Bounding Nonsplitting Enumeration Degrees

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Goal: Introduce a form of \( \Sigma^0_2 \)-permitting for the enumeration degrees.

Till now, density was the only known property that held in all ideals of \( \Sigma^0_2 \)-enumeration degrees.

A is enumeration reducible to \( B \) (\( A \leq_e B \)) if we can enumerate \( A \) given any enumeration of \( B \).

**Definition**

\( A \leq_e B \) iff there is c.e. set \( \Phi \) such that

\[
A = \{ x : \exists \langle x, P \rangle \in \Phi \ (P \text{ finite and } P \subseteq B) \} = \Phi^B
\]
Basic Facts

We can embed the Turing degrees into the enumeration degrees via the embedding \( \iota : \deg_T(A) \mapsto \deg_e(A \oplus \overline{A}) \).

- The image of the Turing degrees under \( \iota \) is known as the “total degrees”.

\[
0_e = \{ W : W \text{ is c.e.} \}.
\]

\[
0'_e = \deg_e(K).
\]

**Theorem** (Cooper, 1984)

A is \( \Sigma^0_2 \) iff \( A \leq_e K \).

**Theorem** (Cooper, 1984)

The \( \Sigma^0_2 \)-enumeration degrees are dense.
The Global and Local Picture

\[ 0_e = \{ W : W \text{ is c.e.} \} \]

\[ 0_e = \deg_e (\overline{K}) \]

\[ 0_e'' = \nu(0_T'') \]

\[ 0_e' = \nu(0_T') \]

\[ 0_e = \nu(0_T) \]

\[ \Sigma_2^0 \]

\[ \text{Total} \]

\[ \text{Partial} \]

\[ \text{Total} \]

\[ \Sigma_2^0 \]

\[ \text{Partial} \]

\[ \text{Partial} \]
Nonsplitting Degrees

Definition
A degree $a$ is nonsplitting if $a > 0_e$ and for every $x, y < a$, $x \lor y < a$.

Theorem (Ahmad 1989 (c.f. Ahmad, Lachlan 1998))
There exists a nonsplitting $\Sigma^0_2$-enumeration degree.

The requirements:

• Nontrivial
  $\mathcal{N}_\Phi$ : $A \neq \Phi$, and

• Nonsplitting
  $\mathcal{S}_{\psi, \Omega_0, \Omega_1}$ : $A = \psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow \exists \Gamma_0, \Gamma_1 [A = \Gamma_0^{\Omega_0^A}$ or $A = \Gamma_1^{\Omega_1^A}]$. 
Bounding Nonsplitting Degrees

Theorem (Kent, Sorbi 2007)

Every nontrivial $\Sigma^0_2$-enumeration degree bounds a nonsplitting degree.

The requirements:

- $A \leq_e B$

  $R : A = \Theta^B$

- Nontrivial

  $N_\Phi : A = \Phi \Rightarrow \exists \Delta (B = \Delta)$, and

- Nonsplitting

  $S_{\psi, \Omega_0, \Omega_1} : A = \psi^{\Omega^A_0 \oplus \Omega^A_1} \Rightarrow \exists \Gamma_0, \Gamma_1 [A = \Gamma^\Omega^A_0 \text{ or } A = \Gamma^\Omega^A_1]$

  or $\exists \Lambda [B = \Lambda]$. 
Some Corollaries

Corollary

The nonsplitting degrees are downwards dense in the $\Delta^0_2$-enumeration degrees.

Corollary

There is a properly $\Sigma^0_2$ nonsplitting enumeration degree.

Corollary

The c.e. Turing degrees are not elementarily equivalent to any ideal of the $\Sigma^0_2$-enumeration degrees.

Question

Are the nonsplitting degrees dense in the $\Sigma^0_2$ or $\Delta^0_2$ enumeration degrees?
1. Pick $x$ and set $x \in A$.
2. If ever $x \in \Phi$, set $x \notin A$. 

$N_\Phi : A \neq \Phi$
N-Requirement - Bounded

\[ N_\Phi : A = \Phi \Rightarrow B = \Delta \]

1. Assume \( x_0, \ldots x_{n-1} \in \Phi \cap A \).
2. Pick \( x_n \).
3. While 1. holds, enumerate \( \langle x_n, B \upharpoonright x_n \rangle \in \Theta \).
4. If ever \( x_n \in \Phi \), stop defining \( x_n \) axioms, and enumerate \( D_n = \bigcap \{ D : \langle x_n, D \rangle \in \Theta \} \) into \( \Delta \).
N-Requirement - Bounded

\[ N_\Phi : A = \Phi \Rightarrow B = \Delta \]

- If infinitely many \( x_i \) are defined and each \( x_i \in A \), then since \( D_0 \subseteq D_1 \subseteq \cdots \subseteq B \), we can conclude \( \Delta = B \).
- If \( x_i \notin A \) then \( x_j \notin A \) for all \( j > i \).
- (Conditional Dumping) While \( x_i \in A \), for all \( y \in S(s_i) \), enumerate \( \langle y, B \upharpoonright y \rangle \) into \( \Theta \).
**S-Requirement - Standard**

\[ S_{\Psi, \Omega_0, \Omega_1} : A = \Psi^{\Omega_0^A \oplus \Omega_1^A} \Rightarrow A = \Gamma_0^{\Omega_0^A} \text{ or } A = \Gamma_1^{\Omega_1^A} \]

1. Pick \( x \) and set \( x \in A \).
2. Wait for \( x \in \Psi^{\Omega_0^A \oplus \Omega_1^A} \) via \( \langle x, F_0 \oplus F_1 \rangle \in \Psi \).
3. Enumerate \( \langle x, F_i \rangle \) into \( \Gamma_i \), \( x \) into \( S(\gamma_0) \) and return to Step 1.
   - Hopefully \( x \in A \) iff \( F_0 \subseteq \Omega_0^A \).
   - If true for co-finitely many \( x \), then \( A = \Gamma_0^{\Omega_0^A} \).
   - Strategies below \( \gamma_0 \) can only use \( x \) which have this property.
S-Requirement - Standard

\[ S_{\Psi, \Omega_0, \Omega_1} : A = \Psi \Omega_0^A \oplus \Omega_1^A \Rightarrow A = \Gamma_0^\Omega_0^A \text{ or } A = \Gamma_1^\Omega_1^A \]

4. If ever see \( x \notin A \) and \( F_0 \subseteq \Omega_0^A \) (hence \( x \in \Gamma_0^\Omega_0^A - A \)), dump \( S(\gamma_0) - \{x\} \) into \( A \), enumerate \( x \) into \( S(\gamma_1) \).

- For this \( x \), \( F_0 \subseteq \Omega_0^{A - \{x\}} \), killing \( \Gamma_0 \).
- Hopefully \( x \in A \) iff \( F_1 \subseteq \Omega_1^A \).
- If true for infinitely many \( x \), then \( A = \Gamma_1^\Omega_1^A \).
- Strategies below \( \gamma_1 \) can only use \( x \) which have this property.
S-Requirement - Standard

\[ S_{\Psi,\Omega_0,\Omega_1} : A = \Psi \Omega_0^A \oplus \Omega_1^A \Rightarrow A = \Gamma_0^\Omega_0^A \text{ or } A = \Gamma_1^\Omega_1^A \]

5. If ever see \( x \notin A \) and \( F_1 \subseteq \Omega_1^A \) (hence \( x \in \Gamma_1^\Omega_1^A - A \)), dump \( S(\gamma_1) \cup S(\gamma_0) - x \) into \( A \), and set \( x \notin A \).
   - For this \( x \), \( F_1 \subseteq \Omega_1^{A-\{x\}} \), killing \( \Gamma_1 \).
   - Not a problem since now \( x \in \Psi \Omega_0^{A-\{x\}} \oplus \Omega_1^{A-\{x\}} \).
As with the $N$-strategy, expand the outcome $s$ to $s_0$, $s_1$, $\ldots$.

If we choose $x_0$, $x_1$, $\ldots$ as possible diagonalization witness, and for all $i$, $x_i \in \Psi \Omega_0^A \oplus \Omega_1^A \cap A$, then $B = \Lambda$.

(Conditional Dumping) While $x_i \in A$, for all $y \in S(s_i)$, enumerate $\langle y, B \upharpoonright y \rangle$ into $\Theta$. 

\[ S_{\Psi, \Omega_0, \Omega_1} : A = \Psi \Omega_0^A \oplus \Omega_1^A \Rightarrow A = \Gamma_0^A \text{ or } A = \Gamma_1^A \text{ or } B = \Lambda \]
S-Requirement - Potential Problem

\[ S_{\Psi,\Omega_0,\Omega_1} : A = \Psi \Omega_0^A \oplus \Omega_1^A \Rightarrow A = \Gamma^\Omega_0^A \text{ or } A = \Gamma^\Omega_1^A \text{ or } B = \Lambda \]

- \( \Gamma_0 \) assumes all elements of \( S(\gamma_1) \) have settled down.
- Possibly there is \( x \in S(\gamma_1) \) and \( y \in S(\gamma_0) \) such that
  - while \( x \in A \), \( y \in A \) iff \( y \in \Gamma^\Omega_0^A \), but
  - while \( x \notin A \), \( y \notin \Gamma^\Omega_0^A \).
  - \( \lim_s A(x) \) does not exist, i.e. \( A \) is \( \Sigma_2^0 \).
Assume $S(\gamma_1) = \{x\}$.

• Construct two enumeration operators: $\Gamma_{0,0}$ and $\Gamma_{0,1}$.
• $\Gamma_{0,0}$ assumes $x \notin A$ and $\Gamma_{0,1}$ assumes $x \in A$.
• Accounts for $\Sigma^0_2$ nature of $A$.
• In general, if $|S(\gamma_1)| = n$, then we construct $2^n$ enumeration operators.
Quasi-Lexicographical Ordering

Definition
Define the quasi-lexicographical ordering $\prec_b$ on $2^{<\omega}$ by $\sigma \prec_b \tau$ if

1. $|\sigma| < |\tau|$, or
2. $|\sigma| = |\tau|$ and there is a $k < |\sigma|$ such that $\sigma(k) = 0$, $\tau(k) = 1$, and for all $i < k$, $\sigma(i) = \tau(i)$.

$0 \prec_b 1 \prec_b$
$00 \prec_b 01 \prec_b 10 \prec_b 11 \prec_b$
$000 \prec_b 001 \prec_b 010 \prec_b 011 \prec_b 100 \prec_b 101 \prec_b \cdots$
The strategy proceeds basically the same as before.
We have added infinitely many outcomes to the tree to account for all possible states of elements to the left of the current outcome.
When an element is moved left through the streams, it must make sure that the assumptions of the new stream are consistent with the assumptions of the previous stream.
S-Requirement - Bounded

\[ S_{\Psi, \Omega_0, \Omega_1} : A = \Psi \Omega_0^A \oplus \Omega_1^A \Rightarrow A = \Gamma_0^\Omega_0^A \text{ or } A = \Gamma_1^\Omega_1^A \text{ or } B = \Lambda \]

- Other technical concerns not covered here include:
  - Other uses of conditional dumping.
  - Local approximations to \( B \).
  - etc.
The End!

Kent, Sorbi “Bounding Nonsplitting Enumeration Degrees,”