Substructural Fuzzy Logics

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Fuzzy Logics / Substructural Logics

Roughly speaking...

- *Fuzzy logics* have truth values in $[0, 1]$ and connectives interpreted by real-valued functions.

- *Substructural logics* are obtained by removing/adding structural rules in Gentzen systems.

...but are fuzzy logics substructural?

t-Norm Fuzzy Logics

The best known fuzzy logics have truth values in $[0,1]$ and interpret conjunction and implication connectives by t-norms and their residua, e.g.

t-NORM	$x * y$	$x \rightarrow_* y$
ŁUKASIEWICZ	$\max(0, x + y - 1)$	$\min(1, 1 - x + y)$
GÖDEL	$\min(x, y)$	y if $x > y$; 1 o/w
PRODUCT	$x \cdot y$	y/x if $x > y$; 1 o/w

Valid formulas are those which always take value 1.

Uninorm Fuzzy Logics

More generally, fuzzy logics can be based on *uninorms* (commutative associative increasing binary functions on $[0,1]$ with a unit element); e.g.

$$x * y \ = \ \begin{cases} \min(x,y) & \text{if } x+y \leq 1 \\ \max(x,y) & \text{otherwise} \end{cases}$$

and their residua, defined as:

$$x \rightarrow_* y = \sup\{z \in [0,1] : x * z \leq y\}$$

Standard Algebras

A fuzzy logic $\mathbf{L}$ is based on "standard L-algebras":

$$\langle [0,1], *, \rightarrow_*, \min, \max, e_*, 0, 1 \rangle$$

where $*$ is a uninorm with residuum $\rightarrow_*$ and unit e_*.

LOGIC	UNINORMS
Uninorm Logic $\mathbf{UL}$	left-continuous uninorms
Monoidal t-norm logic $\mathbf{MTL}$	left-continuous t-norms
Basic Logic $\mathbf{BL}$	continuous t-norms
Gödel Logic $\mathbf{G}$	idempotent t-norms

$\models_{\mathrm{L}} A$ iff A is $\geq e$ in all standard L-algebras.

Sequents

A (single-conclusion) sequent is an ordered pair:

$$\Gamma \Rightarrow \Delta$$

where Γ is a finite multiset of formulas and Δ is a multiset containing at most one formula.

We write Γ, Π and Γ, A for $\Gamma \uplus \Pi$ and $\Gamma \uplus \{A\}$.

A Simple Sequent Calculus

Initial Sequents

$$\frac{}{A \Rightarrow A}\,(\mathrm{id})$$

Logical Rules

$$\frac{\Gamma \Rightarrow A \quad \Pi, B \Rightarrow \Delta}{\Gamma, \Pi, A \rightarrow B \Rightarrow \Delta}\,(\rightarrow\Rightarrow) \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}\,(\Rightarrow\rightarrow)$$

Cut Rule

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Pi \Rightarrow A}{\Gamma, \Pi \Rightarrow \Delta}\,(\mathrm{cut})$$

Some Structural Rules

Weakening

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\,(\mathrm{wl}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\,(\mathrm{wr})$$

Contraction

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\,(\mathrm{cl})$$

Some Substructural Logics

Calculus	Weakening	Contraction
GMAILL		
GAMAILL	$\times$	
GIL	$\times$	$\times$

Hypersequents

A hypersequent $\mathcal{G}$ is a finite multiset of sequents:

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

From Sequents to Hypersequents

The *hypersequent version* of a sequent rule adds a "context side-hypersequent" $\mathcal{G}$; e.g.

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow A \quad \mathcal{G} \mid \Pi, B \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Pi, A \rightarrow B \Rightarrow \Delta}\,(\rightarrow\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma, A \Rightarrow B}{\mathcal{G} \mid \Gamma \Rightarrow A \rightarrow B}\,(\Rightarrow\rightarrow)$$

A Transfer Principle

Take the initial sequents and hypersequent versions of the rules of a sequent calculus, and add:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\,(\text{ew}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\,(\text{ec})$$

and the “communication rule”:

$$\frac{\mathcal{G} \mid \Gamma_1, \Pi_1 \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma_2, \Pi_2 \Rightarrow \Sigma}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta \mid \Pi_1, \Pi_2 \Rightarrow \Sigma} \text{(com)}$$

Transferred Calculi

Sequent Calculus		Hypersequent Calculus
GMAILL	$\Rightarrow$	**GUL**
GAMAILL	$\Rightarrow$	**GMTL**
GIL	$\Rightarrow$	**GG**

Soundness and Completeness

We want to show that for a calculus $\mathbf{GL}$:

$$\vdash_{\mathbf{GL}} \Rightarrow A \quad \text{iff} \quad \models_{\mathrm{L}} A$$

We adopt the following strategy:

(1) Let $\mathbf{GL}^{\mathrm{D}}$ be $\mathbf{GL}$ plus a "density" rule and prove:

$$\vdash_{\mathbf{GL}^{\mathrm{D}}} \Rightarrow A \quad \text{iff} \quad \models_{\mathrm{L}} A$$

(2) Establish "density elimination" for $\mathbf{GL}^{\mathrm{D}}$.

The Takeuti-Titani Density Rule

Let $\mathbf{GL}^{\mathrm{D}}$ be $\mathbf{GL}$ extended with:

$$\frac{\mathcal{G} \mid \Gamma, p \Rightarrow \Delta \mid \Pi \Rightarrow p}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \Delta} \text{(density)}$$

where p does not occur in the conclusion.

Under very general conditions, we get:

$$\vdash_{\mathbf{GL}^{\mathrm{D}}} \Rightarrow A \quad \text{iff} \quad \models_{\mathrm{L}} A$$

Density Elimination

Suppose that we have a proof ending in:

$$\dfrac{\dfrac{\vdots}{\Gamma, p \Rightarrow \Delta \mid \Pi \Rightarrow p}}{\Gamma, \Pi \Rightarrow \Delta}\ (\text{density})$$

Replace p on the *left* by Π; on the *right* by Γ and Δ

$$\dfrac{\dfrac{\vdots}{\Gamma, \Pi \Rightarrow \Delta \mid \Gamma, \Pi \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Delta}\ (\text{ec})$$

Make some adjustments to get a (density)-free proof...

Putting Things Together

$\Rightarrow A$ is derivable in $\mathbf{GL}$...

...iff $\Rightarrow A$ is derivable in $\mathbf{GL}^{\mathbf{D}}$...

...iff A is valid in all standard L-algebras.

Uniform Conditions

Single-conclusion hypersequent calculi with weakening admit cut and density elimination if they have:

(a) *Substitutive rules* (making substitutions in a rule instance gives an admissible rule).

(b) *Reductive logical rules* (applications of (cut) can be shifted upwards over logical rules).

These conditions guarantee standard completeness.

Concluding Remarks

- Many fuzzy logics occur naturally as substructural logics in the framework of hypersequents.

- Syntactic conditions guarantee “standard completeness” via cut and density elimination.

- We are investigating more general conditions for density elimination.

References

Substructural Fuzzy Logics. George Metcalfe and Franco Montagna. To Appear in *Journal of Symbolic Logic*.

Density Elimination and Rational Completeness for First-Order Logics. Agata Ciabattoni and George Metcalfe. In *Proceedings of LFCS 2007, volume 4514 of LNCS, pages 132-146, 2007*.