normal measure on $\boldsymbol{\delta}^1_3$ correspond to the three regular cardinals below $\boldsymbol{\delta}^1_3$, namely ω, ω_1, ω_2.

The three regular cardinals between $\boldsymbol{\delta}^1_3$ and $\boldsymbol{\delta}^1_5$ correspond to $j_\mu(\boldsymbol{\delta}^1_3)$ for μ one of these normal measures.

A description computation as above computes these to be:

$$\begin{aligned} j_{\mu_\omega}(\boldsymbol{\delta}^1_3) &= \boldsymbol{\delta}^1_4 = \aleph_{\omega+2} \\ j_{\mu_{\omega_1}}(\boldsymbol{\delta}^1_3) &= \aleph_{\omega\cdot 2+1} \\ j_{\mu_{\omega_2}}(\boldsymbol{\delta}^1_3) &= \aleph_{\omega^\omega+1} \end{aligned}$$

Also,

$$j_{W_3^n}(\boldsymbol{\delta}_3^1) = \aleph_{\omega^{\omega^n}+1}.$$

So,

$$\lambda_5 = \aleph_{\omega^{\omega^\omega}}, \qquad \boldsymbol{\delta}_5^1 = \aleph_{\omega^{\omega^\omega}+1}$$

We can easily read off the cofinality from the description.

Example. Consider again $((d); W_3^2; S_1^2, S_1^2)$ where $d = \mathcal{L}^{26}(1; (2; \cdot_2)) = (1; (2; \cdot_2, \cdot_1)^s, (2; \cdot_1))^s$.

Ordinals of the form $(g; (d)^s; W_3^2, S_1^2, S_1^2)$ for $g\colon \boldsymbol{\delta}_3^1 \to \boldsymbol{\delta}_3^1$ are cofinal in $((d); W_3^2, S_1^2, S_1^2)$.

Now, $\forall^*_{W_3^n} f \ \ \forall^* h_1, h_2$, we are evaluating g at $f^s((d; h_1, h_2)) = \sup_{\beta < (d; h_1, h_2)} f(\beta)$. This has the same cofinality as $(d; h_1, h_2)$.

Now $(d; h_1, h_2)$ has uniform cofinality $(\alpha_1, \alpha_2) \mapsto \alpha_1$, and so $(d; h_1, h_2)$ has cofinality ω_1.

Thus, $(g;(d)^s;W_3^n,S_1^2,S_1^2)$ depends only on $[g]_{\mu_{\omega_1}}$.

This gives a cofinal embedding from $j_{\mu_{\omega_1}}(\boldsymbol{\delta}_3^1)$ into $((d);W_3^n,S_1^2,S_1^2)$.

Thus, $\mathrm{cof}(\aleph_{\omega^\omega\cdot 2+\omega\cdot 3+1})=\aleph_{\omega\cdot 2+1}$.

Using the "linear" theory (described below), we can efficiently compute the cofinalities of all the cardinals. For example, below $\boldsymbol{\delta}_5^1$ we have:

Theorem. *Suppose $\boldsymbol{\delta}_3^1<\aleph_{\alpha+1}<\boldsymbol{\delta}_5^1$. Let $\alpha=\omega^{\beta_1}+\cdots+\omega^{\beta_n}$, where $\omega^\omega>\beta_1\geq\cdots\geq\beta_n$ be the normal form for α. Then:*

- *If $\beta_n=0$, then $\mathrm{cof}(\kappa)=\boldsymbol{\delta}_4^1=\aleph_{\omega+2}$.*
- *If $\beta_n>0$, and is a successor ordinal, then $\mathrm{cof}(\kappa)=\aleph_{\omega\cdot 2+1}$.*
- *If $\beta_n>0$ and is a limit ordinal, then $\mathrm{cof}(\kappa)=\aleph_{\omega^\omega+1}$.*

In general $\boldsymbol{\delta}^1_{2n+1} = \aleph_{\omega(2n-1)+1}$ where $\omega(0) = 1$ and $\omega(n+1) = \omega^{\omega(n)}$.

There are $2^{n+1}-1$ many regular cardinals below $\boldsymbol{\delta}^1_{2n+1}$. The regular cardinals between $\boldsymbol{\delta}^1_{2n-1}$ and $\boldsymbol{\delta}^1_{2n+1}$ correspond to the ultrapowers of $\boldsymbol{\delta}^1_{2n+1}$ by the normal measures on $\boldsymbol{\delta}^1_{2n+1}$, which correspond to the regular cardinals below $\boldsymbol{\delta}^1_{2n+1}$.

There is a canonical family of measures associated to each regular cardinal.

Families are:

$$W_1^m, S_1^{1,m}, W_3^m, S_3^{1,m}, S_3^{2,m}, S_3^{3,m},$$
$$W_5^m, S_5^{1,m}, \ldots, S_5^{7,m}, W_7^m, \ldots$$

W^m_{2n+1} defined using weak partition relation on $\boldsymbol{\delta}^1_{2n+1}$, function $f\colon = \mathrm{dom}(S^{\ell,m}_{2n-1})$ of the correct type and the measure $S^{\ell,m}_{2n-1}$. Here $\ell = 2^n - 1$.

$S^{1,m}_{2n+1}$ defines as S^m_1 was defined using $\boldsymbol{\delta}^1_{2n+1}$.

$S^{\ell,m}_{2n+1}$ for $\ell > 1$ defined using the strong partition relation on $\boldsymbol{\delta}^1_{2n+1}$ functions $F\colon \boldsymbol{\delta}^1_{2n+1} \to \boldsymbol{\delta}^1_{2n+1}$ of the correct type and the measure μ on $\boldsymbol{\delta}^1_{2n+1}$.

μ is the measure induced by the weak partition relation on $\boldsymbol{\delta}^1_{2n+1}$, functions $f\colon \mathrm{dom}(\nu) \to \boldsymbol{\delta}^1_{2n+1}$ and the measure ν, where ν is the m^{th} measure in the $\ell - 1^{\text{st}}$ family.

General Descriptions:

The general level descriptions are defined inductively, and have *indices* associated to them.

Trivial descriptions are level-1 descriptions. They have empty index. These analyze the measure on $\boldsymbol{\delta}^1_1$ and compute $\boldsymbol{\delta}^1_3$.

Level-2 descriptions are those of the form $d = (k; d_m, d_1, \dots, d_\ell)^{(s)}$ we we defined previously. If $d \in \mathcal{D}_n(\vec{K})$, we associate the index W^n_1 to d i.e., ($d = d^{W^n_1}$). These analyze the measure on λ_3.
The level-3 descriptions are those of the form (d) or $(d)^s$ for d a level-2 description. These analyze the measures on $\boldsymbol{\delta}^1_3$ and compute $\boldsymbol{\delta}^1_5$. They also have index W^n_1.

note: There is not much difference between the level-2 and level-3 desciptions; we could group them together.

Level-4 and 5 descriptions analyze the measures on λ_5 and $\boldsymbol{\delta}^1_5$ respectively. They are defined relative to a sequence of measures $\vec{K}$ where $K_i \in W^m_1, \ldots, S^{3,m}_3$ for level-4, and $(W^m_5, \vec{K})$ for level-5.

They have indices of the form $(W^m_3, S^{1,m_1}_1, \ldots, S^{1,m_t}_1)$ (can use $W^{m_i}_1$).

A basic description at this level is a level-3 description in

$$\mathcal{D}(W^m_3, S^{1,m_1}_1, \ldots, S^{1,m_t}_1).$$

Linear Analysis

We describe the cardinal structure without using descriptions or iterated ultrapowers (with Khafizov, Löwe).

Need the description analysis to prove it works, however.

Need this to show that all descriptions actually represent cardinals.

Definition. Let $\mathfrak{A}_\alpha$ be the free algebra on α generators $\{\mathsf{V}_\beta\}_{\beta<\alpha}$ using the operations $\oplus$, $\otimes$.

$\mathfrak{A} = \bigcup_\alpha \mathfrak{A}_\alpha$.

We assign an ordinal height $o(v)$ to every term in $\mathfrak{A}$ inductively as follows.

Definition.

$$
\begin{aligned}
o(\mathsf{v}_0) &= 0 \\
o(\mathsf{v}_\alpha) &= \mathrm{ht}(\mathfrak{A}_\alpha) = \sup\{o(t)+1\colon t \in \mathfrak{A}_\alpha\} \\
o(s \oplus t) &= o(s) + o(t) \\
o(s \otimes t) &= o(s) \cdot o(t)
\end{aligned}
$$

So, $o(\mathsf{v}_0) = 0$, $o(\mathsf{v}_1) = 1$, $o(\mathsf{v}_2) = \omega$, $o(\mathsf{v}_3) = \omega^\omega$, $o(\mathsf{v}_4) = \omega^{\omega^2}$, $o(\mathsf{v}_n) = \omega^{\omega^{n-2}}$, $o(\mathsf{v}_\omega) = \omega^{\omega^\omega}$.

For $\alpha \geq \omega$, $o(\mathsf{v}_\alpha) = \omega^{\omega^\alpha}$.

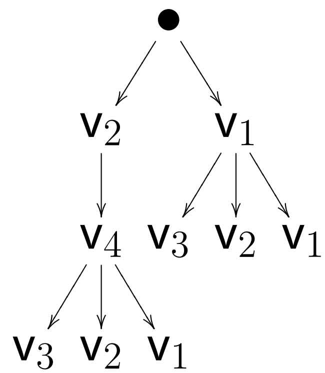
We assign to each generator v_α an order type $\mathrm{ot}(\mathsf{v}_\alpha)$ and a measure $\mu(\mathsf{v}_\alpha)$ on this order-type.

This will generate an assignment of an order-type and a collection of measures ("germ") to each general term as in the following example.

Example. Consider the term

$$t = (((\mathsf{v}_3 \oplus \mathsf{v}_2 \oplus \mathsf{v}_1) \otimes \mathsf{v}_4) \otimes \mathsf{v}_2) \oplus ((\mathsf{v}_3 \oplus \mathsf{v}_2 \oplus \mathsf{v}_1) \otimes \mathsf{v}_1)$$

We can represent this as a tree as follows:



Suppose we know

$$\begin{aligned} o(\mathsf{v}_1) &= 1, & \mu(\mathsf{v}_1) &= \text{principal measure} \\ o(\mathsf{v}_2) &= \omega_1 & \mu(\mathsf{v}_2) &= W_1^1 \\ o(\mathsf{v}_3) &= \omega_2 & \mu(\mathsf{v}_2) &= S_1^1 \\ o(\mathsf{v}_4) &= \omega_3 & \mu(\mathsf{v}_2) &= S_1^2 \end{aligned}$$

Then $\mathrm{ot}(t) = (\omega_2+\omega_1+1)\cdot\omega_3\cdot\omega_1+(\omega_2+\omega_1+1)\cdot 1 = \omega_3\cdot\omega_1+\omega_2+\omega_1$.

We identify $\mathrm{ot}(t)$ the ordering on tuples $\langle i_0, \alpha_0, \ldots, i_k, \alpha_k\rangle$ where $(i_0, \ldots, i_k)$ corresponds to a terminal node in the tree, and for $(i_0, \ldots, i_\ell)$ a node in the tree, $\alpha_\ell < \mathrm{ot}(v))$, where $v = v^{\vec{i}}$ is the variable corresponding to this node.

$\mu(t)$ is the collection of measures $\mu(v^{\vec{i}})$.

We set $\mathrm{ot}(\mathsf{v}_{\omega(2n-1)}) = \boldsymbol{\delta}^1_{2n+1}$, $\mu(\mathsf{v}_{\omega(2n-1)}) = \omega$-cofinal normal measure on $\boldsymbol{\delta}^1_{2n+1}$.

For example, $\mathrm{ot}(\mathsf{v}_\omega) = \boldsymbol{\delta}^1_3$, $\mathrm{ot}(\mathsf{v}_{\omega^{\omega^\omega}}) = \boldsymbol{\delta}^1_5$.

To $\mathsf{v}_{2+\omega(2n-1)+\alpha}$ (where $\alpha < \omega(2n+1)$) we associate the measure defined by the strong partition relation on $\boldsymbol{\delta}^1_{2n+1}$ and the measure ν on $\boldsymbol{\delta}^1_{2n+1}$.

ν is the measure induced by the weak partition relation on $\boldsymbol{\delta}^1_{2n+1}$, functions $f\colon \mathrm{ot}(t) \to \boldsymbol{\delta}^1_{2n+1}$ of the correct type, and the measure (germ) $\mu(t)$.

(note: we order by reverse lexicographic order on the indices, though it turns out this doesn't matter).

Example. $\mu(\mathsf{v}_{\omega+1}) = \omega$-cofinal normal measure on $\boldsymbol{\delta}^1_4$, $\mu(\mathsf{v}_{\omega+2})$ is measure on $\aleph_{\omega+3}$. $\mu(\mathsf{v}_{\omega\cdot 2})$ is the ω-cofinal normal measure on $\aleph_{\omega\cdot 2+1}$.

This assignment describes the cardinal structure as follows.

Theorem. *Let $t \in \mathfrak{A}$ with $ot(t) < \boldsymbol{\delta}^1_{2n+1}$. Then $j_\nu(\boldsymbol{\delta}^1_{2n+1}) = \aleph_{\omega(2n-1)+o(t)+1}$ where $\nu = \nu(t)$ is the measure defined above.*

Example. Consider the term

$$t = (((\mathsf{v}_3 \oplus \mathsf{v}_2 \oplus \mathsf{v}_1) \otimes \mathsf{v}_4) \otimes \mathsf{v}_2) \oplus ((\mathsf{v}_3 \oplus \mathsf{v}_2 \oplus \mathsf{v}_1) \otimes \mathsf{v}_1)$$

above.

Then $j_{\nu(t)}(\boldsymbol{\delta}^1_3) = \aleph_{\omega^{\omega^2}\cdot\omega+\omega^\omega+\omega+2}$.

A Collapsing Result

Theorem. *Assume the non-stationary ideal on ω_1 is ω_2-saturated and there are $\omega+1$ Woodin cardinals in V. Then there is a $\kappa < (\aleph_{\omega_2})^{L(\mathbb{R})}$ such that κ is regular in $L(\mathbb{R})$ but κ is not a cardinal in V.*

Proof combines the determinacy theory of $L(\mathbb{R})$, the Shelah p.c.f. theory of V, and Woodin's theory of the non-stationary ideal.

Woodin Covering Theorem:

Theorem. *(Woodin) Same hypotheses as theorem. If $A \subseteq \lambda < \Theta^{L(\mathbb{R})}$ and $|A| = \omega_1$, then there is a $B \in L(\mathbb{R})$, $|B| = \omega_1$ with $A \subseteq B$.*

A special case of this is the fact that every c.u.b. $C \subseteq \omega_1$ contains a c.u.b. $C_1 \subseteq C$ with $C_1 \in L(\mathbb{R})$.

From Shelah's p.c.f. theory we need the following.

Theorem (ZFC)**.** *Let A be a set of regular cardinals with $|A| < \inf(A)$ and with cof*$(\sup(A)) > \omega$. *Then there is a c.u.b.* $C \subseteq \sup(A)$ *such that* $\max p.c.f.(C^+) = (\sup(A))^+$, *where* $C^+ = \{\kappa^+ \colon \kappa \in C\}$.

From the determinacy theory of $L(\mathbb{R})$ we need the following fact.

Fact. For all $\alpha < \omega_1$, $\boldsymbol{\delta}^1_\alpha < \aleph_{\omega_1}$.

We also need a certain partition property.

Let $\vec{\kappa} = \{\kappa_\alpha\}_{\alpha<\rho}$ be an increasing, discontinuous sequence of regular cardinals.

Let $\vec{\theta} = \{\theta_\alpha\}_{\alpha<\rho}$ be a ρ-sequence of ordinals with $\theta_\alpha \leq \kappa_\alpha$.

We consider block functions f from $\Sigma\theta_\alpha$ to $\sup\kappa_\alpha$.

We say $\vec{\kappa} \longrightarrow (\vec{\kappa})^{\vec{\theta}}$ if for any partition $\mathcal{P}$ of the block functions f into two pieces, there is a blockwise c.u.b. homogeneous set $\vec{H} = \{H_\alpha\}_{\alpha<\rho}$ for $\mathcal{P}$ (for functions blockwise of the correct type).

Let $\mathcal{R} \subseteq \aleph_{\omega_1}$ be the set of cardinals of the form $\boldsymbol{\delta}^1_{\alpha+1}$ for limit α.

For $\kappa \in \mathcal{R}$, there is pointclass $\boldsymbol{\Sigma}_0$ closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee_\omega$ and scale($\boldsymbol{\Sigma}_0$) such that if $\boldsymbol{\Pi}_1 = \forall^{\omega^\omega}\boldsymbol{\Sigma}_0$, then:
$\boldsymbol{\Pi}_1$ is closed under $\forall^{\omega^\omega}$, $\cap_\omega$, $\cup_\omega$, scale($\boldsymbol{\Pi}_1$), and $\boldsymbol{\delta}^1_{\alpha+1} = o(\boldsymbol{\Pi}_1)$.

Theorem (AD + DC)**.** *Let $\vec{\kappa} = \{\kappa_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{R}$. Then for all $\theta < \omega_1$ we have $\vec{\kappa} \longrightarrow (\vec{\kappa})^\theta$.*

In fact, theorem holds for any ω_1 sequence of pointclasses resembling $\boldsymbol{\Pi}^1_1$.

In special case $\vec{\kappa} \subseteq \mathcal{R}$ of theorem, proof is easy as in proof of weak partition relation on ω_1 (special case easier since we can uniformly find universal sets for the $\mathbf{\Pi}_1$ classes).

Theorem ($\mathsf{AD} + \mathsf{DC}$)**.** *Let μ be any measure on ω_1. Let $\vec{\kappa} = \{\kappa_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{R}$. Then $\prod \kappa_\alpha/\mu$ is regular.*

Proof. Let $\delta = \prod \kappa_\alpha/\mu$. Suppose $\pi\colon \lambda \to \delta$ is cofinal, where $\lambda < \delta$. Consider the partition of block functions given by $\mathcal{P}([f]_\mu, [g]_\mu) = 1$ iff there is an element of $\mathrm{ran}(\pi)$ between $[f]_\mu$ and $[g]_\mu$. The homogeneous side must be the 1 side. Let S be block homogeneous for $\mathcal{P}$. For $[f]_\mu < \delta$, define $[f']$ by $f'(\alpha) = f(\alpha)^{\text{th}}$ element of S. Then $A = \{[f']_\mu : [f]_\mu < \delta\}$ has order-type δ and between any two elements of A is an element of $\mathrm{ran}(\pi)$. $\square$

<u>Proof of Theorem</u>:

From the p.c.f. theory, let $C \subseteq \aleph_{\omega_1}$ be a c.u.b. subset of the limit Suslin cardinals such that max pcf$(C^+) = \aleph_{\omega_1+1}$. W.l.o.g. may assume $C \in L(\mathbb{R})$.

Let μ be the normal measure on ω_1 (in $L(\mathbb{R})$), and let $\mathcal{U}$ (in V) be an ultrafilter on ω_1 extending μ.

Let $f : \omega_1 \longrightarrow C^+$ be increasing, $f \in L(\mathbb{R})$, and $f(\alpha) > (\boldsymbol{\delta}^1_\alpha)^+$ almost everywhere.

Thus, $\lambda \doteq [f]_\mu$ is regular in $L(\mathbb{R})$. Assume λ is also regular in V, towards a contradiction. We also assume all elements of $C^+ \subseteq \mathcal{R}$ are regular in V.

Also, $\lambda > \rho \doteq \aleph_{\omega_1+1}$. In V, $\mathrm{cof}(\prod f(\alpha)/\mathcal{U}) = \rho$.

In V we define a cofinal map $\pi\colon \rho \longrightarrow \lambda$ as follows. In V, fix a scale $\{f_\alpha\}_{\alpha<\rho}$ (only need $\mathcal{U}$ unboundedness).

For $\alpha < \rho$, let $\pi(\alpha) = ([g]_\mu)^{L(\mathbb{R})}$, where $g \in L(\mathbb{R})$ represents the μ least equivalence class such that $g \geq f_\alpha$ almost everywhere w.r.t. $\mathcal{U}$. Such a function exists by Woodin's theorem, and the definition is well-defined.

To show π is cofinal, let $\beta < \lambda$. Let $[g_0]_\mu = \beta$. Pick α such that $f_\alpha >_\mathcal{U} g_0$. By definition $\pi(\alpha) = [g]_\mu$ where $g \in L(\mathbb{R})$ and $g >_\mathcal{U} f_\alpha$. So, $g >_\mathcal{U} f_\alpha >_\mathcal{U} g_0$. Since $g, g_0 \in L(\mathbb{R})$, we have $g >_\mu g_0$. Hence, $\pi(\alpha) = [g]_\mu > [g_0]_\mu = \beta$.