Definability in Differential Fields

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General Remarks

- Let $\mathcal{M} = (M, f_i, R_j)$ be a structure.
- Model Theory deals with $\mathcal{M}$-definable subsets of cartesian powers of M and definable interactions between them.
- RM (Morley Rank) is certain dimension on definable sets.
- It is often important to understand the structure of $\mathcal{M}$-definable groups.

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Algebraically Closed Field

Example

- Let $\mathcal{M} = (\mathbb{C}, +, \cdot)$.
- $\mathcal{M}$ has quantifier elimination, i.e. all definable sets are boolean combinations of Algebraic Varieties: solutions of systems of polynomial (algebraic) equations.
- Definable subsets of $\mathbb{C}$ are finite or cofinite, so $\mathrm{RM}(\mathbb{C}) = 1$.
- Definable Groups = Algebraic Groups: algebraic varieties with an algebraic group operation.

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Differentially Closed Field

Example

- Let $\mathcal{M} = (K, \partial)$, where K is a field, ∂ is a derivation and $\mathcal{M}$ is differentially closed.
- $\mathcal{M}$ has quantifier elimination, i.e. all definable sets are boolean combinations of Differential Algebraic Varieties: sets of solutions of systems of differential polynomial equations.
- $\ker(\partial) \subsetneq \ker(\partial^2) \subsetneq \ldots \subsetneq \ker(\partial^n) \subsetneq K$.
- $\mathrm{RM}(\ker(\partial)) = 1, \mathrm{RM}(\ker(\partial^2)) = 2, \ldots, \mathrm{RM}(K) = \omega$.
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Real Closed Field

Example

- Let $\mathcal{M} = (\mathbb{R}, +, \cdot)$.
- $\mathcal{M}$ has quantifier elimination down to $(\mathbb{R}, +, \cdot, <)$, i.e. definable sets are boolean combinations of sets of solutions of systems of polynomial equations and inequalities.
- $\mathbb{R} = \bigcup (n, n+1]$.
- $(0, 1] = \bigcup (\frac{1}{n+1}, \frac{1}{n}]$.
- These intervals can be further definably subdivided. Hence no Morley Rank can be attached to $\mathbb{R}$ or

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Algebraic Groups

- Fix a differential field $(K, +, \cdot, \partial)$ with $C := \ker(\partial)$.
- Fix an algebraic group G over K.

Example

- $G = (K, +)$,
- $G = (K, \cdot)$,
- $G = \mathrm{GL}_n(K)$,
- $G = E$ – an elliptic curve,
- $G = A$ – an abelian variety (e.g. $A = E^n$)

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Differential subgroups of algebraic groups

We are interested in differential subgroups of G.

Example

- $(C, +) < (K, +)$ or $(C^*, \cdot) < (K^*, \cdot)$,
- The same for any algebraic group G defined over C – we can take $G(C)$, the group of its C-points which is a differential subgroup. E.g. $\mathrm{GL}_n(C) < \mathrm{GL}_n(K)$.

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Logarithmic Derivative

Definition

Consider

$$l\partial : (K, \cdot) \to (K, +), \quad l\partial(x) := \frac{\partial x}{x}$$

$l\partial$ is called logarithmic derivative.

Remark

- $\frac{\partial(xy)}{xy} = \frac{\partial(x)y + x\partial(y)}{xy} = \frac{\partial x}{x} + \frac{\partial y}{y}$.
- $l\partial$ is a differential epimorphisms.
- There are no algebraic epimorphisms from $(K, \cdot)$ to $(K, +)$!

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Zilber Trichotomy

Zilber Trichotomy in differential fields

Zilber Trichotomy holds in (K, ∂) i.e. for each definable set X, if $\mathrm{RM}(X) = 1$, then X as a structure is one of the following:

1. Algebraic curve over C,
2. Vector space,
3. Set with no structure.

Definable Groups

Let H be a differential algebraic group of finite RM. Using Zilber Trichotomy, H can be analyzed in terms of groups of the form $G(C)$ and vector spaces.

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Motivations

Why differential algebraic subgroups of algebraic groups are interesting?

1. Diophantine geometry (the next section),
2. Intersections with tori, bad field construction (the third section),
3. Differential Galois theory (not covered).

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From diophantine equation to algebraic groups

- We want to solve a diophantine equation, which means e.g. to find rational solutions of $X^7 + Y^7 = Z^7$.
- We consider a curve $V \subset \mathbb{P}^2(\mathbb{C})$ defined by $X^7 + Y^7 = Z^7$ and want to find $V(\mathbb{Q})$: the set of its rational points.
- V algebraically embeds into $A = J(V)$ – the Jacobian of V, a certain algebraic group.
- $A(\mathbb{Q})$ is finitely generated.
- We are interested in $V(\mathbb{Q}) = V \cap A(\mathbb{Q})$.
- In general, we are interested in intersections of finitely generated subgroups of A with its algebraic subvarieties.

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Diophantine geometry

Set-up in diophantine geometry

- $(K, +, \cdot)$ – algebraically closed field.
- A – commutative algebraic group.
- $\Gamma < A$ – finitely generated subgroup.
- $V \subset A$ – algebraic subvariety.
- We want to analyze $V \cap \Gamma$.

Problem

Γ is not definable in any reasonable fashion.

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Differential Algebraic Groups and Mordell-Lang

Solution

- Expand K by ∂ such that (K, ∂) is differentially closed.
- There is a differential subgroup $G < A$ such that:
  - $\text{RM}(G) < \omega$,
  - $\Gamma \subset G$.
- Using Zilber's trichotomy we can analyze $G \cap V$.
- These ideas were used by Hrushovski in his proof of the geometric Mordell-Lang conjecture.
- After replacing a derivation by its positive characteristic analogue, Hrushovski gave a proof of the positive characteristic Mordell-Lang. This is the only proof known.

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Let $x_1, \ldots, x_n \in \mathbb{C}$ be linearly independent over $\mathbb{Q}$. Then

$$\operatorname{trdeg}_{\mathbb{Q}}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geqslant n.$$

Remark

- Lindemann–Weierstrass: Schanuel Conjecture for $n = 1$.
- Schanuel Conjecture is open for $n \geqslant 2$.
- It is unknown if e and π are algebraically independent, which follows from Schanuel Conjecture for $n = 2$.

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Ax Theorem

Differential Equation of the Exponential Map

Since $\frac{(e^x)'}{e^x} = x'$, there was a hope that Schanuel Conjecture is related with the set of solutions of the differential equation $\frac{\partial y}{y} = \partial x$ in a differential field (K, ∂).

Ax Theorem

Let $x_1, y_1, \ldots, x_n, y_n \in K$ such that:

$$\partial(x_1) = \frac{\partial(y_1)}{y_1}, \ldots, \partial(x_n) = \frac{\partial(y_n)}{y_n}$$

and $\partial(x_1), \ldots, \partial(x_n)$ are $\mathbb{Q}$-linearly independent. Then

$$\mathrm{trdeg}_C(x_1, \ldots, x_n, y_1, \ldots, y_n) \geqslant n + 1.$$

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Consequences of Ax Theorem

- Unfortunately Ax Theorem does not imply Schanuel Conjecture.
- But it implies Weak CIT – a finiteness statement about intersecting tori with algebraic varieties.
- Weak CIT was crucial in the constructions of bad field.
- Bad field is a field K of Morley Rank 2 having a definable subgroup of K^* of Morley Rank 1 (Poizat; A. Baudisch, M. Hills, A. Martin-Pizarro and F. Wagner).

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Generalizations of Ax Theorem

- Ax Theorem is a theorem about solutions of the differential equation related to $\exp : \mathbb{C} \to \mathbb{C}^*$.
- Kirby and later Bertrand proved generalizations of Ax theorem for $\exp : \mathrm{Lie}(A) \to A$ for certain algebraic groups A.
- I generalized it further to any local analytic "very non-algebraic" map between A, B – algebraic groups over $\mathbb{C}$. It includes:
  - $\exp : \mathrm{Lie}(A) \to A$ as in Bertrand, Kirby.
  - $\mathbb{C}^* \ni x \mapsto x^\alpha \in \mathbb{C}^*$ (α non-algebraic).

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Positive Characteristic Project

- The statement of Ax also makes sense in positive characteristic. One has to:
  - Replace derivations with HS-derivations (a positive characteristic analogue of a derivation).
  - Replace local analytic maps with formal maps (forget about convergence of power series)
- It still needs a proof.
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Travel grants will be paid today in cash after lunch break. It refers to:

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Social Events

You can still sign up for social events!! Especially the excursion and sightseeing.

- Sightseeing: Tuesday, 13.00 Departure from the conference site (before lunch).
- Excursion: Tuesday, 14.00 Departure by bus (after lunch).
- Boat Party: Tuesday, 18.30 The boat will depart from the marina near Hala Targowa (have a look at pictures at the conference web page).
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