Bounding Nonsplitting Enumeration Degrees

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Goal: Introduce a form of Σ^0_2-permitting for the enumeration degrees.

Till now, density was the only known property that held in all ideals of Σ^0_2-enumeration degrees.

A is enumeration reducible to B ($A \leq_e B$) if we can enumerate A given any enumeration of B.

Definition

$A \leq_e B$ iff there is c.e. set Φ such that
$A = \{x : \exists \langle x, P \rangle \in \Phi \ (P \text{ finite and } P \subseteq B)\} = \Phi^B$

Basic Facts

We can embed the Turing degrees into the enumeration degrees via the embedding $\iota : \deg_T(A) \mapsto \deg_e(A \oplus \overline{A})$.

- The image of the Turing degrees under ι is known as the "total degrees".

$\mathbf{0}_e = \{W : W \text{ is c.e.}\}$.

$\mathbf{0}'_e = \deg_e(\overline{K})$.

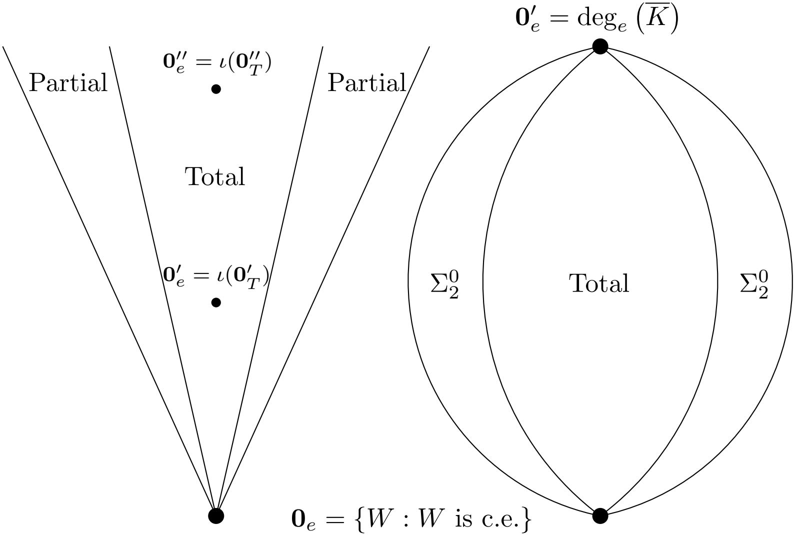
Theorem (Cooper, 1984)

A is Σ^0_2 iff $A \leq_e \overline{K}$.

Theorem (Cooper, 1984)

The Σ^0_2-enumeration degrees are dense.

The Global and Local Picture



Nonsplitting Degrees

Definition

A degree $\mathbf{a}$ is nonsplitting if $\mathbf{a} > \mathbf{0}_e$ and for every $\mathbf{x}, \mathbf{y} < \mathbf{a}$, $\mathbf{x} \vee \mathbf{y} < \mathbf{a}$.

Theorem (Ahmad 1989 (c.f. Ahmad, Lachlan 1998))

There exists a nonsplitting Σ^0_2-enumeration degree.

The requirements:

- Nontrivial

$\mathcal{N}_\Phi \quad : \ A \neq \Phi$, and

- Nonsplitting

$\mathcal{S}_{\Psi,\Omega_0,\Omega_1} \quad : \ A = \Psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow \exists \Gamma_0, \Gamma_1 \left[A = \Gamma_0^{\Omega_0^A} \text{ or } A = \Gamma_1^{\Omega_1^A} \right].$

Bounding Nonsplitting Degrees

Theorem (Kent, Sorbi 2007)

Every nontrivial Σ^0_2-enumeration degree bounds a nonsplitting degree.

The requirements:

- $A \leq_e B$

$\mathcal{R} \quad : A = \Theta^B$

- Nontrivial

$\mathcal{N}_\Phi \quad : A = \Phi \Rightarrow \exists \Delta (B = \Delta)$, and

- Nonsplitting

$$\mathcal{S}_{\Psi,\Omega_0,\Omega_1} \quad : A = \Psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow \exists \Gamma_0, \Gamma_1 \left[A = \Gamma_0^{\Omega_0^A} \text{ or } A = \Gamma_1^{\Omega_1^A} \right] \text{ or } \exists \Lambda [B = \Lambda].$$

Some Corollaries

Corollary

The nonsplitting degrees are downwards dense in the Δ^0_2-enumeration degrees.

Corollary

There is a properly Σ^0_2 nonsplitting enumeration degree.

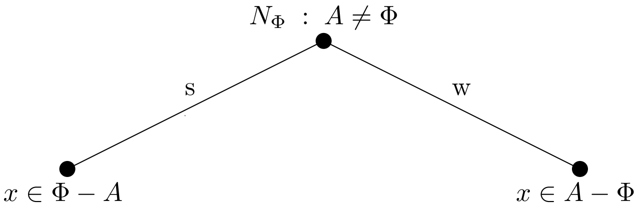
Corollary

The c.e. Turing degrees are not elementarily equivalent to any ideal of the Σ^0_2-enumeration degrees.

Question

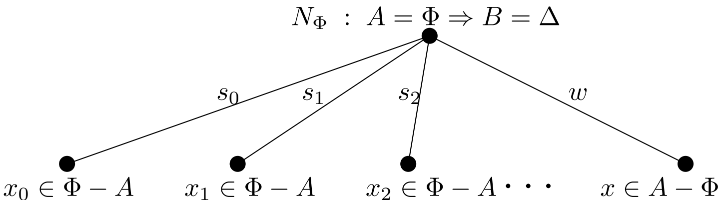
Are the nonsplitting degrees dense in the Σ^0_2 or Δ^0_2 enumeration degrees?

N-Requirement - Standard



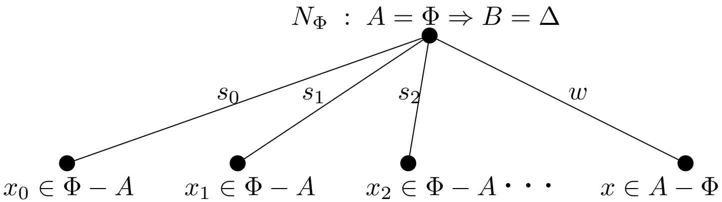
1. Pick x and set $x \in A$.
2. If ever $x \in \Phi$, set $x \notin A$.

N-Requirement - Bounded



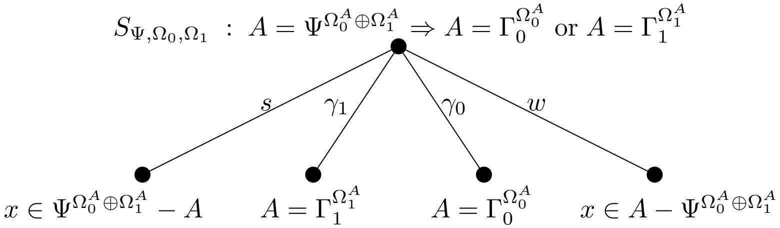
1. Assume $x_0, \ldots x_{n-1} \in \Phi \cap A$.
2. Pick x_n.
3. While 1. holds, enumerate $\langle x_n, B \upharpoonright x_n \rangle \in \Theta$.
4. If ever $x_n \in \Phi$, stop defining x_n axioms, and enumerate $D_n = \bigcap\{D : \langle x_n, D\rangle \in \Theta\}$ into Δ.

N-Requirement - Bounded



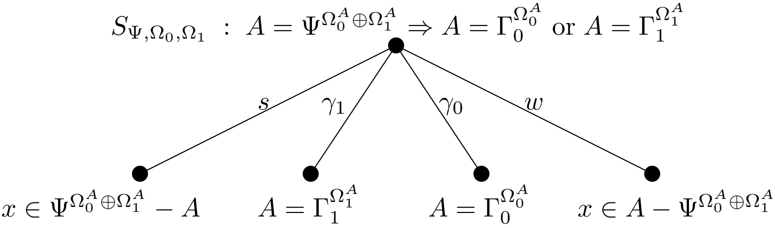
- If infinitely many x_i are defined and each $x_i \in A$, then since $D_0 \subseteq D_1 \subseteq \cdots \subseteq B$, we can conclude $\Delta = B$.
- If $x_i \notin A$ then $x_j \notin A$ for all $j > i$.
- (Conditional Dumping) While $x_i \in A$, for all $y \in S(s_i)$, enumerate $\langle y, B \restriction y \rangle$ into Θ.

S-Requirement - Standard



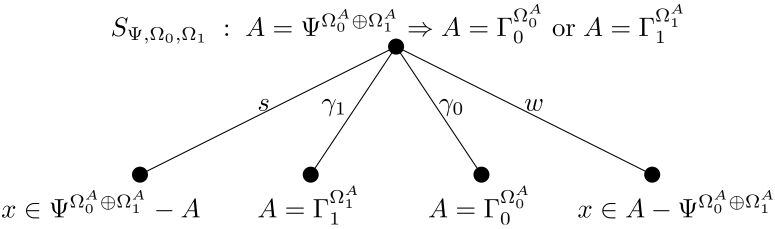
1. Pick x and set $x \in A$.
2. Wait for $x \in \Psi^{\Omega_0^A \oplus \Omega_1^A}$ via $\langle x, F_0 \oplus F_1 \rangle \in \Psi$.
3. Enumerate $\langle x, F_i \rangle$ into Γ_i, x into $S(\gamma_0)$ and return to Step 1.
    - Hopefully $x \in A$ iff $F_0 \subseteq \Omega_0^A$.
    - If true for co-finitely many x, then $A = \Gamma_0^{\Omega_0^A}$.
    - Strategies below γ_0 can only use x which have this property.

S-Requirement - Standard



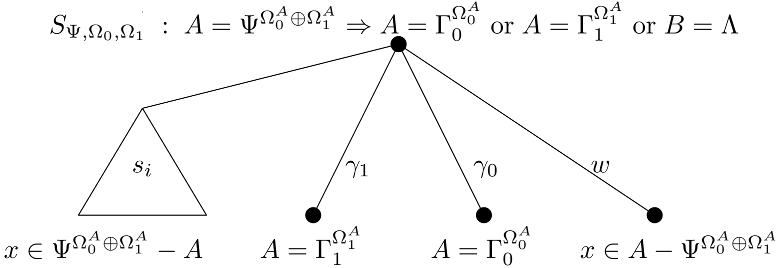
4. If ever see $x \notin A$ and $F_0 \subseteq \Omega_0^A$ (hence $x \in \Gamma_0^{\Omega_0^A} - A$), dump $S(\gamma_0) - \{x\}$ into A, enumerate x into $S(\gamma_1)$.
   - For this x, $F_0 \subseteq \Omega_0^{A-\{x\}}$, killing Γ_0.
   - Hopefully $x \in A$ iff $F_1 \subseteq \Omega_1^A$.
   - If true for infinitely many x, then $A = \Gamma_1^{\Omega_1^A}$.
   - Strategies below γ_1 can only use x which have this property.

S-Requirement - Standard



5. If ever see $x \notin A$ and $F_1 \subseteq \Omega_1^A$ (hence $x \in \Gamma_1^{\Omega_1^A} - A$), dump $S(\gamma_1) \cup S(\gamma_0) - x$ into A, and set $x \notin A$.
   - For this x, $F_1 \subseteq \Omega_1^{A-\{x\}}$, killing Γ_1.
   - Not a problem since now $x \in \Psi^{\Omega_0^{A-\{x\}} \oplus \Omega_1^{A-\{x\}}}$.

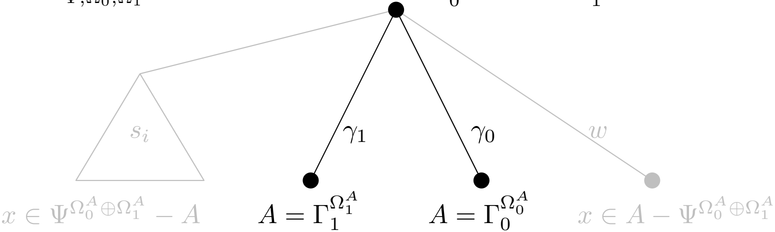
S-Requirement - Bounded (v. 1.0)



- As with the N-strategy, expand the outcome s to $s_0, s_1, \ldots$.
- If we choose $x_0, x_1, \ldots$ as possible diagonalization witness, and for all i, $x_i \in \Psi^{\Omega_0^A \oplus \Omega_1^A} \cap A$, then $B = \Lambda$.
- (Conditional Dumping) While $x_i \in A$, for all $y \in S(s_i)$, enumerate $\langle y, B \upharpoonright y \rangle$ into Θ.

S-Requirement - Potential Problem

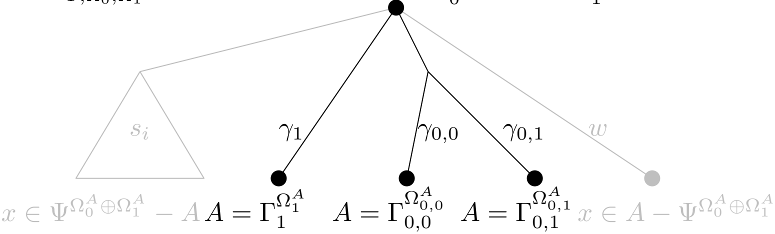
$$S_{\Psi,\Omega_0,\Omega_1} \; : \; A = \Psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow A = \Gamma_0^{\Omega_0^A} \text{ or } A = \Gamma_1^{\Omega_1^A} \text{ or } B = \Lambda$$



- Γ_0 assumes all elements of $S(\gamma_1)$ have settled down.
- Possibly there is $x \in S(\gamma_1)$ and $y \in S(\gamma_0)$ such that
  - while $x \in A$, $y \in A$ iff $y \in \Gamma_0^{\Omega_0^A}$, but
  - while $x \notin A$, $y \notin \Gamma_0^{\Omega_0^A}$.
  - $\lim_s A(x)$ does not exist, i.e. A is Σ_2^0.

S-Requirement - Solution

$$S_{\Psi,\Omega_0,\Omega_1} : A = \Psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow A = \Gamma_0^{\Omega_0^A} \text{ or } A = \Gamma_1^{\Omega_1^A} \text{ or } B = \Lambda$$



- Assume $S(\gamma_1) = \{x\}$.
- Construct two enumeration operators: $\Gamma_{0,0}$ and $\Gamma_{0,1}$.
- $\Gamma_{0,0}$ assumes $x \notin A$ and $\Gamma_{0,1}$ assumes $x \in A$.
- Accounts for Σ_2^0 nature of A.
- In general, if $|S(\gamma_1)| = n$, then we construct 2^n enumeration operators.

Quasi-Lexicographical Ordering

Definition

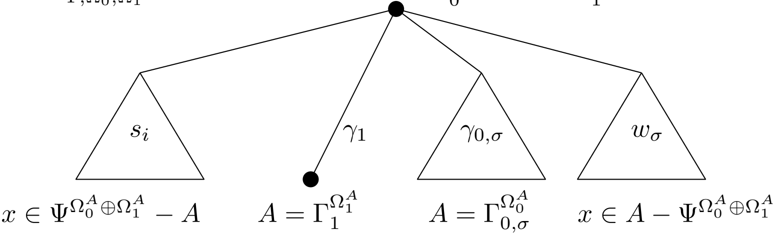
Define the quasi-lexicographical ordering $<_b$ on $2^{<\omega}$ by $\sigma <_b \tau$ if

1. $|\sigma| < |\tau|$, or
2. $|\sigma| = |\tau|$ and there is a $k < |\sigma|$ such that $\sigma(k) = 0$, $\tau(k) = 1$, and for all $i < k$, $\sigma(i) = \tau(i)$.

$$\begin{array}{lllllllllllll}
0 & <_b & 1 & <_b \\
00 & <_b & 01 & <_b & 10 & <_b & 11 & <_b \\
000 & <_b & 001 & <_b & 010 & <_b & 011 & <_b & 100 & <_b & 101 & <_b & \cdots
\end{array}$$

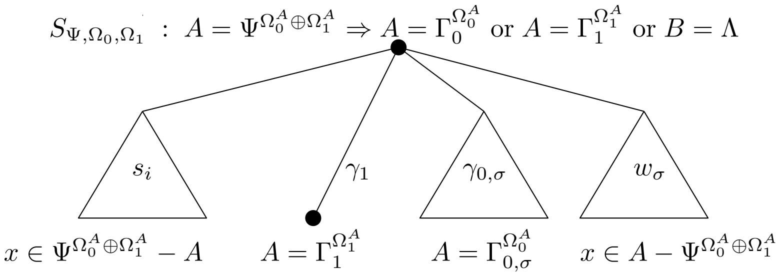
S-Requirement - Bounded

$$S_{\Psi,\Omega_0,\Omega_1} : A = \Psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow A = \Gamma_0^{\Omega_0^A} \text{ or } A = \Gamma_1^{\Omega_1^A} \text{ or } B = \Lambda$$



- The strategy proceeds basically the same as before.
- We have added infinitely many outcomes to the tree to account for all possible states of elements to the left of the current outcome.
- When an element is moved left through the streams, it must make sure that the assumptions of the new stream are consistent with the assumptions of the previous stream.

S-Requirement - Bounded



- Other technical concerns not covered here include:
  - Other uses of conditional dumping.
  - Local approximations to B.
  - etc.

The End!

Kent, Sorbi “Bounding Nonsplitting Enumeration Degrees,”

to appear in The Journal of Symbolic Logic.