Around splitting and reaping number for partitions of ω

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Motivation

- The combinatorial structure of $\left(\wp(\omega)/fin, \leq_{fin}\right)$ is described by cardinal invariants of the continuum.

- We can define analogous cardinal invariants describing properties of $(\wp(\omega)/\mathcal{I}, \leq_{\mathcal{I}})$, $(Dense(\mathbb{Q})/nwd \leq_{nwd})$, $((\omega)^{\omega}, \leq^{*})$ or $((\omega)^{\omega}, {}^{*}\geq)$.

Program: Compare these cardinal invariants to investigate the similarities and differences between these structures.

In this talk, we focus on the reaping number and splitting number on $\left(\wp(\omega)/fin, \leq_{fin}\right)$ and $((\omega)^{\omega}, \leq^{*})$.

$\boxed{(\omega)}$

Definition 1.

$X \subset \wp(\omega)$ *is a* **partition** *of* ω *if* $\forall x, y \in X(x \neq y \to x \cap y = \emptyset)$ *and* $\bigcup X = \omega$.

$$(\omega) = \{X \subset \wp(\omega) : X \text{ is a partition of } \omega\}.$$

For $X, Y \in (\omega)$ X *is* **coarser** *than* Y $(X \leq Y)$ *if*

$$\forall x \in X \exists Y' \subset Y(x = \bigcup Y').$$

Proposition 2. $((\omega), \leq)$ *is a lattice.*

$$X \wedge Y = \textbf{infinimum} \text{ of } X \text{ and } Y.$$

$\boxed{(\omega)^\omega}$

$$\begin{aligned}(\omega)^\omega &= \{X \in (\omega) : |X| = \aleph_0\} \\ (\omega)^{<\omega} &= \{X \in (\omega) : |X| < \aleph_0\}.\end{aligned}$$

Definition 3. *Let* $X, Y \in (\omega)^\omega$.

X *is* **almost coarser** *than* Y $(X \leq^* Y)$ *if* $\forall^\infty x \in X \exists Y'(x = \bigcup Y')$.

X *and* Y *are* **compatible** $(X \| Y)$ *if* $X \wedge Y \in (\omega)^\omega$.

X *and* Y *are* **incompatible** $(X \perp Y)$ *if* $X \wedge Y \in (\omega)^{<\omega}$

$\boxed{\text{splitting family and reaping family}}$

Definition 4.

For $X, Y \in (\omega)^\omega$ X **dual-splits** Y *if* $X \wedge Y \in (\omega)^\omega$ *and* $Y \not\leq^* X$.

$\mathcal{S} \subset (\omega)^\omega$ *is a* **dual-splitting family** *if* $\forall Y \in (\omega)^\omega \exists X \in \mathcal{S}(X$ *dual-splits* $Y)$.

$$\mathfrak{s}_d = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a dual-splitting family}\}.$$

$\mathcal{R} \subset (\omega)^\omega$ *is a* **dual-reaping family** *if*

$$\forall Y \in (\omega)^\omega \exists X \in \mathcal{R}\ (Y \text{ doesn't dual-split } X).$$

$$\mathfrak{r}_d = \min\{|\mathcal{R}| : \mathcal{R} \text{ is a dual-reaping family}\}.$$

$\boxed{\text{Cichoń's diagram}}$

Let $\mathcal{M}$ be the meager ideal. Let $\mathcal{N}$ be the null ideal. Then the following relation holds.

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[d] & \mathrm{non}(\mathcal{M}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathcal{M}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathcal{N}) \arrow[l] \arrow[d] \\
\mathrm{add}(\mathcal{N}) & \mathfrak{b} \arrow[l] \arrow[d] & \mathfrak{d} \arrow[l] \arrow[d] & \\
 & \mathrm{add}(\mathcal{M}) \arrow[l] & \mathrm{cov}(\mathcal{M}) \arrow[l] & \mathrm{non}(\mathcal{N}) \arrow[l]
\end{tikzcd} .$$

Cichoń's diagram

$\kappa \to \lambda$ means $\kappa \geq \lambda$ is provable in ZFC.

$\mathfrak{r}_d$, $\mathfrak{s}_d$ and cardinal invariants in Cichoń's diagram

Theorem 5.

(i) (Cichoń, Krawczyk, Majcher-Iwanow, Węglorz)

$$\mathfrak{s}_d \geq cov(\mathcal{M}).$$

(ii) $\mathfrak{s}_d \geq cov(\mathcal{N}).$ $\mathfrak{r}_d \leq non(\mathcal{M}), non(\mathcal{N}).$

(iii) (Kamo) $\mathfrak{r}_d \leq \mathfrak{d}.$ $\mathfrak{s}_d \geq \mathfrak{b}.$

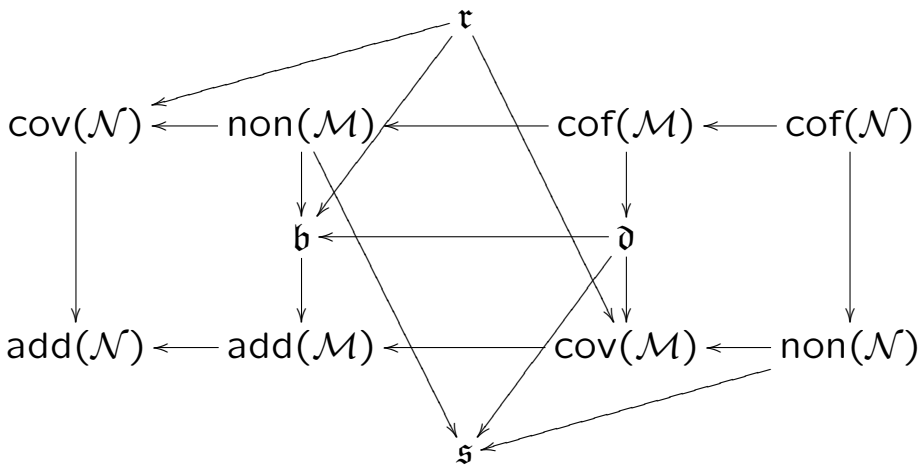
$\mathfrak{r}_d$, $\mathfrak{s}_d$ and Cichoń's diagram

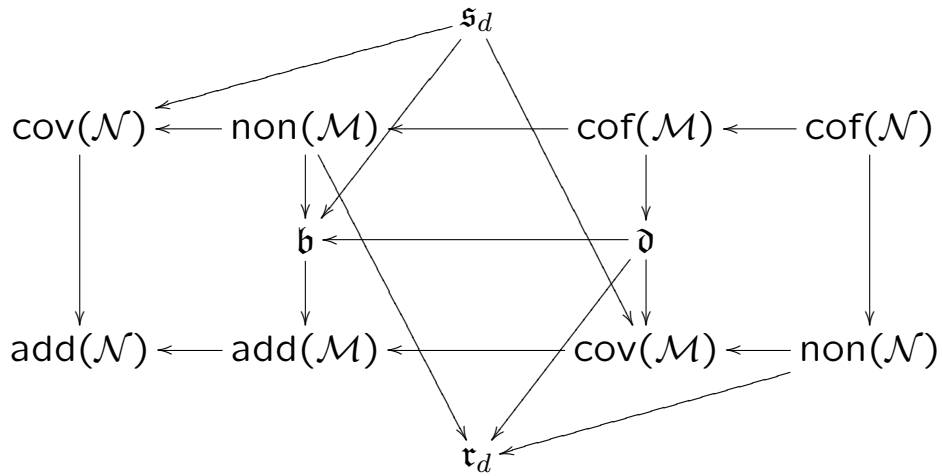
$$
\begin{tikzcd}
 & & \mathfrak{s}_d \arrow[dll] \arrow[dl] \arrow[ddr] & & \\
\mathrm{cov}(\mathcal{N}) \arrow[dd] & \mathrm{non}(\mathcal{M}) \arrow[l] \arrow[d] \arrow[ddr] & & \mathrm{cof}(\mathcal{M}) \arrow[ll] \arrow[d] & \mathrm{cof}(\mathcal{N}) \arrow[l] \arrow[dd] \\
 & \mathfrak{b} \arrow[d] & & \mathfrak{d} \arrow[ll] \arrow[d] \arrow[dl] & \\
\mathrm{add}(\mathcal{N}) & \mathrm{add}(\mathcal{M}) \arrow[l] & & \mathrm{cov}(\mathcal{M}) \arrow[ll] & \mathrm{non}(\mathcal{N}) \arrow[l] \arrow[dll] \\
 & & \mathfrak{r}_d \arrow[ull] & &
\end{tikzcd}
$$

$\kappa \to \lambda$ means $\kappa \geq \lambda$ is provable in ZFC.

$\mathfrak{r}$, $\mathfrak{s}$ and Cichoń's diagram

Let $\mathfrak{r}$ and $\mathfrak{s}$ be the reaping number and splitting number for $\left(\wp(\omega)/fin, \leq_{fin}\right)$ respectively.





$\boxed{\mathfrak{r},\ \mathfrak{s},\ \mathfrak{r}_d,\ \mathfrak{s}_d \text{ and Cichoń's diagram}}$

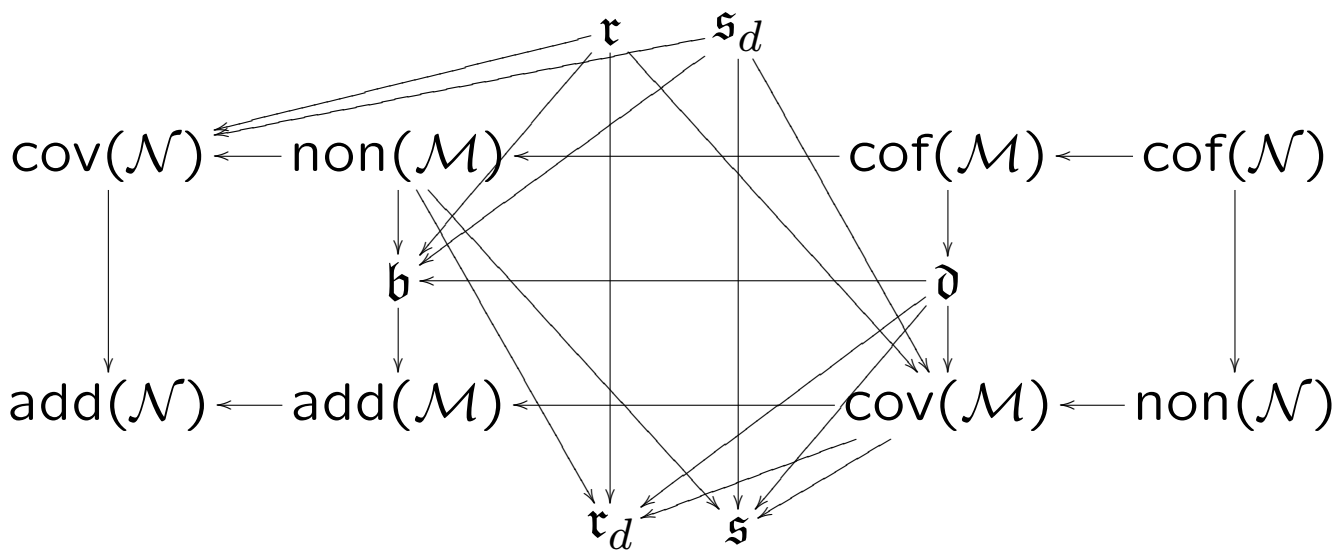
Theorem 6.

(1) (Cichoń, Krawczyk, Majcher-Iwanow, Węglorz)

$$\mathfrak{r}_d \leq \mathfrak{r}.$$

(2) (Kamburelis and Węglorz)

$$\mathfrak{s}_d \geq \mathfrak{s}.$$



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$\boxed{\mathfrak{s},\ \mathfrak{r},\ \mathfrak{s}_d \text{ and } \mathfrak{r}_d}$

Theorem 7 (Blass and Shelah). $Con(\mathfrak{u} < \mathfrak{s})$.

So $Con(\mathfrak{r}_d < \mathfrak{s})$. $Con(\mathfrak{r} < \mathfrak{s}_d)$.

Let $\mathbb{DS}$ be a c.c.c forcing notion introduced by Cichoń, Krawczyk, Majcher-Iwanow and Węglorz.

Theorem 8. *(Cichoń, Krawczyk, Majcher-Iwanow and Węglorz) If $V \models CH$, then $V^{\mathbb{DS}_{\omega_2}} \models \mathfrak{r}_d = \mathfrak{c}$.*

Since $\mathbb{DS}$ is a Suslin c.c.c, $V^{\mathbb{DS}_{\omega_2}} \models \mathfrak{s} = \omega_1$. Hence $Con(\mathfrak{s} < \mathfrak{r}_{\mathfrak{d}})$.

Theorem 9. *(Minami) If $V \models MA$, then $V^{\mathbb{DS}_{\omega 1}} \models \mathfrak{r} > \mathfrak{s}_d$. So $Con(\mathfrak{r} > \mathfrak{s}_d)$.*

Question 9.1. $Con(\mathfrak{s}_d < \mathfrak{r}_d)$?

$\mathfrak{r}_d$, $\mathfrak{s}_d$ and Cichoń's diagram 2

$$
\begin{tikzcd}
 & & & \mathfrak{s}_d \arrow[dll] \arrow[dl] \arrow[dr] & & \\
\mathrm{cov}(\mathcal{N}) \arrow[dd] & \mathrm{non}(\mathcal{M}) \arrow[l] \arrow[d] \arrow[ddr] & & & \mathrm{cof}(\mathcal{M}) \arrow[ll] \arrow[d] & \mathrm{cof}(\mathcal{N}) \arrow[l] \arrow[dd] \\
 & \mathfrak{b} \arrow[d] & & & \mathfrak{d} \arrow[ll] \arrow[d] \arrow[dl] & \\
\mathrm{add}(\mathcal{N}) & \mathrm{add}(\mathcal{M}) \arrow[l] \arrow[ddr] & & & \mathrm{cov}(\mathcal{M}) \arrow[ll] & \mathrm{non}(\mathcal{N}) \arrow[l] \arrow[dll] \\
 & & & \mathfrak{r}_d & &
\end{tikzcd}
$$

Theorem 10 (Brendle).
If $V \models CH$, *then* $V^{\mathbb{DS}_{\omega_2}} \models \mathfrak{r}_d > \mathfrak{b}$.
If $V \models MA$, *then* $V^{\mathbb{DS}_{\omega_1}} \models \mathfrak{s}_d < \mathfrak{d}$.

Question 10.1. $\mathfrak{s}_d \leq$ *cof*$(\mathcal{M})$*?* $\mathfrak{r}_d \geq$ *add*$(\mathcal{M})$*?*

Hechler forcing and $\mathfrak{r}_d$ and $\mathfrak{s}_d$

By $\mathbb{D}$ we denote the Hechler forcing.

Theorem 11. *(Minami) Let $\mathbb{D}_\alpha$ be the α-stage finite support iteration of Hechler forcing.*

If $V \models CH$, then $V^{\mathbb{D}_{\omega_2}} \models \mathfrak{r}_d = \omega_1 < add(\mathcal{M}) = \omega_2$.

If $V \models MA$, then $V^{\mathbb{D}_{\omega_1}} \models \mathfrak{s}_d = \omega_2 > cof(\mathcal{M}) = \omega_1$.

So $Con(\mathfrak{r}_d < add(\mathcal{M}))$ and $Con(\mathfrak{s}_d > cof(\mathcal{M}))$.