Large cardinals and forcing-absoluteness

July 15, 2007

Theorem 1 (Shoenfield). *If ϕ is a unary Σ^1_2 formula, $x \subset \omega$ and M is a model of ZFC containing ω_1^V, then $M \models \phi(x)$ if and only if $\phi(x)$.*

Theorem 2 (Levy-Shoenfield). *For any Σ_1 sentence σ, σ holds in L if and only if it holds in V.*

Example: "$x \in L$" is Σ^1_2.

"There is a nonconstructible real" is Σ^1_3 and not forcing-absolute in ZFC.

3 Definition. A cardinal κ is *measurable* if there is a κ-complete ultrafilter on κ.

Theorem 4 (Solovay)**.** *If κ is a measurable cardinal, ϕ is a unary Σ^1_3 formula, $x \subset \omega$ and M is a forcing extension of V by a partial order of cardinality less than κ, then $M \models \phi(x)$ if and only if $\phi(x)$.*

5 Definition. A regular cardinal δ is *Woodin* if for every function

$$f\colon \delta \to \delta$$

there is a elementary embedding

$$j\colon V \to M$$

such that κ is closed under f and

$$V_{j(f)(\kappa)} \subset M,$$

where κ is the critical point of j.

Theorem 6 (Woodin). *If δ is a Woodin cardinal, there are partial orders*

$$(\mathbb{Q}_{<\delta}; Coll(\omega_1, {<}\delta) * (\mathcal{P}(\omega_1)/NS_{\omega_1}))$$

which add an elementary embedding $j\colon V \to M$ with critical point ω_1^V such that M is countably closed in the forcing extension.

Theorem 7 (Martin-Steel)**.** *Let n be an integer. Suppose that*

$$\delta_1 < \ldots < \delta_n$$

are Woodin cardinals, and $\kappa > \delta_n$ is measurable. Let ϕ be a unary Σ^1_{n+3} formula, fix $x \subset \omega$ and suppose that M is a forcing extension of V by a partial order of cardinality less than δ_1. Then $M \models \phi(x)$ if and only if $\phi(x)$.

8 Definition. A *tower of measures* is a sequence $\langle \mu_i : i < \omega \rangle$ such that each μ_i is an ultrafilter on $[Z]^i$, for some fixed underlying set Z.

9 Definition. A tower of measures

$$\langle \mu_i : i < \omega \rangle$$

is *countably complete* if for every sequence

$$\langle A_i : i < \omega \rangle$$

such that each $A_i \in \mu_i$, there exists an $f : \omega \to Z$ such that for all i,

$$f | i \in A_i$$

(where Z is the underlying set).

10 Definition. Given a cardinal κ a set $A \subset \omega^\omega$ is κ-*homogeneously Suslin* if there is a collection of κ-complete ultrafilters

$$\{\mu_s : s \in \omega^{<\omega}\}$$

such that A is the set of $f \in \omega^\omega$ such that

$$\langle \mu_{f|i} : i < \omega \rangle$$

is a countably complete tower.

11 Definition. Given a cardinal κ, a set $A \subset \omega^\omega$ is κ-*weakly homogeneously Suslin* if there is a κ-homogeneously Suslin set $B \subset \omega^\omega \times \omega^\omega$ such that

$$A = \{x \mid \exists y (x, y) \in B\}.$$

Theorem 12 (Martin). *If κ is a measurable cardinal, Π^1_1 sets are κ-homogeneously Suslin.*

Theorem 13 (Martin-Steel). *If δ is a Woodin cardinal and $A \subset \omega^\omega$ is δ^+-weakly homogeneously Suslin, then $\omega^\omega \backslash A$ is $<\delta$-homogeneously Suslin*

14 Definition. If $S \subset (\omega \times Z)^{<\omega}$ is a tree (for some set Z),

$$p[S] = \{f \in \omega^\omega \mid \exists g \in Z^\omega \forall i \in \omega (f|i, g|i) \in S\}$$

15 Definition. Given a cardinal κ, a set $A \subset \omega^\omega$ is κ-*universally Baire* if there are trees S,T such that

$$p[S] = A,$$

$$p[T] = \omega^\omega \setminus A$$

and

$$p[S] = \omega^\omega \setminus p[T]$$

in all forcing extensions by partial orders of cardinality less than or equal to κ.

Theorem 16 (Martin-Solovay). *If $A \subset \omega^\omega$ is κ-weakly homogeneously Suslin, then A is $<\kappa$-universally Baire.*

Theorem 17 (Woodin). *If δ is a Woodin cardinal, then every δ-universally Baire set of reals is $<\delta$-weakly homogeneously Suslin.*

Theorem 18 (Woodin)**.** *Suppose that δ is a Woodin cardinal. Fix $A \subset \omega^\omega$. Suppose that for every $r \in \omega^\omega$ which is generic over V for a partial order in V_δ, either*

- *for every $\mathbb{Q}_{<\delta}$-embedding $j\colon V \to M$, if $r \in M$ then $r \in j(A)$; or*
- *for every $\mathbb{Q}_{<\delta}$-embedding $j\colon V \to M$, if $r \in M$ then $r \notin j(A)$.*

Then A is $<\delta$-universally Baire.

Theorem 19 (Woodin). *If δ is a limit of Woodin cardinals, and there is a measurable cardinal above δ, then the theory of $L(\mathbb{R})$ cannot be changed by forcing with partial orders of cardinality less than δ.*

Theorem 20 (Woodin). *If A is universally Baire and there exist proper class many Woodin cardinals, then the theory of $L(A, \mathbb{R})$ cannot be changed by set forcing.*

21 Definition. A cardinal κ is *supercompact* if for every λ there is an elementary embedding $j\colon V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and M is closed under λ-sequences.

Theorem 22 (Woodin)**.** *Suppose that κ is a supercompact cardinal and that there exist proper class many Woodin cardinals, and let M be a forcing extension of V in which $\mathcal{P}(\kappa)^V$ is countable. Then the theory of $L(\Gamma_{uB})$ cannot be changed by set forcing over M.*

The logic $L(Q)$ is the extension of first order logic with the quantifier $\exists^{\aleph_1}$ with the intended meaning "there exists uncountably many."

A forcing-absoluteness version of Keisler's L(Q)-completeness theorem: the truth value of statements of the form "there exists a correct L(Q) model of T", for T a theory in $L(Q)$, cannot be changed by forcing.

An alternate proof:

Given a countably complete ideal I on ω_1, forcing with the Boolean algebra

$$\mathcal{P}(\omega_1)/I$$

adds a V-ultrafilter on ω_1, and an ultrapower embedding with critical point ω_1^V into some (possibly illfounded) class model.

If the ultrapower is always wellfounded, we say that I is *precipitous*.

Given a countable model M satisfying enough of ZFC (ZFC*) to carry out the ultrapower construction, and an ideal I in M on ω_1^M, we can repeat this process ω_1 times, taking direct limits at limit stages.

$$M \to M_1 \to M_2 \to \ldots \to M_\omega \to M_{\omega+1} \to \ldots \to M_{\omega_1}$$

This is called an *iteration* of (M, I).

Given $\alpha < \omega_1$ and

$$j_{\alpha,\alpha+1} \colon M_\alpha \to M_{\alpha+1},$$

for each $x \in M_\alpha$, $j_{\alpha,\alpha+1}(x) = j_{\alpha,\alpha+1}[x]$ if and only if

$$M_\alpha \models \text{“}x \text{ is countable.”}$$

It follows that M_{ω_1} is correct about uncountability.

So, if it consistent with ZFC* that there is a model of T which is correct about uncountability, then there is such a model.

A separable topological space X is *countable dense homogeneous* if for any any countable dense subsets D, D' of X there is a homeomorphism of X taking D to D'.

Theorem 23 (Farah-Hrušák-Ranero)**.** *There is a countable dense homogeneous set of reals of cardinality* $\aleph_1$*.*

Proof strategy: allow predicates for Borel sets in Keisler's theorem.

In ZFC, one cannot add predicates for analytic sets.

If ω_1 is not strongly inaccessible in L, then there is a real x such that $L[x]$ has uncountably many reals. For any real x,

"there are uncountably many reals in $L[x]$"

is an L(Q) sentence with an analytic set as a predicate which is forced to be false by collapsing ω_1.

$\mathbb{P}_{max}$: a homogeneous partial order in $L(\mathbb{R})$. Conditions are (roughly) iterable pairs (M, I). the order is (roughly) embeddability by iterations.

Theorem 24 (Woodin)**.** *If δ is a limit of Woodin cardinals and there is a measurable cardinal above δ, then every Π_2 sentence for*

$$\langle H(\omega_2); NS_{\omega_1}, A : A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})\rangle$$

which can be forced by a partial order of cardinality less than δ holds in the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$.

It follows that the truth values of Σ_1 sentences for $H(\omega_2)$ cannot be changed by forcing.

A simpler argument gives forcing absoluteness for Σ_1 sentences.

A pair (M, I) is A-*iterable* if $A \cap M \in M$ and

$$j(A \cap M) = A \cap j(\mathbb{R} \cap M)$$

for all iterations j of (M, I).

For A the complete Π^1_1 set, this just means that all iterates are wellfounded (in which case we say that (M, I) is *iterable*).

Suppose that

$$\delta < \kappa$$

are Woodin cardinals, and that

$$A \subset \omega^\omega$$

and

$$\omega^\omega \setminus A$$

are κ-universally Baire. Fix

$$X \prec V_\kappa$$

and let M be the transitive collapse of X Then if M^* is any forcing extension of M and I is any precipitous ideal on $\omega_1^{M^*}$ in M^*, then

$$(M^*, I)$$

is A-iterable.

If (M, NS_{ω_1}) is iterable, one can also iterate to make the final model correct about stationarity.

Corollary 25. *If $\delta < \kappa$ are a Woodin cardinals and*

$$A \subset \omega^\omega$$

and

$$\omega^\omega \setminus A$$

are δ^+-weakly homogeneously Suslin, the truth values of Σ_1 sentences with predicates for A and NS_{ω_1} cannot be changed by forcing with partial orders of cardinality less than δ.

Theorem 26 (Todorcevic). *If $B \subset \omega_1$ and*

$$\omega_1^{L[B]} = \omega_1,$$

then there is in $L[B]$ a partition of ω_1 into infinitely many pieces all stationary in V.

Theorem 27 (Larson). *For any $B \subset \omega_1$ such that $\omega_1^{L[B]} = \omega_1$, there is a partial order forcing that there is no partition in $L[B]$ of ω_1 into uncountably many pieces all stationary in V.*

Corollary 28. *If there is a measurable cardinal above a Woodin cardinal, there is a $B \subset \omega_1$ such that $\omega_1^{L[B]} = \omega_1$ and such that there is no partition in $L[B]$ of ω_1 into uncountably many pieces all stationary in V.*

Theorem 29 (Steel). *Suppose that there exist infinitely many Woodin cardinals below a measurable cardinal, and let $T \subset \mathbb{R}^{<\omega_1}$ be a tree in $L(\mathbb{R})$. Then exactly one of the following holds.*

- *T has an uncountable branch in every model of ZFC containing $L(\mathbb{R})$;*

- *there is a function $f \in L(\mathbb{R})$ which assigns to each $p \in T^+$ a wellordering of ω of length $dom(p)$.*

Farah-Ketchersid-Larson: extension to all universally Baire trees.

Key points:

- (Martin) Under AD, the cones generate an ultrafilter onthe Turing degrees;

- (Woodin) Under AD, for every set of ordinals Z, for a cone of reals x, $\omega_2^{L[Z,x]}$ is a Woodin cardinal in $\mathrm{HOD}_{\{Z\}}^{L[Z,x]}$.

Theorem 30 (Woodin). *If δ is a measurable Woodin cardinal, then every Σ_1^2 sentence forceable by a partial order in V_δ holds in all forcing extensions satisfying CH by partial orders in V_δ.*

Theorem 31 (A. Miller)**.** *The continuum hypothesis implies that there is a MAD family which is a σ-set.*

Theorem 32 (Zapletal)**.** *If there exists a measurable Woodin cardinal and CH holds, then every projective forcing which does not collapse ω_1 is proper.*

Very brief sketch of proof: Since δ is a measurable Woodin cardinal, there are a Woodin cardinal $\lambda < \delta$ and a condition $a \in \mathbb{P}_\delta$ such that a forces that $\lambda = \omega_1^{V[G]}$ and V_ζ is in the image model, where ζ is the least strongly inaccessible cardinal above λ, and that there is a "fast" club through the Woodin cardinals below λ.

Let g be V_ζ-generic for P, in M. Successively choose generic filters H_κ for each $\mathbb{Q}_{<\kappa}^{V_\zeta[g]}$ for each κ in C, extending one another, such that each real in M is in some model of the form $V_\zeta[g][H_\kappa]$.

Woodin: If (M, I) is iterable and M^* is an iterate of M by I, then $M \notin H(\omega_1)^{M^*}$.

33 Question. Is iterability needed for this fact?

Restatement of Woodin's Σ_1^2 absoluteness theorem:

Suppose that there exist proper class many Woodin cardinals, and that δ is a measurable Woodin cardinal. Let T be a theory in the expanded language with predicates for NS_{ω_1} and each universally Baire set of reals. Suppose that some partial order of cardinality less than δ forces that there exists a correct model of T containing the reals. Then for any set of reals X of cardinality $\aleph_1$, there is a correct model of T containing X.

Key point: (Steel) $\mathbb{Q}_{<\kappa}$-embeddings map the Martin-Solovay tree for the complement of a λ-weakly homogeneously Suslin set (when $\lambda > \kappa$) to itself.

34 Definition. $\diamond$ is the statement that there exists a sequence

$$\langle \sigma_\alpha : \alpha < \omega_1 \rangle$$

such that for every $A \subset \omega_1$ the set

$$\{\alpha < \omega_1 \mid A \cap \alpha = \sigma_\alpha\rangle$$

is stationary.

35 Question (Steel). Is $\diamond$ a Σ^2_2 invariant from some large cardinal assumption?

Theorem 36 (Larson-Yorioka)**.** *Assume that $\diamond$ holds. If M is a countable transitive model of ZFC***, I is an ideal on ω_1^M and (M, I) is iterable, then there is an iterate M^* of M such that for all partial orders $P \in M^*$,*

$$M^* \models P \text{ is c.c.c.}$$

if and only if P is c.c.c.

Magidor-Malitz logic is the extension of first order logic with quantifiers of the form "there exists an uncountable X such that all n-tuples from X satisfy ϕ," for each integer n.

A forcing-absoluteness statement of the Magidor-Malitz completeness theorem for this logic: if a statement of the form "there exists a correct model of T", for T a theory in Magidor-Malitz logic, can be forced, then it follows from $\diamond$.

Say that a model M is *correct about partitions* if for every set $X \in M$ consisting of finite sets of ordinals, if there is an uncountable set of ordinals whose finite subsets are all in X, there there is such a set in M.

Theorem 37 (Farah-Larson-Magidor). *Assume that $\diamond$ holds. If M is a countable transitive model of ZFC*, I is an ideal on ω_1^M and (M, I) is iterable, then there is an iterate M^* of M such that M^* is correct about partitions.*

Proof strategy: Let $\langle \sigma_\alpha : \alpha < \omega \rangle$ witness $\diamond$. For each $\alpha < \omega_1$, let Φ_α be the set of formulas with constants in M_α satisfied by every member of $\sigma_{\omega_1^{M_\alpha}}$. Whenever possible, don't let any new elements satisfy these types.

Corollary 38. *If $\diamond$ holds and there exist a proper class of Woodin cardinals and T is a theory in the expanded language with predicates for NS_{ω_1} and each universally Baire set of reals and it is possible to force the existence of a correct model of T correct about partitions, then such a model exists already.*

Examples: Each of

"there exists a Suslin tree"

and

$$Cov(Null) = \aleph_1$$

follows from the existence of a correct model satisfying it.

The continuum hypothesis does not follow from the absoluteness principle in Corollary 38.

Theorem 39 (Todorcevic)**.** *For each* $S \subset \omega_1$ *there is a partition* $K \subset [\omega_1]^2$ *such that if there an uncountable* X *with* $[X]^2 \subset K$ *then* S *contains a club, and if* S *contains a club then some proper forcing adds such an* X.

One can get correctness about partitions with or without correctness for NS_{ω_1}.

Say that a model M is correct about trees of height and cardinality ω_1 if for every such tree $T \in M$, if T has an uncountable path then it has one in M.

Another version of Woodin's Σ^2_1 absoluteness theorem, using $\diamond$:

Assume $\diamond$. Suppose that there exist proper class many Woodin cardinals, and that δ is a measurable Woodin cardinal. Let T be a theory in the expanded language with predicates for NS_{ω_1} and each universally Baire set of reals. Suppose that some partial order of cardinality less than δ forces that there exists a correct model of T which is correct about trees of height and cardinality ω_1. Then such a model exists already.

Theorem 40 (Woodin). *If δ is a measurable Woodin cardinal and there exists a Woodin cardinal above δ, then in a forcing extension there is a model satisfying all Σ^2_2 sentences ϕ such that ϕ + CH can be forced over V by a partial order in V_δ.*

The model: a δ-symmetric extension followed by a Cohen-generic subset of ω_1.

If there exist proper class many Woodin cardinals, then this model satisfies any Σ^2_2 sentence forceable by a partial order in V_δ, even allowing for parameters for any universally Baire set of reals.

However, if one allows a predicate for the nonstationary ideal on ω_1, Σ^2_2 maximality if false.

41 Definition. $\diamondsuit^*$ is the statement that there exists a sequence

$$\langle \sigma_\alpha : \alpha < \omega_1 \rangle$$

such that each σ_α is a countable set, and for every $A \subset \omega_1$ the set

$$\{\alpha < \omega_1 \mid A \cap \alpha \in \sigma_\alpha\rangle$$

contains a club.

$\diamondsuit^*$ and "$\exists A \in NS^+_{\omega_1}$ such that the restriction of NS_{ω_1} to A is $\aleph_1$-dense" are each $\Sigma^2_2(NS_{\omega_1})$ statements consistent (from large cardinals) with CH but not with each other.

The *Stationary Set Splitting Game* is a game of length ω_1 in which players I and II build subsets of ω_1, A and B, respectively.

I wins if A is stationary and one of $A \cap B$ and $A \setminus B$ is not stationary.

(Larson-Shelah) Each of I and II can be forced to have a winning strategy, along with $\diamond$ holding on every stationary subset of ω_1