

Complexity of the isomorphism relation for countable models of ω -stable theories

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Motivation: Intuitive notion of complexity of a theory

Example: Vector spaces over a fixed field versus arbitrary graphs

We compare two notions of complexity for a theory:

- Shelah's Classification Theory (stability hierarchy, NDOP, depth)
- Borel reducibility (H. Friedman, L. Stanley): a notion coming from descriptive set theory

Borel reducibility

- A *Borel space* is a set X equipped with a σ -algebra \mathcal{B}
- $f : (X, \mathcal{B}) \rightarrow (X', \mathcal{B}')$ is a *Borel map* if for all $A \in \mathcal{B}'$, $f^{-1}[A] \in \mathcal{B}$.
- To a topological space (X, \mathcal{T}) we associate a Borel space (X, \mathcal{B}) : \mathcal{B} is the smallest σ -algebra containing \mathcal{T} .
- (X, \mathcal{T}) is *polish* iff it is completely metrisable and separable. The associated Borel spaces are called standard Borel spaces.
- Examples: finite or countable spaces with discrete topology, \mathbb{R} , $\mathcal{C} = 2^\omega$, $\mathcal{N} = \omega^\omega$

Proposition 1. *Let (X, \mathcal{B}) , (X', \mathcal{B}') be standard. Then*

- (1) *X is finite, countable or of cardinality 2^{\aleph_0}*
- (2) *if $|X| = |Y|$ then X and Y are Borel isomorphic*
- (3) *The category of standard Borel spaces is closed under countable products*

For X, Y standard and $E \subset X \times X$ and $F \subset Y \times Y$ equivalence relations, we define $E \leq_B F$ iff there is $f : X \rightarrow Y$ Borel s.t. for all $a, b \in X$, $a E b$ iff $f(a) F f(b)$.

Connection to model theory

Let L be countable. X_L , the set of L -structures living on ω is standard Borel. Example: If $L = \{c, R, f\}$ then

$$X_L = \omega \times 2^{\omega \times \dots \times \omega} \times \omega^{\omega \times \dots \times \omega}$$

If $\sigma \in L_{\omega_1\omega}$ then $\text{Mod}(\sigma) \subset X_L$ is *invariant Borel* and thus standard (the converse is also true).

Notation: $\cong_\sigma = \cong_L \upharpoonright \text{Mod}(\sigma)^2$, $=_n$ ($n \leq \omega$), $=_{\mathbb{R}}$

- E is *Borel* if it is a Borel subset of $X \times X$
- E is *smooth* if $E \leq_{B=\mathbb{R}}$
- E is *countable* if all E -equivalence classes are
- E is *essentially countable* if $E \leq_B F$ for some countable F

ω -stability

A complete first-order theory T is ω -stable if $|S_1(A)| = |A|$ for all infinite A .

We then have

- Morley rank of definable sets
- non-forking extensions of types (Morley rank does not decrease)
- a notion of independence: $A \downarrow_C B$ (i.e. $t(A/BC) \supset t(A/C)$ is non-forking)
- strongly regular types (for which dimension is well-defined)
- prime models over any set

And we can define the property NDOP and depth for T

We have to adapt these notions to the case of countable models:

- ENI types (those which can have *finite* dimension)
- ENI-NDOP
- eni-depth

First results: the extremes

Let T a complete, ω -stable first order theory with infinite models.

Theorem 2. (*Laskowski-Shelah*)

(1) If T has ENI-DOP, then $\cong_T \approx_B \cong_{\text{graphs}}$

(2) If T has ENI-NDOP and is eni-deep, then $\cong_T \approx_B \cong_{\text{graphs}}$

Proposition 3. If T has $\kappa \leq \aleph_0$ countable models, then $\cong_T \approx_B \cong_n$

Theorem 4. If T has ENI-NDOP and eni-depth 1, then \cong_T is smooth.

A cofinal sequence of increasing complexity

Theorem 5. *There exists a sequence $(T_\alpha)_{1 \leq \alpha < \omega_1}$ of ω -stable theories having ENI-NDOP with the following properties : for all $\alpha < \omega_1$,*

- T_α has depth and eni-depth α
- \cong_{T_α} est Borel
- for all β with $\alpha < \beta < \omega_1$, $\cong_{T_\alpha} <_B \cong_{T_\beta}$ (i.e. $\cong_{T_\alpha} \leq_B \cong_{T_\beta}$ and $\cong_{T_\beta} \not\leq_B \cong_{T_\alpha}$)

Moreover, $(T_\alpha)_{1 \leq \alpha < \omega_1}$ is Borel-cofinal in the sense that for each countable L and $\sigma \in L_{\omega_1 \omega}$, if \cong_σ is Borel, then $\cong_\sigma \leq_B \cong_{T_\alpha}$ for some $\alpha < \omega_1$.

A non-Borel theory of depth 2

Theorem 6. *There is a complete first order ω -stable theory having ENI-NDOP, of (eni-) depth 2 whose isomorphism relation is not Borel.*

Let $L = \{\pi_i^j, S_i\}_{i < \omega, j \leq i+1}$ be the language with sorts U, V_i, C_i ($i < \omega$) and

$$\pi_i^j : V_i \rightarrow C_j \text{ for } j \leq i$$

$$\pi_i^{i+1} : V_i \rightarrow U$$

$$S_i : V_i \rightarrow V_i$$

and let T be the L -theory that states

- (1) $|U| = \infty$ and $|C_i| = 2$ for all $i < \omega$
- (2) $\pi_i : V_i \rightarrow C_0 \times C_1 \times \cdots \times C_i \times U$ is onto, where $\pi_i(x) = (\pi_i^0(x), \pi_i^1(x), \dots, \pi_i^{i+1}(x))$
- (3) for all $i < \omega$, S_i is a successor function on V_i
- (4) $\pi_i \circ S_i = \pi_i$

Description of the isomorphism relation

We fix for all $i < \omega$ $C_i = \{a_i^0, a_i^1\}$.

Essentially, the 1-types are the following:

- $r(x) = \{U(x)\}$
- for $s \in 2^{<\omega} \setminus \{\emptyset\}$ and $b \in U$:
 $p_s^b(x) = \{\pi_{|s|}(x) = (a_{s(0)}, a_{s(1)}, \dots, a_{|s|-1}, b)\}$

Let $\mathcal{A} = \{f \mid f : 2^{<\omega} \setminus \{\emptyset\} \rightarrow \omega + 1\}$ and define an action of \mathcal{C} on \mathcal{A} by

$$\text{for } \sigma \in \mathcal{C}, \delta \in \mathcal{A}, \quad \sigma\delta(s) = \delta(s + \sigma \upharpoonright |s|)$$

For $M \models T$ countable and $b \in U(M)$ we define $\delta_b^M \in \mathcal{A}$ by $\delta_b^M(s) = \dim_M(p_s^b) - 1$. Then we can prove

Proposition 7. *Countable $M, N \models T$ are isomorphic if and only if there exists $\sigma \in \mathcal{C}$ and bijective $f : U(M) \rightarrow U(N)$ such that for all $b \in U(M)$, $\delta_b^M = \sigma\delta_{f(b)}^N$.*

So, roughly speaking, isomorphism types are countable sets of countably coloured complete binary trees up to “simultaneous flips” of levels.

An idea of the proof of non-Borelness (part I)

(1) We show that $\text{SH}=\infty$ (“Scott Height”), which is equivalent to non-Borelness. Goal : define for all $\alpha < \omega_1$ models M, N such that $M \not\cong N$ and $M \equiv_\alpha N$. Recall that

- $(M, \bar{a}) \equiv_0 (N, \bar{b})$ iff \bar{a} and \bar{b} have same quantifier-free type
- $(M, \bar{a}) \equiv_\lambda (N, \bar{b})$ iff $\forall \beta < \lambda (M, \bar{a}) \equiv_\beta (N, \bar{b})$
- $(M, \bar{a}) \equiv_{\alpha+1} (N, \bar{b})$ iff $\forall x \in M \exists y \in N (M, \bar{a} \frown x) \equiv_\alpha (N, \bar{b} \frown y)$ and vice versa

(2) What pairs of models $((M, \bar{u}), (N, \bar{v}))$ we consider :

- Fix $(X_i)_{i < \omega}$ independent
- Define trees δ_i with stabiliser $\mathcal{C}^{X_i} = \{\sigma \in \mathcal{C} \mid \forall n \notin X_i \sigma(n) = 0\}$
- M, N realise only orbits $o_i = \mathcal{C}\delta_i$
- For all $i < \omega$, $M^{o_i} \cong N^{o_i}$
- \bar{u} and \bar{v} have same type
- $\{\sigma \in \mathcal{C} \mid t(\bar{u}/\text{acl}(\emptyset)), t(\bar{v}/\sigma \text{acl}(\emptyset)) \text{ are conjugate}\} \neq \emptyset$

(3) Define configurations : $c = (D, X, d)$ with

- $D : \omega \rightarrow \mathcal{P}(\mathcal{C})$
- $D : \omega \rightarrow \mathcal{P}(\omega)$
- $d \in \mathcal{P}(\omega)$

An idea of the proof of non-Borelness (part II)

- (4) Assign $((M, \bar{u}), (N, \bar{v})) \mapsto c^{((M, \bar{u}), (N, \bar{v}))} = (D, X, d) :$
- $D(i) = \{\sigma \mid \sigma \text{ allows } M^{o_i} \cong N^{o_i}\}$
 - $X(i) = X_i$
 - $d = \{\sigma \in \mathcal{C} \mid t(\bar{u}/\text{acl}(\emptyset)), t(\bar{v}/\sigma\text{acl}(\emptyset)) \text{ are conjugate}\}$
- (5) Define *thin* configurations (for $\not\cong$): $\bigcap_{i \in I} D(i) \cap d = \emptyset$ for all $I \subset \omega$ infinite
- Define α -*rich* configurations (for $\equiv_{\alpha+\omega}$)
e.g. 0-rich : $\forall i < \omega \ D(i) \cap d \neq \emptyset$
- (6) Construction of α -rich configurations for all $\alpha < \omega_1$ and show they are thin.

Some open questions

- Is the non-Borel depth 2 theory as complicated as graphs?
- Are there non-smooth first-order theories which are essentially countable?
- Are there “simple” eni-depth α theories, e.g. smooth ones for $\alpha > 2$?
- What can be said about the superstable case?