Complexity of the isomorphism relation for countable models of $\omega$-stable theories

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July 2, 2007

Motivation: Intuitive notion of complexity of a theory

Example: Vector spaces over a fixed field versus arbitrary graphs

We compare two notions of complexity for a theory:

- Shelah’s Classification Theory (stability hierarchy, NDOP, depth)
- Borel reducibility (H. Friedman, L. Stanley): a notion coming from descriptive set theory
Borel reducibility

- A *Borel space* is a set $X$ equipped with a $\sigma$-algebra $\mathcal{B}$
- $f : (X, \mathcal{B}) \rightarrow (X', \mathcal{B}')$ is a *Borel map* if for all $A \in \mathcal{B}'$, $f^{-1}[A] \in \mathcal{B}$.
- To a topological space $(X, \mathcal{T})$ we associate a Borel space $(X, \mathcal{B})$: $\mathcal{B}$ is the smallest $\sigma$-algebra containing $\mathcal{T}$.
- $(X, \mathcal{T})$ is *polish* iff it is completely metrisable and separable. The associated Borel spaces are called standard Borel spaces.
- Examples: finite or countable spaces with discrete topology, $\mathbb{R}$, $\mathcal{C} = 2^\omega$, $\mathcal{N} = \omega^\omega$

**Proposition 1.** Let $(X, \mathcal{B})$, $(X', \mathcal{B}')$ be standard. Then

1. $X$ is finite, countable or of cardinality $2^{\aleph_0}$
2. if $|X| = |Y|$ then $X$ and $Y$ are Borel isomorphic
3. The category of standard Borel spaces is closed under countable products

For $X, Y$ standard and $E \subset X \times X$ and $F \subset Y \times Y$ equivalence relations, we define $E \leq_B F$ iff there is $f : X \rightarrow Y$ Borel s.t. for all $a, b \in X$, $a \ E \ b$ iff $f(a) \ F \ f(b)$.  

Connection to model theory

Let $L$ be countable. $X_L$, the set of $L$-structures living on $\omega$ is standard Borel. Example: If $L = \{c, R, f\}$ then

$$X_L = \omega \times 2^{\omega \times \cdots \times \omega} \times \omega^{\omega \times \cdots \times \omega}$$

If $\sigma \in L_{\omega_1 \omega}$ then $\text{Mod}(\sigma) \subset X_L$ is invariant Borel and thus standard (the converse is also true).

Notation: $\cong_{\sigma} = \cong_L \upharpoonright \text{Mod}(\sigma)^2$, $=_{n} (n \leq \omega)$, $=_{\mathbb{R}}$

- $E$ is Borel if it is a Borel subset of $X \times X$
- $E$ is smooth if $E \leq_{B=\mathbb{R}}$
- $E$ is countable if all $E$-equivalence classes are
- $E$ is essentially countable if $E \leq B F$ for some countable $F$
ω-stability

A complete first-order theory $T$ is ω-stable if $|S_1(A)| = |A|$ for all infinite $A$.

We then have

- Morley rank of definable sets
- non-forking extensions of types (Morley rank does not decrease)
- a notion of independence: $A \downarrow B$ (i.e. $t(A/BC) \supset t(A/C)$ is non-forking)
- strongly regular types (for which dimension is well-defined)
- prime models over any set

And we can define the property NDOP and depth for $T$

We have to adapt these notions to the case of countable models:

- ENI types (those which can have finite dimension)
- ENI-NDOP
- eni-depth
First results: the extremes

Let $T$ a complete, $\omega$-stable first order theory with infinite models.

**Theorem 2.** *(Laskowski-Shelah)*

1. If $T$ has ENI-DOP, then $\cong_T \cong_B \cong_{\text{graphs}}$
2. If $T$ has ENI-NDOP and is eni-deep, then $\cong_T \cong_B \cong_{\text{graphs}}$

**Proposition 3.** If $T$ has $\kappa \leq \aleph_0$ countable models, then $\cong_T \cong_B \cong_n$

**Theorem 4.** If $T$ has ENI-NDOP and eni-depth 1, then $\cong_T$ is smooth.
Theorem 5. There exists a sequence \((T_\alpha)_{1 \leq \alpha < \omega_1}\) of \(\omega\)-stable theories having ENI-NDOP with the following properties: for all \(\alpha < \omega_1\),

- \(T_\alpha\) has depth and eni-depth \(\alpha\)
- \(\cong_{T_\alpha}\) est Borel
- for all \(\beta\) with \(\alpha < \beta < \omega_1\), \(\cong_{T_\alpha} <_B \cong_{T_\beta}\) (i.e. \(\cong_{T_\alpha} \leq_B \cong_{T_\beta}\) and \(\cong_{T_\beta} \not<_B \cong_{T_\alpha}\))

Moreover, \((T_\alpha)_{1 \leq \alpha < \omega_1}\) is Borel-cofinal in the sense that for each countable \(L\) and \(\sigma \in L_{\omega_1^\omega}\), if \(\cong_\sigma\) is Borel, then \(\cong_\sigma \leq_B \cong_{T_\alpha}\) for some \(\alpha < \omega_1\).
A non-Borel theory of depth 2

**Theorem 6.** There is a complete first order ω-stable theory having ENI-NDOP, of (eni-) depth 2 whose isomorphism relation is not Borel.

Let \( L = \{\pi^j_i, S_i\}_{i<\omega, j \leq i+1} \) be the language with sorts \( U, V_i, C_i \) \((i < \omega)\) and

\[
\pi^j_i : V_i \to C_j \text{ for } j \leq i \\
\pi^{i+1}_i : V_i \to U \\
S_i : V_i \to V_i
\]

and let \( T \) be the \( L \)-theory that states

1. \( |U| = \infty \) and \( |C_i| = 2 \) for all \( i < \omega \)
2. \( \pi_i : V_i \to C_0 \times C_1 \times \cdots \times C_i \times U \) is onto, where \( \pi_i(x) = (\pi^0_i(x), \pi^1_i(x), \ldots, \pi^{i+1}_i(x)) \)
3. for all \( i < \omega \), \( S_i \) is a successor function on \( V_i \)
4. \( \pi_i \circ S_i = \pi_i \)
Description of the isomorphism relation

We fix for all $i < \omega$ $C_i = \{a_i^0, a_i^1\}$.

Essentially, the 1-types are the following:

- $r(x) = \{U(x)\}$
- for $s \in 2^{<\omega} \setminus \{\emptyset\}$ and $b \in U$:
  $$p_s^b(x) = \{\pi_{|s|}(x) = (a_{s(0)}, a_{s(1)}, \ldots, a_{|s|−1}, b)\}$$

Let $A = \{f \mid f : 2^{<\omega} \setminus \{\emptyset\} \to \omega + 1\}$ and define an action of $C$ on $A$ by

$$\text{for } \sigma \in C, \delta \in A, \quad \sigma \delta(s) = \delta(s + \sigma \upharpoonright |s|)$$

For $M \models T$ countable and $b \in U(M)$ we define $\delta_b^M \in A$ by $\delta_b^M(s) = \dim_M(p_s^b) - 1$. Then we can prove

**Proposition 7.** Countable $M, N \models T$ are isomorphic if and only if there exists $\sigma \in C$ and bijective $f : U(M) \to U(N)$ such that for all $b \in U(M)$, $\delta_b^M = \sigma \delta_{f(b)}^N$.

So, roughly speaking, isomorphism types are countable sets of countably coloured complete binary trees up to “simultaneous flips” of levels.
An idea of the proof of non-Borelness (part I)

(1) We show that \( SH=\infty \) ("Scott Height"), which is equivalent to non-Borelness. Goal: define for all \( \alpha < \omega_1 \) models \( M, N \) such that \( M \not\equiv N \) and \( M \equiv_{\alpha} N \). Recall that

- \((M, \bar{a}) \equiv_0 (N, \bar{b})\) iff \( \bar{a} \) and \( \bar{b} \) have the same quantifier-free type
- \((M, \bar{a}) \equiv_\lambda (N, \bar{b})\) iff \( \forall \beta < \lambda (M, \bar{a}) \equiv_\beta (N, \bar{b}) \)
- \((M, \bar{a}) \equiv_{\alpha+1} (N, \bar{b})\) iff \( \forall x \in M \exists y \in N (M, \bar{a} \prec x) \equiv_\alpha (N, \bar{b} \prec y) \) and vice versa

(2) What pairs of models \(((M, \bar{u}), (N, \bar{v}))\) we consider:

- Fix \((X_i)_{i<\omega}\) independent
- Define trees \( \delta_i \) with stabiliser
  \[ C_{X_i} = \{ \sigma \in C | \forall n \notin X_i \sigma(n) = 0 \} \]
- \( M, N \) realise only orbits \( o_i = C_{\delta_i} \)
- For all \( i < \omega, M^{o_i} \cong N^{o_i} \)
- \( \bar{u} \) and \( \bar{v} \) have the same type
- \( \{ \sigma \in C | t(\bar{u}/acl(\emptyset)), t(\bar{v}/\sigma acl(\emptyset)) \text{ are conjugate} \} \neq \emptyset \)

(3) Define configurations: \( c = (D, X, d) \) with

- \( D : \omega \to \mathcal{P}(\mathcal{C}) \)
- \( D : \omega \to \mathcal{P}(\omega) \)
- \( d \in \mathcal{P}(\omega) \)
An idea of the proof of non-Borelness (part II)

(4) Assign \(((M, \bar{u}), (N, \bar{v})) \mapsto c^{((M, \bar{u}), (N, \bar{v}))} = (D, X, d)\):

- \(D(i) = \{\sigma|\sigma\text{ allows } M^{o_i} \cong N^{o_i}\}\)
- \(X(i) = X_i\)
- \(d = \{\sigma \in C|t(\bar{u}/\text{acl}(\emptyset)), t(\bar{v}/\sigma\text{acl}(\emptyset))\text{ are conjugate}\}\)

(5) Define thin configurations (for \(\not\equiv\)): \(\bigcap_{i \in I} D(i) \cap d = \emptyset\) for all \(I \subset \omega\) infinite

Define \(\alpha\)-rich configurations (for \(\equiv_{\alpha+\omega}\))
e.g. 0-rich : \(\forall i < \omega \; D(i) \cap d \neq \emptyset\)

(6) Construction of \(\alpha\)-rich configurations for all \(\alpha < \omega_1\) and show they are thin.
Some open questions

- Is the non-Borel depth 2 theory as complicated as graphs?
- Are there non-smooth first-order theories which are essentially countable?
- Are there “simple” eni-depth $\alpha$ theories, e.g. smooth ones for $\alpha > 2$?
- What can be said about the superstable case?