Interpretation of Arithmetic in certain finitely presented groups

Alexey Muranov (joint with Tuna Altınel)

Université Lyon 1, France

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Abstract

The Arithmetic $(\mathbb{N}, +, \times)$ is parametrically interpreted in all groups of Richard Thompson and Graham Higman, and also in some other groups of piecewise linear permutations of $[0, 1)$.
In Tompson's group F, Arithmetic is interpreted without parameters.

Outline

Preliminaries

- Groups of Thompson and Higman
- Interpretation of Arithmetic in groups

Interpretatability

- Definable copies of $\mathbb{Z} \wr \mathbb{Z}$ and undecidability
- Interpretation of Arithmetic in F without parameters

Questions

Basics of Thompson and Higman's groups

Thompson's groups:

- appeared in 1965, now are denoted F, T, and V;
- $F \subset T \subset V$;
- T, V, and $[F,F]$ are simple, $F/[F,F] \cong \mathbb{Z} \oplus \mathbb{Z}$;
- all are finitely presented.

Higman's groups:

- appeared in 1974, are denoted $G_{n,r}$, $n = 2, 3, \dots$, $r = 1, 2, 3, \dots$;
- $G_{2,1} \cong V$;
- if n is even, then $G_{n,r}$ is simple;
  if n is odd, then $[G_{n,r}, G_{n,r}]$ is simple, $G_{n,r}/[G_{n,r}, G_{n,r}] \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$;
- all are finitely presented.

Can be realized by piecewise linear permutations of $[0;1)$.
Elements of F are represented by piecewise linear homeomorphisms $[0;1) \to [0;1)$.

Representation by piecewise linear permutations

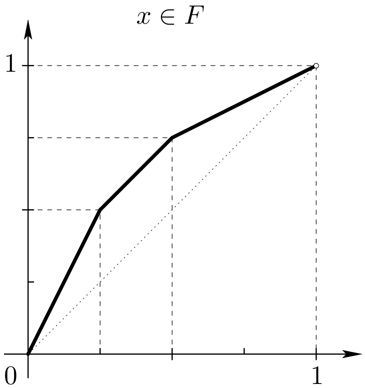


Figure: An element of *F* and an element of *V* as piecewise linear permutations of [0,1).

Arithmetic in virtually solvable groups

Theorem (Noskov, 1983)

A finitely generated virtually solvable group parametrically interprets Arithmetic if and only if it is not virtually abelian.

Remark

In his paper, Noskov actually does not claim to prove this.

F interprets Arithmetic

Theorem (Bardakov, Tolstykh, 2007, `arXiv:math/0701748`)

F has a parametrically definable subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.
Therefore, F parametrically interprets Arithmetic, and $\mathrm{Th}(F)$ is hereditarily undecidable.

The proof is based on results of Yu. Ershov, G. Noskov, R. Robinson.
The idea: find in F a parametrically definable subgroup isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.

($\mathbb{Z} \wr \mathbb{Z}$ is a semi-direct product of $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$ with $\mathbb{Z}$.)

$\mathbb{Z} \wr \mathbb{Z}$ and undecidability

Theorem 1 (Altınel, M.)

All groups of Thompson and Higman have parametrically definable subgroups isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.

Corollary

The elementary theories of these groups are undecidable.

Arithmetic in F

Theorem 2 (Altınel, M.)

The following map $\mathbb{N} \to F/[F,F]$ is an interpretation of Arithmetic in F without parameters:

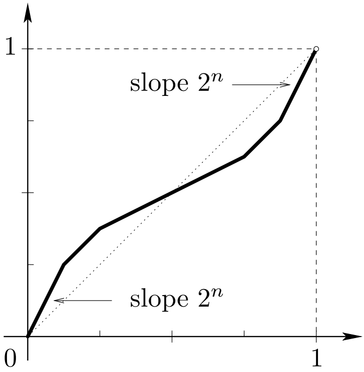
$n \mapsto$  mod $[F,F]$

Figure: Interpretation of $(\mathbb{N},+,\times)$ in $(F,\times)$.

We do not know:

Question 1

Is F parametrically definable in V?

Question 2

Is Arithmetic parametrically bi-interpretable with F?

Remark

F is interpretable in Arithmetic since the word problem for F is decidable.

Remark

Thomas Scanlon established bi-interpretability of infinite finitely generated fields with Arithmetic, and used it to prove Pop's conjecture:

> *two finitely generated fields having the same first-order theories in the language of rings are isomorphic.*