

Sets and the Concept of Set

Donald A. Martin

UCLA

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Outline of talk

- (1) The concept of set.
- (2) Axioms for set theory and the negative thesis.
- (3) The positive thesis and its consequences.

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Basic Question. Does the concept of set determine what object the set with given members has to be?

The iterative set hierarchy

Stage 0 $V_0 = \mathbb{N}$

Stage 1 $V_1 = \mathbb{N} \cup \mathcal{P}(\mathbb{N}) = V_0 \cup \mathcal{P}(V_0)$

Stage 2 $V_2 = V_1 \cup \mathcal{P}(V_1)$

Etc. Through the transfinite

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Informal Separation Axiom. Let x be a set. For any property P , there is a set whose members are those members of x that have property P .

Power Set Axiom. Let x be a set. There is a set whose members are the subsets of x .

Union Axiom. Let x be a set. There is a set whose members are the members of members of x .

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0th Thesis. *The iterative concept of set is not—or, at least, is not merely—a concept of what it is for an object to be a set. It is a concept of what it is to be a universe of sets.*

1st Thesis. *The Informal Separation Axiom is sharp enough to guarantee that any two candidates for V_1 (for the natural numbers and the sets of natural numbers) are isomorphic, that any two candidates for V_2 are isomorphic, and so one for V_3 , V_4 , etc. Indeed, the iterative concept of set is sharp enough that any two instantiations of it are isomorphic.*

2nd Thesis. *In our present state of knowledge, it is an open question whether the Informal Separation Axiom (or even the iterative concept of set) is sharp enough to determine a truth-value for the Continuum Hypothesis.*

What about the the Natural Numbers?

Basic Question for Natural Numbers No

0th Thesis for Natural Numbers Yes

1st Thesis for Natural Numbers Yes

2nd Thesis for Natural Numbers No

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ZFC axiom systems

- ▶ Informal ZFC axioms
- ▶ First-order ZFC axioms
- ▶ Second order ZFC axioms

ZFC axioms based on plural logic

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Large cardinal axioms

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- ▶ Large cardinal axioms imply nothing about the Continuum Hypothesis, which is a statement about V_2 .

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Isomorphism. any two structures instantiating the concept of set have definably isomorphic V_2 's.

$$\mathfrak{M}_1 \quad \mathfrak{M}_2$$

$$V_0(\mathfrak{M}_1) \cong^{\pi_0} V_0(\mathfrak{M}_2)$$

$$V_1(\mathfrak{M}_1) \cong^{\pi_1} V_1(\mathfrak{M}_2)$$

$$V_2(\mathfrak{M}_1) \cong^{\pi_2} V_2(\mathfrak{M}_2)$$

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1st Thesis for V_2 + 2nd Thesis \Rightarrow

We do not at present know that there is a structure instantiating the concept of set.